# A LOCAL MONOTONICITY FORMULA FOR AN INHOMOGENEOUS SINGULAR PERTURBATION PROBLEM AND APPLICATIONS. PART II

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ABSTRACT. In this paper we continue with our work in [7] where we developed a local monotonicity formula for solutions to an inhomogeneous singular perturbation problem of interest in combustion theory. There we proved local monotonicity formulae for solutions $u^\varepsilon$ to the singular perturbation problem and for $u = \lim u^\varepsilon$, assuming that both $u^\varepsilon$ and $u$ were defined in an arbitrary domain $\mathcal{D}$ in $\mathbb{R}^{N+1}$.

In the present work we obtain global monotonicity formulae for limit functions $u$ that are globally defined, while $u^\varepsilon$ are not. We derive such global formulae from a local one that we prove here.

In particular, we obtain a global monotonicity formula for blow up limits $u_0$ of limit functions $u$ that are not globally defined. As a consequence of this formula, we characterize blow up limits $u_0$ in terms of the value of a density at the blow up point.

We also present applications of the results in this paper to the study of the regularity of $\partial\{u > 0\}$ (the flame front in combustion models).

The fact that our results hold for the inhomogeneous singular perturbation problem allows a very wide applicability, for instance to problems with nonlocal diffusion and/or transport.

## 1. INTRODUCTION

In this paper we continue with our work in [7] where we developed a local monotonicity formula for solutions to an inhomogeneous singular perturbation problem of interest in combustion theory. That formula was inspired on a global monotonicity formula that G. S. Weiss developed for solutions of the global homogeneous problem (see [10]).

As in [7], the problem under consideration here is the following: for $\varepsilon > 0$ we let $u^\varepsilon$ be a solution to

$$(P_\varepsilon(f_\varepsilon)) \qquad \Delta u^\varepsilon - u_t^\varepsilon = \beta_\varepsilon(u^\varepsilon) + f_\varepsilon \quad \text{in } \mathcal{D}$$

where $\varepsilon > 0$, $\mathcal{D}$ is a domain in $\mathbb{R}^{N+1}$, $f_\varepsilon \in L^\infty(\mathcal{D})$, $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon})$ with $\beta$ a Lipschitz continuous function, $\beta(s) > 0$ for $0 < s < 1$ and $\beta(s) = 0$ otherwise. This type of reaction term appears in the study of the propagation of deflagration flames. In that context $\varepsilon$ represents the inverse of the activation energy (see, for instance, [1], [2], [9] and the references therein).

We are looking at the inhomogeneous equation —this is, we allow $f_\varepsilon \not\equiv 0$— which makes the applicability of our results much wider. In particular, our results apply to more general equations that include nonlocal diffusion and/or transport (see [6], [7], [8] for a discussion and applications).

In [10], Weiss obtained a *global* monotonicity formula for solutions $u^\varepsilon$ of the global homogeneous version of problem $P_\varepsilon(f_\varepsilon)$ (i.e., with $u^\varepsilon$ defined in $\mathbb{R}^N \times (0, T)$ and $f_\varepsilon \equiv 0$), as well as an analogous formula for $u = \lim u^\varepsilon$ ($\varepsilon \to 0$).

In [7], we proved *local* monotonicity formulae for solutions $u^\varepsilon$ to the inhomogeneous problem and for $u = \lim u^\varepsilon$, assuming that both $u^\varepsilon$ and $u$ were defined in an arbitrary domain $\mathcal{D}$ in $\mathbb{R}^{N+1}$.

In the present work we deal with the inhomogeneous problem and we obtain *global* monotonicity formulae for limit functions $u$ that are globally defined (i.e., defined in the whole region $\mathbb{R}^N \times (0, T)$) while $u^\varepsilon$ are not. Such formulae cannot be derived from the ones in the previous papers [10] and [7] that were described above.

In fact, we obtain the first of our *global* monotonicity formulae in Theorem 2.2. We derive such *global* formula from a *local* one that we prove in Theorem 2.1.

We also obtain a *global* monotonicity formula for blow up limits $u_0$ of limit functions $u$ that are not globally defined (here $u_0 = \lim_{\lambda_n \to 0} \frac{1}{\lambda_n} u(x_0 + \lambda_n x, t_0 + \lambda_n^2 t)$ with $(x_0, t_0) \in \partial\{u > 0\}$ and $u = \lim u^\varepsilon$). See Corollary 2.1.

As a consequence of this last formula, we are able to characterize blow up limits $u_0$ in terms of the value of a density at the blow up point $(x_0, t_0)$. Namely, in terms of

$$(1.1) \qquad \delta(x_0, t_0) = \lim_{r \to 0} \frac{1}{r^2} \int_{t_0 - 4r^2}^{t_0 - r^2} \int_{\mathbb{R}^N} \left( |\nabla(u\psi)|^2 + 2\psi^2\chi + \frac{1}{2}\frac{(u\psi)^2}{t - t_0} \right) G(x - x_0, t_0 - t)\, dx\, dt,$$

where $G(x, t) = \frac{1}{(4\pi t)^{N/2}} \exp\left(-\frac{|x|^2}{4t}\right)$, $\chi = \lim B_\varepsilon(u^\varepsilon)$ and $\psi = \psi(x) \in C_0^\infty$, $0 \le \psi \le 1$, $\psi \equiv 1$ in a neighborhood of $x_0$ (this limit exists and it is finite and independent of the cut off function $\psi$, by the local monotonicity formula we proved in [7]).

More precisely, in the stationary case, we prove that if $\delta(x_0, t_0) = 3M$ and $u_0$ is a blow up limit at $(x_0, t_0)$, then $u_0 = \alpha x_1^+ - \gamma x_1^-$ for some $\alpha > 0$ and $\gamma \ge 0$, in some coordinate system (Theorem 3.2). In addition, we show that if $\delta(x_0, t_0) = 6M$, then $u_0 = \alpha|x_1|$ for some $\alpha \ge 0$ (Theorem 3.1). Here $M = \int_0^1 \beta(s)\, ds$. See also Remark 3.2.

Furthermore, we prove that $3M \le \delta(x_0, t_0) \le 6M$ and that $\delta(x_0, t_0)$ is $3M$ or $6M$ almost everywhere on the free boundary. Moreover, in dimension 2, $\delta(x_0, t_0)$ is $3M$ or $6M$ everywhere on the free boundary (Propositions 3.1, 3.2 and 3.3).

In Section 4 we present applications of Theorems 3.2 and 3.1 to the study of the regularity of the boundary of $\{u > 0\}$ for $u = \lim u^\varepsilon$ (the flame front in combustion models). We proved these regularity results in [8].

Let us remark that our global monotonicity formula proven in Theorem 2.2 allows us to show, in the particular case that $\|f_\varepsilon\|_{L^\infty} \to 0$, that limit functions $u$ that are globally defined satisfy inequality (2.12), which is the same one proven in [10] for limit functions of the global homogeneous problem —even though in our case $u^\varepsilon$ are not globally defined and $f_\varepsilon \not\equiv 0$.

In particular, we obtain this same inequality for blow up limits $u_0$ of limit functions $u$ that are not globally defined (see inequality (2.14) in Corollary 2.1).

On the other hand, let us mention that the results on characterization of blow up limits described above are similar to the ones obtained in [10] for the global homogeneous problem but for a different density. The density in [10] —unlike the one given by (1.1)— is defined only for global functions.

We also remark that the applications given in [10] to these results (namely, classification of points in $\partial\{u > 0\}$ and rectifiability of the singular set) are different from the applications we are presenting in Section 4.

We finally want to point out that all the results in this paper are new when $f_\varepsilon \not\equiv 0$, even in the case that $u^\varepsilon$ are globally defined. Moreover, our results are also new when $f_\varepsilon \equiv 0$, in case $u^\varepsilon$ are not globally defined.

An outline of the paper is as follows. In Section 2 we prove the monotonicity formulae. In Section 3 we use these results to characterize blow up limits in terms of the density at the blow up point and Section 4 contains applications to regularity results of $\partial\{u > 0\}$ for $u = \lim u^\varepsilon$.

**Notation.**

Given a point $(\bar{x}, \bar{t}) \in \mathbb{R}^{N+1}$ and $R, R_0 > 0$, we will denote

$$Q_{R,R_0}(\bar{x}, \bar{t}) := B_R(\bar{x}) \times (\bar{t} - 4R_0^2, \bar{t}].$$

We will be considering rescalings of functions in a neighborhood of $(\bar{x}, \bar{t})$ and we will denote

$$v_r(x,t) = \frac{1}{r} v(\bar{x} + rx, \bar{t} + r^2 t) \quad \text{and} \quad v^r(x,t) = v(\bar{x} + rx, \bar{t} + r^2 t).$$

We will say that a function $v$ is in the class $Lip\,(1, 1/2)$ in a domain $\mathcal{D} \subset \mathbb{R}^{N+1}$, if $v$ is bounded and there exists a constant $L = L(\mathcal{D})$ such that

$$|v(x,t) - v(y,\tau)| \le L(|x - y| + |t - \tau|^{\frac{1}{2}})$$

for every $(x,t)$, $(y,\tau)$ in $\mathcal{D}$. The norm in $Lip\,(1, 1/2)$ in $\mathcal{D}$ is

$$\|v\|_{Lip\,(1,1/2)} = \|v\|_{L^\infty(\mathcal{D})} + \sup_{(x,t),(y,\tau)\in\mathcal{D}} \frac{|v(x,t) - v(y,\tau)|}{|x - y| + |t - \tau|^{1/2}}.$$

We will denote by

$$|v|_{Lip\,(1,1/2)} = \sup_{(x,t),(y,\tau)\in\mathcal{D}} \frac{|v(x,t) - v(y,\tau)|}{|x - y| + |t - \tau|^{1/2}}$$

the $Lip\,(1, 1/2)$ seminorm in $\mathcal{D}$.

Finally, we will denote

$$B_\varepsilon(r) = \int_0^r \beta_\varepsilon(s)\,ds, \qquad M = \int_0^1 \beta(s)\,ds$$

and $G(x,t) = \frac{1}{(4\pi t)^{N/2}} \exp(-\frac{|x|^2}{4t})$.

## 2. Monotonicity formulae

In this section we prove monotonicity formulae for solutions $u^\varepsilon$ of problem $P_\varepsilon(f_\varepsilon)$ and for $u = \lim u^\varepsilon$ $(\varepsilon \to 0)$.

In fact, in Theorem 2.1 we prove a *local* monotonicity formula for solutions $u^\varepsilon$ to problem $P_\varepsilon(f_\varepsilon)$ that are defined in bounded domains of $\mathbb{R}^{N+1}$. This formula is an improvement of the one we obtained in Theorem 2.1 in [7].

As a consequence we obtain, in Theorem 2.2, a *global* monotonicity formula for limit functions $u$ that are globally defined (i.e., defined in the whole region $\mathbb{R}^N \times (0, T)$) while $u^\varepsilon$ are not.

In particular, in Corollary 2.1 we obtain a *global* monotonicity formula for blow up limits $u_0$ of limit functions $u$ that are not globally defined (here $u_0 = \lim_{\lambda_n \to 0} \frac{1}{\lambda_n} u(x_0 + \lambda_n x, t_0 + \lambda_n^2 t)$ with $(x_0, t_0) \in \partial\{u > 0\}$ and $u = \lim u^\varepsilon$).

We first prove

**Theorem 2.1** ($\varepsilon$ -**Local Monotonicity Formula**). *Let $u^\varepsilon$ be a solution to $P_\varepsilon(f_\varepsilon)$ in $Q_{R,R_0}(\bar{x},\bar{t})$ where $R_0 \leq R$ and $R > 1$. Let $\psi = \psi(x) \in C_0^\infty(B_R(\bar{x}))$, $0 \leq \psi \leq 1$ and $\psi \equiv 1$ in $B_{R/2}(\bar{x})$. Assume that $f_\varepsilon \in L^\infty(Q_{R,R_0}(\bar{x},\bar{t}))$ and*

$$(2.1) \quad |u^\varepsilon(x,t)| \leq A_1\big(1 + |x - \bar{x}| + |t - \bar{t}|^{1/2}\big) \text{ in } Q_{R,R_0}(\bar{x},\bar{t}), \quad |u^\varepsilon|_{Lip_{(1,1/2)}(Q_{R,R_0}(\bar{x},\bar{t}))} \leq A_2$$

$$\|\nabla\psi\|_{L^\infty(B_R(\bar{x}))} \leq \frac{A_4}{R}, \quad \|D^2\psi\|_{L^\infty(B_R(\bar{x}))} \leq A_4.$$

*For $0 < r \leq R_0$, let*

$$(2.2) \quad \begin{aligned} \mathcal{W}^\varepsilon_{(\bar{x},\bar{t})}(r) &= \mathcal{W}^\varepsilon_{(\bar{x},\bar{t})}(r, u^\varepsilon, \psi) = \\ &= \frac{1}{r^2} \int_{\bar{t}-4r^2}^{\bar{t}-r^2} \int_{\mathbb{R}^N} \Big(|\nabla(u^\varepsilon\psi)|^2 + 2\psi^2 B_\varepsilon(u^\varepsilon) + \frac{1}{2}\frac{(u^\varepsilon\psi)^2}{t - \bar{t}}\Big) G(x - \bar{x}, \bar{t} - t)\, dx\, dt. \end{aligned}$$

*Then,*

$$(2.3) \quad \begin{aligned} \frac{\partial\mathcal{W}^\varepsilon_{(\bar{x},\bar{t})}}{\partial r}(r) &\geq \int_{-4}^{-1} \int_{\mathbb{R}^N} \big(\partial_r w_r^\varepsilon\big)^2 \frac{rG(x,-t)}{-t}\, dx\, dt \\ &\quad - C\|f_\varepsilon\|_{L^\infty(Q_{R,R_0}(\bar{x},\bar{t}))}\Big(1 + r\|f_\varepsilon\|_{L^\infty(Q_{R,R_0}(\bar{x},\bar{t}))}\Big) \\ &\quad - C\|f_\varepsilon\|_{L^\infty(Q_{R,R_0}(\bar{x},\bar{t}))}\frac{|u^\varepsilon(\bar{x},\bar{t})|}{r} - C\Big(1 + \|f_\varepsilon\|_{L^\infty(Q_{R,R_0}(\bar{x},\bar{t}))}\Big)R^2 e^{-C'\frac{R^2}{r^2}}. \end{aligned}$$

*Here $w^\varepsilon(x,t) = \psi(x)u^\varepsilon(x,t)$ and $w_r^\varepsilon(x,t) = \frac{1}{r}w^\varepsilon(\bar{x} + rx, \bar{t} + r^2t)$.*

*The constant $C$ in (2.3) depends only on $A_i$, the dimension $N$, $M_1 = \max_{0 \leq s \leq 1} s\beta(s)$ and $M = \int_0^1 \beta(s)\, ds$. The constant $C'$ is universal.*

*Proof.* By rescaling we get, for $0 < r \leq R_0$,

$$(2.4) \quad \mathcal{W}^\varepsilon_{(\bar{x},\bar{t})}(r) = \int_{-4}^{-1} \int_{\mathbb{R}^N} \Big(|\nabla w_r^\varepsilon|^2 + 2(\psi^r)^2 B_\varepsilon(ru_r^\varepsilon) + \frac{1}{2}\frac{(w_r^\varepsilon)^2}{t}\Big) G(x,-t)\, dx\, dt.$$

Proceeding as in Theorem 2.1 in [7], we get

$$(2.5) \quad \begin{aligned} \frac{\partial\mathcal{W}^\varepsilon_{(\bar{x},\bar{t})}}{\partial r}(r) &\geq \int_{-4}^{-1} \int_{\mathbb{R}^N} (\partial_r w_r^\varepsilon)^2 \frac{rG(x,-t)}{-t}\, dx\, dt+ \\ &\quad + \int_{-4}^{-1} \int_{\mathbb{R}^N} (\partial_r w_r^\varepsilon)(-2\psi^r r f_\varepsilon{}^r)G(x,-t)\, dx\, dt+ \\ &\quad + \int_{-4}^{-1} \int_{\mathbb{R}^N} (\partial_r w_r^\varepsilon)(-2u_r^\varepsilon\Delta\psi^r - 4\nabla\psi^r\nabla u_r^\varepsilon)G(x,-t)\, dx\, dt+ \\ &\quad + \int_{-4}^{-1} \int_{\mathbb{R}^N} \big(-2\psi^r\beta_\varepsilon(ru_r^\varepsilon)ru_r^\varepsilon\partial_r\psi^r + 4\psi^r\partial_r\psi^r B_\varepsilon(ru_r^\varepsilon)\big)G(x,-t)\, dx\, dt = \\ &= \int_{-4}^{-1} \int_{\mathbb{R}^N} (\partial_r w_r^\varepsilon)^2 \frac{rG(x,-t)}{-t}\, dx\, dt + I + II + III. \end{aligned}$$

Now,

$$\partial_r w_r^\varepsilon(x,t) = -\frac{w^\varepsilon(\bar{x} + rx, \bar{t} + r^2t)}{r^2} + \frac{\nabla w^\varepsilon(\bar{x} + rx, \bar{t} + r^2t)}{r} \cdot x + 2tw_t^\varepsilon(\bar{x} + rx, \bar{t} + r^2t).$$

There holds,

$$|w^\varepsilon(\bar{x} + rx, \bar{t} + r^2t)| \leq |w^\varepsilon(\bar{x},\bar{t})| + (A_1A_4 + A_2)\,(|x| + |t|^{1/2})\,r,$$

and

$$\left|\left(-\frac{w^\varepsilon(\bar{x}+rx,\bar{t}+r^2t)}{r^2}+\frac{\nabla w^\varepsilon(\bar{x}+rx,\bar{t}+r^2t)}{r}\cdot x\right)\left(-2\psi^r r f_\varepsilon{}^r\right)G(x,-t)\right|\le$$

$$\le\left(2\frac{|u^\varepsilon(\bar{x},\bar{t})|}{r}+2(5A_1A_4+2A_2)\left(|x|+|t|^{1/2}\right)\right)\|f_\varepsilon\|_{L^\infty}\,G(x,-t).$$

So that,

$$\int_{-4}^{-1}\int_{\mathbb{R}^N}\left(-\frac{w^\varepsilon(\bar{x}+rx,\bar{t}+r^2t)}{r^2}+\frac{\nabla w^\varepsilon(\bar{x}+rx,\bar{t}+r^2t)}{r}\cdot x\right)(-2\psi^r r f_\varepsilon{}^r))G(x,-t)\,dx\,dt\ge$$

$$\ge-C_2\|f_\varepsilon\|_{L^\infty(Q_{R,R_0}(\bar{x},\bar{t}))}\frac{|u^\varepsilon(\bar{x},\bar{t})|}{r}-\bar{C}_1\|f_\varepsilon\|_{L^\infty(Q_{R,R_0}(\bar{x},\bar{t}))}.$$

Now, as in [7],

$$\left|\int_{-4}^{-1}\int_{\mathbb{R}^N}\left(2tw_t^\varepsilon(\bar{x}+rx,\bar{t}+r^2t)(-2\psi^r r f_\varepsilon{}^r)\right)G(x,-t)\,dx\,dt\right|\le$$

$$\le 32\|f_\varepsilon\|_{L^\infty(Q_{R,R_0}(\bar{x},\bar{t}))}\left(\int_{-4}^{-1}\int_{\mathbb{R}^N}|\partial_t(u_r^\varepsilon)|^2(\psi^r)^2G(x,-t)\,dx\,dt\right)^{1/2}.$$

Let $v=u_r^\varepsilon$. Then $v$ is a solution to $\Delta v-v_t=\beta_{\varepsilon/r}(v)+rf_\varepsilon{}^r$. Thus,

$$\int_{-4}^{-1}\int_{\mathbb{R}^N}v_t^2(\psi^r)^2G(x,-t)\,dx\,dt=\int_{-4}^{-1}\int_{\mathbb{R}^N}v_t\Delta v(\psi^r)^2G(x,-t)\,dx\,dt-$$

$$-\int_{-4}^{-1}\int_{\mathbb{R}^N}\beta_{\varepsilon/r}(v)\,v_t\,(\psi^r)^2G(x,-t)\,dx\,dt-\int_{-4}^{-1}\int_{\mathbb{R}^N}rf_\varepsilon{}^r v_t(\psi^r)^2G(x,-t)\,dx\,dt=$$

$$=i)+ii)+iii).$$

There holds,

$$i)=-\int_{-4}^{-1}\int_{\mathbb{R}^N}\nabla v_t\nabla v\,(\psi^r)^2G(x,-t)\,dx\,dt-2\int_{-4}^{-1}\int_{\mathbb{R}^N}v_t\nabla v\,\psi^r\nabla\psi^r G(x,-t)\,dx\,dt-$$

$$-\int_{-4}^{-1}\int_{\mathbb{R}^N}v_t\nabla v\,(\psi^r)^2\nabla G\,dx\,dt.$$

Arguing as in [7], but using that in the present case $r\le R_0\le R$ and $|\nabla\psi|\le\frac{A_4}{R}$, we get, for $0<\eta<1$,

$$i)\le\eta\int_{-4}^{-1}\int_{\mathbb{R}^N}v_t^2(\psi^r)^2G(x,-t)\,dx\,dt+C_\eta(A_2,A_4).$$

Proceeding in a similar way,

$$iii)\le\eta\int_{-4}^{-1}\int_{\mathbb{R}^N}v_t^2(\psi^r)^2G(x,-t)\,dx\,dt+C_\eta r^2\|f_\varepsilon\|_{L^\infty(Q_{R,R_0}(\bar{x},\bar{t}))}^2.$$

Using that $\beta_{\varepsilon/r}(v)\,v_t=\partial_t B_{\varepsilon/r}(v)$ and $0\le B_{\varepsilon/r}(s)\le M$ we get for ii)

$$ii)\le C(M).$$

Thus,

$$(2.6)\qquad\left|\int_{-4}^{-1}\int_{\mathbb{R}^N}(\partial_t u_r^\varepsilon)^2(\psi^r)^2G(x,-t)\,dx\,dt\right|\le C\left(1+r^2\|f_\varepsilon\|_{L^\infty(Q_{R,R_0}(\bar{x},\bar{t}))}^2\right).$$

Summing up,

$$I \geq -C\|f_\varepsilon\|_{L^\infty(Q_{R,R_0}(\bar{x},\bar{t}))} \frac{|u^\varepsilon(\bar{x},\bar{t})|}{r} - C\|f_\varepsilon\|_{L^\infty(Q_{R,R_0}(\bar{x},\bar{t}))}\Big(1 + r\|f_\varepsilon\|_{L^\infty(Q_{R,R_0}(\bar{x},\bar{t}))}\Big).$$

On the other hand, since

(2.7)
$$\partial_r w_r^\varepsilon(x,t) = -\frac{\psi(\bar{x}+rx)u^\varepsilon(\bar{x}+rx,\bar{t}+r^2t)}{r^2} + \frac{\psi(\bar{x}+rx)\nabla u^\varepsilon(\bar{x}+rx,\bar{t}+r^2t)}{r}\cdot x$$
$$+ \frac{u^\varepsilon(\bar{x}+rx,\bar{t}+r^2t)\nabla\psi(\bar{x}+rx)}{r}\cdot x + 2t\psi(\bar{x}+rx)\partial_t u^\varepsilon(\bar{x}+rx,\bar{t}+r^2t),$$

then, for $x$ in the support of $\psi^r$ and $-4 \leq t \leq -1$, there holds

$$|\partial_r w_r^\varepsilon| \leq \frac{4A_1 R}{r^2} + \frac{A_2}{r}\frac{R}{r} + + \frac{4A_1 R}{r}\frac{A_4}{R}\frac{R}{r} + \frac{8}{r}|\psi^r \partial_t u_r^\varepsilon| = C\frac{R}{r^2} + \frac{C}{r}|\psi^r \partial_t u_r^\varepsilon|$$

and

$$|-2u_r^\varepsilon \Delta\psi^r - 4\nabla\psi^r \nabla u_r^\varepsilon| \leq \frac{8A_1 R}{r}r^2 A_4 + 4r\frac{A_4}{R}A_2 \leq CR^2$$

(we have used that $r \leq R_0 \leq R$ and $R > 1$). Since $\psi \equiv 1$ in $B_{R/2}(\bar{x})$, it follows that

$$|II| \leq \frac{CR^3}{r^2}\int_{-4}^{-1}\int\!\!\!\!\!\!-_{\frac{R}{2r}\leq|x|\leq\frac{R}{r}} G(x,-t)\,dx\,dt$$
$$+ \frac{CR^2}{r}\Big(\int_{-4}^{-1}\int_{\mathbb{R}^N}(\partial_t u_r^\varepsilon)^2(\psi^r)^2 G(x,-t)\,dx\,dt\Big)^{1/2}\Big(\int_{-4}^{-1}\int\!\!\!\!\!\!-_{\frac{R}{2r}\leq|x|\leq\frac{R}{r}} G(x,-t)\,dx\,dt\Big)^{1/2}.$$

Now, observing that

(2.8)
$$\int_{-4}^{-1}\int\!\!\!\!\!\!-_{\frac{R}{2r}\leq|x|\leq\frac{R}{r}} G(x,-t)\,dx\,dt \leq Ce^{-C'\frac{R^2}{r^2}}$$

and recalling (2.6), we obtain

$$II \geq -C\Big(1 + \|f_\varepsilon\|_{L^\infty(Q_{R,R_0}(\bar{x},\bar{t}))}\Big)R^2 e^{-C''\frac{R^2}{r^2}}.$$

Since $0 \leq s\,\beta_\varepsilon(s) \leq M_1$ and $0 \leq B_\varepsilon(s) \leq M$, we have

$$|-2\psi^r \beta_\varepsilon(ru_r^\varepsilon)\,ru_r^\varepsilon\,\partial_r\psi^r + 4\psi^r\,\partial_r\psi^r\,B_\varepsilon(ru_r^\varepsilon)| \leq 2M_1\frac{A_4}{R}|x| + 4\frac{A_4}{R}|x|M \leq \frac{CR}{r},$$

for $x$ in the support of $\psi^r$ and $-4 \leq t \leq -1$. Then, using again (2.8) we conclude that

$$II + III \geq -\bar{C}\Big(1 + \|f_\varepsilon\|_{L^\infty(Q_{R,R_0}(\bar{x},\bar{t}))}\Big)R^2 e^{-C'''\frac{R^2}{r^2}}.$$

The theorem is proved.                                                          $\square$

As a consequence of Theorem 2.1, we obtain

**Theorem 2.2 (Global Monotonicity Formula).** *Let $u^{\varepsilon_j}$ be a family of solutions to $P_{\varepsilon_j}(f_{\varepsilon_j})$ in $Q_{R_j,R_0}(\bar{x},\bar{t})$ with $R_j \to \infty$. Let $\psi_j = \psi_j(x) \in C_0^\infty(B_{R_j}(\bar{x}))$, $0 \leq \psi_j \leq 1$, $\psi_j \equiv 1$ in $B_{R_j/2}(\bar{x})$. Assume that*

(2.9)
$$|u^{\varepsilon_j}(x,t)| \leq A_1\big(1 + |x-\bar{x}| + |t-\bar{t}|^{1/2}\big) \text{ in } Q_{R_j,R_0}(\bar{x},\bar{t}), \quad |u^{\varepsilon_j}|_{Lip_{(1,1/2)}(Q_{R_j,R_0}(\bar{x},\bar{t}))} \leq A_2$$
$$\|\nabla\psi_j\|_{L^\infty(B_{R_j}(\bar{x}))} \leq \frac{A_4}{R_j}, \quad \|D^2\psi_j\|_{L^\infty(B_{R_j}(\bar{x}))} \leq A_4.$$

*Let $u = \lim u^{\varepsilon_j}$ uniformly on compact sets of $\mathbb{R}^N \times (\bar{t} - 4R_0^2, \bar{t}]$, $\chi = \lim B_{\varepsilon_j}(u^{\varepsilon_j})$ *−weakly in $L_{loc}^{\infty}(\mathbb{R}^N \times (\bar{t} - 4R_0^2, \bar{t}])$, $A \geq \|f_{\varepsilon_j}\|_{L^{\infty}(Q_{R_j,R_0}(\bar{x},\bar{t}))}$ and $\varepsilon_j \to 0$.*

*For $0 < r < R_0$, let*

$$\begin{aligned}
\mathcal{W}_{(\bar{x},\bar{t})}(r) &= \mathcal{W}_{(\bar{x},\bar{t})}(r, u, \chi) \\
&= \frac{1}{r^2} \int_{\bar{t}-4r^2}^{\bar{t}-r^2} \int_{\mathbb{R}^N} \left( |\nabla u|^2 + 2\chi + \frac{1}{2}\frac{u^2}{t-\bar{t}} \right) G(x - \bar{x}, \bar{t} - t)\, dx\, dt.
\end{aligned} \tag{2.10}$$

*Then, for $R_0 > \rho_1 > \rho_2 > 0$,*

$$\begin{aligned}
\mathcal{W}_{(\bar{x},\bar{t})}(\rho_1) - \mathcal{W}_{(\bar{x},\bar{t})}(\rho_2) &\geq \int_{\rho_2}^{\rho_1} \int_{-4}^{-1} \int_{\mathbb{R}^N} \left( \partial_r u_r \right)^2 \frac{rG(x,-t)}{-t}\, dx\, dt\, dr \\
&\quad - C\,A\,(\rho_1 - \rho_2)\big(1 + (\rho_1 + \rho_2)A\big) - C\,A\,|u(\bar{x},\bar{t})| \log\left(\frac{\rho_1}{\rho_2}\right)
\end{aligned} \tag{2.11}$$

*where $C$ is as in Theorem 2.1 and $u_r(x,t) = \frac{1}{r}u(\bar{x} + rx, \bar{t} + r^2 t)$.*

*In addition, if $\|f_{\varepsilon_j}\|_{L^{\infty}(Q_{R_j,R_0}(\bar{x},\bar{t}))} \to 0$ as $j \to \infty$ there holds that, for $R_0 > \rho_1 > \rho_2 > 0$,*

$$\mathcal{W}_{(\bar{x},\bar{t})}(\rho_1) - \mathcal{W}_{(\bar{x},\bar{t})}(\rho_2) \geq \int_{\rho_2}^{\rho_1} \int_{-4}^{-1} \int_{\mathbb{R}^N} \left( \partial_r u_r \right)^2 \frac{rG(x,-t)}{-t}\, dx\, dt\, dr. \tag{2.12}$$

*Proof.* First of all, it is clear that the second assertion follows immediately from the first one since, when $\|f_{\varepsilon_j}\|_{L^{\infty}(Q_{R_j,R_0}(\bar{x},\bar{t}))} \to 0$ as $j \to \infty$, we may take as $A$ an arbitrarily small constant.

In order to obtain (2.11), we will apply Theorem 2.1 to our family and pass to the limit.

In fact, integrating equation (2.3) for $j$ fixed and bounding $\|f_{\varepsilon_j}\|_{L^{\infty}(Q_{R_j,R_0}(\bar{x},\bar{t}))}$ by $A$, we get

$$\begin{aligned}
\mathcal{W}_{(\bar{x},\bar{t})}^{\varepsilon_j}(\rho_1) - \mathcal{W}_{(\bar{x},\bar{t})}^{\varepsilon_j}(\rho_2) &\geq \int_{\rho_2}^{\rho_1} \int_{-4}^{-1} \int_{\mathbb{R}^N} \left( \partial_r w_r^{\varepsilon_j} \right)^2 \frac{rG(x,-t)}{-t}\, dx\, dt\, dr \\
&\quad - CA \int_{\rho_2}^{\rho_1} \left(1 + rA\right) dr \\
&\quad - CA|u^{\varepsilon_j}(\bar{x},\bar{t})| \log\left(\frac{\rho_1}{\rho_2}\right) - C\,(1 + A)\,R_j^2 \int_{\rho_2}^{\rho_1} e^{-C'\frac{R_j^2}{r^2}}\, dr = I - II - III - IV.
\end{aligned} \tag{2.13}$$

It is easy to see that, as $j \to \infty$,

$$II = C\,A(\rho_1 - \rho_2)\left(1 + A\frac{\rho_1 + \rho_2}{2}\right),$$

$$III \to C\,A\,|u(\bar{x},\bar{t})| \log\left(\frac{\rho_1}{\rho_2}\right),$$

and

$$0 \leq IV \leq C\,(1 + A)\,R_j^2\, e^{-C'\frac{R_j^2}{\rho_1^2}} (\rho_1 - \rho_2) \to 0.$$

On the other hand, we recall that, by (2.7), $\partial_r w_r^{\varepsilon_j}(x,t)$ is the sum of four terms,

$$\partial_r w_r^{\varepsilon_j}(x,t) = i) + ii) + iii) + iv).$$

Now, observe that $\psi_j \to 1$ and $\nabla \psi_j \to 0$ uniformly on compact sets of $\mathbb{R}^N$. Moreover, we know that $\nabla u^{\varepsilon_j} \to \nabla u$ strongly in $L_{loc}^2$ (this convergence was proved in [6] for nonnegative

functions, but the same proof holds in the present case). Thus, by taking a subsequence, we obtain in $\mathbb{R}^N \times (-4, -1)$,

$$i) \rightarrow -\frac{u(\bar{x} + rx, \bar{t} + r^2 t)}{r^2} \quad a.e. \qquad \text{and}$$

$$|i)| \leq \frac{A_1\big(1 + r(|x| + |t|^{1/2})\big)}{r^2},$$

$$ii) \rightarrow \frac{\nabla u(\bar{x} + rx, \bar{t} + r^2 t)}{r} \cdot x \quad a.e. \qquad \text{and}$$

$$|ii)| \leq \frac{A_2 |x|}{r},$$

$$iii) \rightarrow 0 \quad a.e. \qquad \text{and}$$

$$|iii)| \leq \frac{2 A_1 A_4 |x|}{r}.$$

So that,

$$\int_{\rho_2}^{\rho_1} \int_{-4}^{-1} \int_{\mathbb{R}^N} \Big( -\frac{\psi_j(\bar{x} + rx) u^{\varepsilon_j}(\bar{x} + rx, \bar{t} + r^2 t)}{r^2} + \frac{\psi_j(\bar{x} + rx) \nabla u^{\varepsilon_j}(\bar{x} + rx, \bar{t} + r^2 t)}{r} \cdot x$$
$$+ \frac{u^{\varepsilon_j}(\bar{x} + rx, \bar{t} + r^2 t) \nabla \psi_j(\bar{x} + rx)}{r} \cdot x \Big)^2 \frac{rG(x, -t)}{-t} \, dx \, dt \, dr$$
$$\rightarrow \int_{\rho_2}^{\rho_1} \int_{-4}^{-1} \int_{\mathbb{R}^N} \Big( -\frac{u(\bar{x} + rx, \bar{t} + r^2 t)}{r^2} + \frac{\nabla u(\bar{x} + rx, \bar{t} + r^2 t)}{r} \cdot x \Big)^2 \frac{rG(x, -t)}{-t} \, dx \, dt \, dr.$$

Next, we use the convergence of $u^{\varepsilon_j}$, estimate (2.6) and the fact that $\psi_j u_t^{\varepsilon_j} = \partial_t(\psi_j u^{\varepsilon_j})$ to deduce that

$$\psi_j(\bar{x} + rx) \partial_t u^{\varepsilon_j}(\bar{x} + rx, \bar{t} + r^2 t) \rightharpoonup u_t(\bar{x} + rx, \bar{t} + r^2 t)$$

weakly in $L^2\big(\mathbb{R}^N \times (-4, -1), (-t)G(x, -t) \, dx \, dt\big)$. Then,

$$\liminf_{j \to \infty} \int_{\rho_2}^{\rho_1} \int_{-4}^{-1} \int_{\mathbb{R}^N} \Big( 2t \psi_j(\bar{x} + rx) \partial_t u^{\varepsilon_j}(\bar{x} + rx, \bar{t} + r^2 t) \Big)^2 \frac{rG(x, -t)}{-t} \, dx \, dt \, dr$$
$$\geq \int_{\rho_2}^{\rho_1} \int_{-4}^{-1} \int_{\mathbb{R}^N} \Big( 2t u_t(\bar{x} + rx, \bar{t} + r^2 t) \Big)^2 \frac{rG(x, -t)}{-t} \, dx \, dt \, dr$$

and

$$\int_{\rho_2}^{\rho_1} \int_{-4}^{-1} \int_{\mathbb{R}^N} \Big( 2t \psi_j(\bar{x} + rx) \partial_t u^{\varepsilon_j}(\bar{x} + rx, \bar{t} + r^2 t) \Big)$$
$$\Big( -\frac{\psi_j(\bar{x} + rx) u^{\varepsilon_j}(\bar{x} + rx, \bar{t} + r^2 t)}{r^2} + \frac{\psi_j(\bar{x} + rx) \nabla u^{\varepsilon_j}(\bar{x} + rx, \bar{t} + r^2 t)}{r} \cdot x$$
$$+ \frac{u^{\varepsilon_j}(\bar{x} + rx, \bar{t} + r^2 t) \nabla \psi_j(\bar{x} + rx)}{r} \cdot x \Big) \frac{rG(x, -t)}{-t} \, dx \, dt \, dr$$
$$\rightarrow \int_{\rho_2}^{\rho_1} \int_{-4}^{-1} \int_{\mathbb{R}^N} \Big( 2t u_t(\bar{x} + rx, \bar{t} + r^2 t) \Big)$$
$$\Big( -\frac{u(\bar{x} + rx, \bar{t} + r^2 t)}{r^2} + \frac{\nabla u(\bar{x} + rx, \bar{t} + r^2 t)}{r} \cdot x \Big) \frac{rG(x, -t)}{-t} \, dx \, dt \, dr.$$

Returning to (2.13) we conclude that

$$\liminf_{j \to \infty} I \geq \int_{\rho_2}^{\rho_1} \int_{-4}^{-1} \int_{\mathbb{R}^N} \left( \partial_r u_r \right)^2 \frac{rG(x, -t)}{-t} \, dx \, dt \, dr.$$

Similarly, we can prove that

$$\frac{1}{r^2} \int_{\bar{t}-4r^2}^{\bar{t}-r^2} \int_{\mathbb{R}^N} \left( \left| \nabla(\psi_j u^{\varepsilon_j}) \right|^2 + \frac{1}{2} \frac{(\psi_j u^{\varepsilon_j})^2}{t - \bar{t}} \right) G(x - \bar{x}, \bar{t} - t) \, dx \, dt$$

$$\to \frac{1}{r^2} \int_{\bar{t}-4r^2}^{\bar{t}-r^2} \int_{\mathbb{R}^N} \left( |\nabla u|^2 + \frac{1}{2} \frac{u^2}{t - \bar{t}} \right) G(x - \bar{x}, \bar{t} - t) \, dx \, dt.$$

Finally, given $\sigma > 0$, let $R > 0$ be such that

$$2M \int_{\bar{t}-4r^2}^{\bar{t}-r^2} \int_{|x - \bar{x}| > R} G(x - \bar{x}, \bar{t} - t) \, dx \, dt < \frac{\sigma}{2}.$$

Then, for $j$ large,

$$\left| \int_{\bar{t}-4r^2}^{\bar{t}-r^2} \int_{\mathbb{R}^N} \left\{ \psi_j^2 B_{\varepsilon_j}(u^{\varepsilon_j}) - \chi \right\} G(x - \bar{x}, \bar{t} - t) \, dx \, dt \right|$$

$$\leq \left| \int_{\bar{t}-4r^2}^{\bar{t}-r^2} \int_{|x - \bar{x}| < R} \left\{ B_{\varepsilon_j}(u^{\varepsilon_j}) - \chi \right\} G(x - \bar{x}, \bar{t} - t) \, dx \, dt \right|$$

$$+ 2M \left| \int_{\bar{t}-4r^2}^{\bar{t}-r^2} \int_{|x - \bar{x}| > R} G(x - \bar{x}, \bar{t} - t) \, dx \, dt \right|$$

$$< \left| \int_{\bar{t}-4r^2}^{\bar{t}-r^2} \int_{|x - \bar{x}| < R} \left\{ B_{\varepsilon_j}(u^{\varepsilon_j}) - \chi \right\} G(x - \bar{x}, \bar{t} - t) \, dx \, dt \right| + \frac{\sigma}{2}$$

$$< \sigma \qquad \text{if } j \geq j_0(R).$$

Therefore, $\mathcal{W}_{(\bar{x}, \bar{t})}^{\varepsilon_j}(r) \to \mathcal{W}_{(\bar{x}, \bar{t})}(r)$ as $j \to \infty$ and the result follows. $\qquad \square$

We apply Theorem 2.2 to derive

**Corollary 2.1** (**Global Monotonicity Formula for Blow up limits**). *Let $u^{\varepsilon_j}$ be a family of solutions to $P_{\varepsilon_j}(f_{\varepsilon_j})$ in a domain $\mathcal{D} \subset \mathbb{R}^{N+1}$, uniformly bounded in $Lip\,(1, 1/2)$ norm with $f_{\varepsilon_j}$ uniformly bounded in $L^\infty$ norm in $\mathcal{D}$. Assume $u^{\varepsilon_j} \to u$ uniformly on compact subsets of $\mathcal{D}$ and $B_{\varepsilon_j}(u^{\varepsilon_j}) \to \chi$ *–weakly in $L^\infty(\mathcal{D})$ with $\varepsilon_j \to 0$. Let $(x_0, t_0) \in \mathcal{D} \cap \partial\{u > 0\}$, $u_{\lambda_n}(x, t) = \frac{1}{\lambda_n} u(x_0 + \lambda_n x, t_0 + \lambda_n^2 t)$, $\chi^{\lambda_n}(x, t) = \chi(x_0 + \lambda_n x, t_0 + \lambda_n^2 t)$ and $\lambda_n \to 0$. Assume $u_{\lambda_n} \to u_0$ uniformly on compact sets of $\mathbb{R}^{N+1}$, $\chi^{\lambda_n} \to \chi_0$ *–weakly in $L^\infty_{loc}(\mathbb{R}^{N+1})$. Let $(\bar{x}, \bar{t}) \in \mathbb{R}^{N+1}$. Then, for $\mathcal{W}_{(\bar{x}, \bar{t})}(r) = \mathcal{W}_{(\bar{x}, \bar{t})}(r, u_0, \chi_0)$ and $\rho_1 > \rho_2 > 0$ there holds that*

$$(2.14) \qquad \mathcal{W}_{(\bar{x}, \bar{t})}(\rho_1) - \mathcal{W}_{(\bar{x}, \bar{t})}(\rho_2) \geq \int_{\rho_2}^{\rho_1} \int_{-4}^{-1} \int_{\mathbb{R}^N} \left( \partial_r (u_0)_r \right)^2 \frac{rG(x, -t)}{-t} \, dx \, dt \, dr$$

*where $(u_0)_r(x, t) = \frac{1}{r} u_0(\bar{x} + rx, \bar{t} + r^2 t)$.*

*Proof.* We will apply Theorem 2.2 to the functions $u^{\delta_n}(x, t) := \frac{1}{\lambda_n} u^{\varepsilon_{j_n}}(x_0 + \lambda_n x, t_0 + \lambda_n^2 t)$ where $j_n$ is chosen such that (see for instance [7], proof of Theorem 3.1),

- $\delta_n := \frac{\varepsilon_{j_n}}{\lambda_n} \to 0$.
- $u^{\delta_n} \to u_0$ uniformly on compact sets of $\mathbb{R}^{N+1}$.
- $B_{\delta_n}(u^{\delta_n}) \to \chi_0$ $*$–weakly in $L^\infty_{loc}(\mathbb{R}^{N+1})$.
- $|\frac{u(x_0,t_0)-u^{\varepsilon_{j_n}}(x_0,t_0)}{\lambda_n}| \le 1$.

Now, let $\sigma > 0$ such that $B_{2\sigma}(x_0) \times [t_0 - 4\sigma^2, t_0 + 4\sigma^2] \subset \mathcal{D}$. Let $R_n = \sigma/\lambda_n$ and let $R_0 > 0$ be fixed. Then, for $(x,t) \in Q_{R_n,R_0}(\bar{x},\bar{t})$ we have that $|\lambda_n x| \le \lambda_n|\bar{x}| + \lambda_n|x - \bar{x}| \le \lambda_n|\bar{x}| + \sigma < 2\sigma$ and $-4\sigma^2 \le \lambda_n^2(\bar{t} - 4R_0^2) < \lambda_n^2 t < \lambda_n^2\bar{t} \le 4\sigma^2$ if $n$ is large enough. Thus, $u^{\delta_n}$ is defined in $Q_{R_n,R_0}(\bar{x},\bar{t})$ for $n$ large.

Let $\psi \in C_0^\infty(B_\sigma(0))$, $0 \le \psi \le 1$, $\psi \equiv 1$ in $B_{\sigma/2}(0)$, and $\psi_n(x) = \psi(\lambda_n(x - \bar{x}))$. Then, $|\nabla \psi_n| \le A_4 \frac{\lambda_n}{\sigma}$, $|D^2\psi_n| \le A_4$ in $B_{R_n}(\bar{x})$ for a certain constant $A_4$, $\psi_n \in C_0^\infty(B_{R_n}(\bar{x}))$, $\psi_n \equiv 1$ in $B_{R_n/2}(\bar{x})$.

Observe that $u^{\delta_n}$ are solutions to $P_{\delta_n}(f_{\delta_n})$ with $f_{\delta_n}(x,t) = \lambda_n f_{\varepsilon_{j_n}}(x_0 + \lambda_n x, t_0 + \lambda_n^2 t)$. Moreover, $\|f_{\delta_n}\|_{L^\infty(Q_{R_n,R_0}(\bar{x},\bar{t}))} \to 0$ as $n \to \infty$.

In order to apply Theorem 2.2, let $\rho_1 > \rho_2 > 0$ arbitrary and then, take $R_0 > \rho_1$. We only need to show that the hypotheses (2.9) are satisfied. In fact, the bounds of $\psi_n$ and its derivatives follow immediately by construction as observed above. On the other hand, taking $L > 0$ such that $|u^{\varepsilon_{j_n}}|_{Lip_{(1,1/2)}(\mathcal{D})} \le L$ we get, for $(x,t) \in Q_{R_n,R_0}(\bar{x},\bar{t})$,

$$|u^{\delta_n}(x,t)| = \frac{1}{\lambda_n}|u^{\varepsilon_{j_n}}(x_0 + \lambda_n x, t_0 + \lambda_n^2 t)| \le |\frac{u^{\varepsilon_{j_n}}(x_0,t_0)}{\lambda_n}| + L(|x| + |t|^{1/2})$$

$$\le |\frac{u(x_0,t_0) - u^{\varepsilon_{j_n}}(x_0,t_0)}{\lambda_n}| + L(|\bar{x}| + |\bar{t}|^{1/2}) + L(|x - \bar{x}| + |t - \bar{t}|^{1/2})$$

$$\le A_1(1 + |x - \bar{x}| + |t - \bar{t}|^{1/2})$$

for a certain constant $A_1$ depending on $\bar{x}$ and $\bar{t}$ but independent of $n$. Moreover,

$$|u^{\delta_n}|_{Lip_{(1,1/2)}(Q_{R_n,R_0}(\bar{x},\bar{t}))} \le L.$$

Thus, (2.14) follows and the corollary is proved. $\qquad\square$

## 3. Characterization of blow up limits in terms of the density at the blow up point

In this section we apply the results of Section 2 to characterize blow up limits $u_0$ in terms of the value of a density at the the blow up point, in the stationary case.

In fact, we consider a family $u^{\varepsilon_j}$ of stationary solutions to $P_{\varepsilon_j}(f_{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ with $\|f_{\varepsilon_j}\|_{L^\infty(\Omega)} \le C$, $\|u^{\varepsilon_j}\|_{L^\infty(\Omega)} \le C'$ and $\varepsilon_j \to 0$. By the results in [8], it follows that $u^{\varepsilon_j}$ are locally uniformly bounded in $Lip$ norm in $\Omega$, so that the results of Section 2 apply to this family.

Let $u = \lim u^{\varepsilon_j}$ uniformly on compact subsets of $\Omega$ and $\chi = \lim B_{\varepsilon_j}(u^{\varepsilon_j})$ $*$–weakly in $L^\infty(\Omega)$. For $x_0 \in \Omega \cap \partial\{u > 0\}$, we consider

$$(3.1) \quad \delta(x_0) := \delta(x_0, 0) = \lim_{r \to 0} \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} \left(|\nabla(\psi u)|^2 + 2\psi^2\chi + \frac{(\psi u)^2}{2t}\right)G(x - x_0, -t)dx\, dt,$$

where $\psi \in C_0^\infty\big(B_\sigma(x_0)\big)$, $0 \le \psi \le 1$, $\psi \equiv 1$ in $B_{\sigma/2}(x_0)$ and $B_\sigma(x_0) \subset\subset \Omega$ (this limit exists and it is finite and independent of the cut off function $\psi$, by Theorem 2.2 and Corollary 2.1 in [7]).

First, we prove Theorems 3.1 and 3.2, where we characterize blow up limits $u_0$ at free boundary points $x_0$ when $\delta(x_0) \in \{3M, 6M\}$ (here $u_0 = \lim_{\lambda_n \to 0} \frac{1}{\lambda_n} u(x_0 + \lambda_n x)$, with $x_0 \in \Omega \cap \partial\{u > 0\}$ and $M = \int_0^1 \beta(s)\, ds$). See Remark 3.2 for the reciprocal of these results.

Then, we show that $\delta(x_0) \in [3M, 6M]$. Moreover, $\delta(x_0) \in \{3M, 6M\}$ for $\mathcal{H}^{N-1}$-almost every $x_0 \in \Omega \cap \partial\{u > 0\}$ and also for every $x_0 \in \Omega \cap \partial\{u > 0\}$, if $N = 2$ (Propositions 3.1, 3.2 and 3.3).

We start with

**Theorem 3.1.** *Let $u^{\varepsilon_j}$ be stationary solutions to $P_{\varepsilon_j}(f_{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ with $\|f_{\varepsilon_j}\|_{L^\infty(\Omega)} \le C$, $\|u^{\varepsilon_j}\|_{L^\infty(\Omega)} \le C'$ and $\varepsilon_j \to 0$. Let $u = \lim u^{\varepsilon_j}$ uniformly on compact subsets of $\Omega$ and $\chi = \lim B_{\varepsilon_j}(u^{\varepsilon_j})$ $*$–weakly in $L^\infty(\Omega)$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ and $\delta(x_0)$ as in (3.1).*

*Assume $\delta(x_0) = 6M$. Let $\lambda_n \to 0$ be such that there exists $u_0(x) = \lim_{n \to \infty} \frac{1}{\lambda_n} u(x_0 + \lambda_n x)$ uniformly on compact sets of $\mathbb{R}^N$. Then, there exists $\alpha \ge 0$ such that, in a certain coordinate system,*

$$u_0(x) = \alpha |x_1|.$$

*Proof.* By taking a subsequence, we may assume that $\chi^{\lambda_n}(x) = \chi(x_0 + \lambda_n x) \to \chi_0(x)$ $*$–weakly in $L_{loc}^\infty(\mathbb{R}^N)$. By the results of [7] (Corollaries 2.1 and 2.2) we know that, for $r > 0$,

$$(3.2)\quad \delta(x_0) = \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} \Big(|\nabla u_0|^2 + 2\chi_0 + \frac{u_0^2}{2t}\Big) G(x, -t)\, dx\, dt = \int_{-4}^{-1} \int_{\mathbb{R}^N} 2\chi_0(x) G(x, -t)\, dx\, dt.$$

Since $0 \le \chi_0 \le M$ there holds that $0 \le \delta(x_0) \le 6M$. Thus,

$$\delta(x_0) = 6M \qquad \Rightarrow \qquad \chi_0 \equiv M \quad \text{a. e. in} \quad \mathbb{R}^N.$$

On the other hand, we know that $u_0$ is homogeneous (see Corollary 2.1 in [7]). This is,

$$u_0(rx) = r u_0(x) \quad \text{for every } r > 0, \quad x \in \mathbb{R}^N.$$

Since $u_0 = \lim u^{\delta_n}$ and $\chi_0 = \lim B_{\delta_n}(u^{\delta_n})$ with $\delta_n$ and $u^{\delta_n}$ as in Corollary 2.1, we can apply the results in Lemma 3.1 in [8] to deduce that $\chi_0(x) \in \{0, M\}$ for almost every $x \in \mathbb{R}^N$, $\chi_0 = 0$ in $\{u_0 < 0\}$, $\chi_0 = M$ in $\{u_0 > 0\}$ and $\chi_0$ is constant on every connected component of $\{u_0 \le 0\}^\circ$. In particular, since $\chi_0 \equiv M$, we have that $u_0 \ge 0$.

If $u_0 \equiv 0$ in $\mathbb{R}^N$, then $u_0(x) = \alpha |x_1|$ with $\alpha = 0$ and the theorem is proved in this case.

Thus, we may assume that $u_0 > 0$ somewhere.

Now, let us show that the theorem holds when $u_0$ depends only on 1 variable. In fact, if $u_0 = u_0(x_1)$ depends on 1 variable, the only possible components of $\{u_0 > 0\}$ are $\{x_1 > 0\}$ and $\{x_1 < 0\}$.

If $u_0 > 0$ in $\{x_1 > 0\}$ there holds that $u_0(x_1) = \alpha x_1$ in $\{x_1 > 0\}$ with $\alpha > 0$ since it is harmonic in this set and $u_0(0) = 0$.

Assume $u_0 = 0$ in $\{x_1 < 0\}$. Using that $u_0 = \lim u^{\delta_n}$, $\chi_0 = \lim B_{\delta_n}(u^{\delta_n})$ and that $u^{\delta_n}$ are solutions to $P_{\delta_n}(f_{\delta_n})$ with $f_{\delta_n} \to 0$ uniformly on compact subsets of $\mathbb{R}^N$, we proceed as in Proposition 3.2 in [8] (see also Proposition 5.1 in [6] and Proposition 5.2 in [4]) and deduce that

$$\frac{\alpha^2}{2} = M - M = 0,$$

which is a contradiction.

By similar arguments, if $u_0 > 0$ in $\{x_1 < 0\}$ there exists $\gamma > 0$ such that $u_0(x_1) = -\gamma x_1$ in $\{x_1 < 0\}$.

This time we proceed as in Proposition 3.3 in [8] (see also Proposition 5.3 in [4]), and deduce that

$$\frac{\alpha^2}{2} - \frac{\gamma^2}{2} = M - M = 0.$$

So that, $\gamma = \alpha > 0$ and $u_0(x_1) = \alpha|x_1|$.

Therefore, the theorem is proved when $u_0$ depends on 1 variable.

We will devote the rest of the proof to showing that this is necessarily the case. The proof will follow from a dimension reduction argument.

Let us first observe that if $u_0$ depends on $k$ variables, then $\chi_0$ depends on the same $k$ variables. In fact, assume that $u_0$ does not depend on a direction $\bar{\nu}$. We will show that $\chi_0$ does not depend on $\bar{\nu}$. For simplicity, we assume that $\bar{\nu} = e_1$.

Multiplying equation $P_{\delta_n}(f_{\delta_n})$ by $u_{x_1}^{\delta_n} \varphi$, where $\varphi \in C_0^\infty(\mathbb{R}^N)$, and integrating by parts, we get

$$-\frac{1}{2} \int |\nabla u^{\delta_n}|^2 \varphi_{x_1} + \int \nabla u^{\delta_n} \nabla \varphi \, u_{x_1}^{\delta_n} = \int B_{\delta_n}(u^{\delta_n})\varphi_{x_1} - \int f_{\delta_n} u_{x_1}^{\delta_n} \varphi.$$

Passing to the limit, we obtain

$$(3.3) \qquad\qquad -\frac{1}{2} \int |\nabla u_0|^2 \varphi_{x_1} = \int \chi_0 \varphi_{x_1}.$$

We observe the left hand side of (3.3) vanishes because $u_0$ does not depend on $x_1$ and thus, this equality implies that $\chi_0$ does not depend on $x_1$, as we claimed.

Next, in order to develop the dimension reduction argument, let us assume that $u_0$ depends on $k$ variables. Thus, we will assume that $u_0 = u_0(x_1, \cdots, x_k)$ and correspondingly, $\chi_0 = \chi_0(x_1, \cdots, x_k)$.

Since the rest of the proof relies on the definition of the functional $\mathcal{W}_{(\bar{x},0)}(r) = \mathcal{W}_{(\bar{x},0)}(r, u_0, \chi_0)$ (more precisely, on (2.14) and (3.2)), we see that we can assume that we are working in $\mathbb{R}^k$ instead of $\mathbb{R}^N$.

We will show that there is a direction $\nu$ in $\mathbb{R}^k$ such that $u_0$ does not depend on this direction. This is, we will show that $\nabla u_0 \cdot \nu = 0$ in $\mathbb{R}^k$, thus deducing that $u_0$ actually depends on $k-1$ variables. Iterating this argument we finally get that $u_0$ only depends on 1 variable.

Let us assume that $k \geq 2$.

In order to find $\nu$ we first observe that there exists $\tilde{x} \in \mathbb{R}^k \setminus \{0\}$ such that $u_0(\tilde{x}) = 0$. In fact, if this is not the case we have

$$\begin{cases} \Delta u_0 = 0 & \text{in } \mathbb{R}^k \setminus \{0\} \\ |u_0(x) - u_0(y)| \leq L|x - y| \end{cases}$$

for some $L > 0$, and we deduce that $\Delta u_0 = 0$ in $\mathbb{R}^k$, $u_0 \geq 0$, $u_0(0) = 0$ which is a contradiction.

Therefore, there exists $\tilde{x} \neq 0$ such that $u_0(\tilde{x}) = 0$. Since $u_0$ is homogeneous there holds that $u_0(\lambda \tilde{x}) = 0$ for every $\lambda > 0$. By rotating this line we find in $\mathbb{R}^k$ a point $0 \neq \bar{x} \in \partial\{u_0 > 0\}$. If not, we would have $u_0 \equiv 0$ and this is not the case.

Let us apply the monotonicity formula (2.14) at $(\bar{x}, \bar{t})$ with $\bar{t} = 0$. We have, for $\rho_1 > \rho_2 > 0$,

$$0 \le \int_{\rho_2}^{\rho_1} \int_{-4}^{-1} \int_{\mathbb{R}^k} \left(\partial_r (u_0)_r\right)^2 \frac{rG(x, -t)}{-t} \, dx \, dt \, dr \le \mathcal{W}_{(\bar{x},0)}(\rho_1) - \mathcal{W}_{(\bar{x},0)}(\rho_2)$$

where $(u_0)_r(x) = \frac{1}{r} u_0(\bar{x} + rx)$ and $\mathcal{W}_{(\bar{x},0)}(r) = \mathcal{W}_{(\bar{x},0)}(r, u_0, \chi_0)$.

Now let $R > 0$ be fixed. Letting $\rho_1 \to R^-$ and $\rho_2 \to 0^+$ (the limits exist by the monotonicity of $\mathcal{W}_{(\bar{x},0)}(r)$ shown in Corollary 2.1) we get

$$0 \le \int_0^R \int_{-4}^{-1} \int_{\mathbb{R}^k} \left(\partial_r (u_0)_r\right)^2 \frac{rG(x, -t)}{-t} \, dx \, dt \, dr \le \mathcal{W}_{(\bar{x},0)}(R) - \mathcal{W}_{(\bar{x},0)}(0^+).$$

Now, it is easy to see from the rescaling invariance of $\mathcal{W}_{(\bar{x},0)}(r)$, by arguments similar to those used in Corollary 2.2 in [7], that

$$\mathcal{W}_{(\bar{x},0)}(0^+) = \int_{-4}^{-1} \int_{\mathbb{R}^k} 2\chi_{00} \, G(x, -t) \, dx \, dt,$$

where $\chi_{00}(x) = \lim_{j\to\infty} \chi_0(\bar{x} + \mu_j x)$ for a certain sequence $\mu_j \to 0$.

Since $\chi_0 \equiv M$, the same is true for $\chi_{00}$. Thus, $\mathcal{W}_{(\bar{x},0)}(0^+) = 6M$.

On the other hand, we know that $\mathcal{W}_{(0,0)}(r)$ is constant (recall (3.2)). Therefore,

$$\mathcal{W}_{(\bar{x},0)}(0^+) = 6M = \delta(x_0) = \mathcal{W}_{(0,0)}(R).$$

Thus,

$$(3.4) \qquad 0 \le \int_0^R \int_{-4}^{-1} \int_{\mathbb{R}^k} \left(\partial_r (u_0)_r\right)^2 \frac{rG(x, -t)}{-t} \, dx \, dt \, dr \le \mathcal{W}_{(\bar{x},0)}(R) - \mathcal{W}_{(0,0)}(R).$$

Let us see that the right hand side converges to 0 as $R \to \infty$. In fact, since $u_0$ is homogeneous,

$$\mathcal{W}_{(\bar{x},0)}(R) - \mathcal{W}_{(0,0)}(R) = \frac{1}{R^2} \int_{-4R^2}^{-R^2} \int_{\mathbb{R}^k} \left(|\nabla u_0|^2 + 2\chi_0 + \frac{u_0^2}{2t}\right) \left(G(x - \bar{x}, -t) - G(x, -t)\right) dx \, dt$$

$$= \frac{1}{R^2} \int_{-4R^2}^{-R^2} \int_{\mathbb{R}^k} \left(\left|\nabla u_0\left(\frac{x}{R}\right)\right|^2 + 2\chi_0(x) + \frac{u_0^2\left(\frac{x}{R}\right)}{2\frac{t}{R^2}}\right) \left(G(x - \bar{x}, -t) - G(x, -t)\right) dx \, dt$$

$$= \int_{-4}^{-1} \int_{\mathbb{R}^k} \left(|\nabla u_0(y)|^2 + 2\chi_0(Ry) + \frac{u_0^2(y)}{2s}\right) \left(G\left(y - \frac{\bar{x}}{R}, -s\right) - G(y, -s)\right) dy \, ds$$

$$\le \int_{-4}^{-1} \int_{\mathbb{R}^k} \left(L^2(1 + |y|^2) + 2M\right) \frac{\left|e^{\frac{|y - \frac{\bar{x}}{R}|^2}{4s}} - e^{\frac{|y|^2}{4s}}\right|}{(-4\pi s)^{k/2}} dy \, ds$$

$$= \int_{-4}^{-1} \int_{\mathbb{R}^k} F_R(y, s) \, dy \, ds,$$

with $F_R(y, s) \to 0$ as $R \to \infty$ and $|F_R(y, s)| \le C(1 + |y|^2) e^{-C' |y|^2}$ if $R$ is large. Thus,

$$\int_{-4}^{-1} \int_{\mathbb{R}^k} F_R(y, s) \, dy \, ds \quad \to \quad 0 \qquad \text{as } R \to \infty$$

and, passing to the limit as $R \to \infty$ in (3.4), we deduce that for a.e. $r > 0$

$$\int_{-4}^{-1} \int_{\mathbb{R}^k} \left(\partial_r (u_0)_r\right)^2 \frac{rG(x, -t)}{-t} \, dx \, dt = 0.$$

Therefore, $\partial_r(u_0)_r = 0$ so that, for a.e. $x$,

$$0 = \partial_r\Big(\frac{u_0(\bar{x} + rx)}{r}\Big) = -\frac{u_0(\bar{x} + rx)}{r^2} + \frac{x}{r} \cdot \nabla u_0(\bar{x} + rx)$$

or equivalently,

$$-\frac{u_0(x)}{r^2} + \frac{x - \bar{x}}{r^2} \cdot \nabla u_0(x) \equiv 0.$$

But, since $u_0(rx) = ru_0(x)$, there holds that

$$\nabla u_0(x) \cdot x - u_0(x) \equiv 0.$$

Therefore,

$$\nabla u_0(x) \cdot \bar{x} \equiv 0.$$

Now, if $\nu = \frac{\bar{x}}{|\bar{x}|}$, there holds that $u_0$ is independent of the direction $\nu$ and the theorem is proved. $\qquad\square$

We next obtain

**Theorem 3.2.** *Let $u^{\varepsilon_j}$ be stationary solutions to $P_{\varepsilon_j}(f_{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ with $\|f_{\varepsilon_j}\|_{L^\infty(\Omega)} \leq C$, $\|u^{\varepsilon_j}\|_{L^\infty(\Omega)} \leq C'$ and $\varepsilon_j \to 0$. Let $u = \lim u^{\varepsilon_j}$ uniformly on compact subsets of $\Omega$, and $\chi = \lim B_{\varepsilon_j}(u^{\varepsilon_j})$ $*$–weakly in $L^\infty(\Omega)$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ and $\delta(x_0)$ as in (3.1).*

*Assume $\delta(x_0) = 3M$. Let $\lambda_n \to 0$ be such that there exists $u_0(x) = \lim_{n\to\infty} \frac{1}{\lambda_n} u(x_0 + \lambda_n x)$ uniformly on compact sets of $\mathbb{R}^N$. Then, there exist $\alpha > 0$ and $\gamma \geq 0$ such that, in a certain coordinate system,*

$$(3.5)\qquad\qquad u_0(x) = \alpha x_1^+ - \gamma x_1^-.$$

*Proof.* As in the proof of Theorem 3.1, we will first show that the theorem holds when $u_0$ depends only on one variable, and then proceed by a dimension reduction argument.

In fact, we may assume that $\chi^{\lambda_n}(x) = \chi(x_0 + \lambda_n x) \to \chi_0(x)$ $*$–weakly in $L^\infty_{loc}(\mathbb{R}^N)$, with $\chi_0(x) \in \{0, M\}$ for almost every $x \in \mathbb{R}^N$, $\chi_0 \equiv M$ in $\{u_0 > 0\}$, $\chi_0 \equiv 0$ in $\{u_0 < 0\}$, $\chi_0$ constant (either $0$ or $M$) on any connected component of $\{u_0 \leq 0\}^\circ$. In addition, the bounds in the proof of Lemma 3.1 in [8] imply that $\chi_0 \in BV_{loc}(\mathbb{R}^N)$. In particular, $\chi_0 = M\chi_{\{\chi_0>0\}}$ and thus, $\{\chi_0 > 0\}$ is a set of locally finite perimeter (see, for instance, [5]).

So let us now show that, if $u_0$ depends only on one variable ($u_0 = u_0(x_1)$ in a certain coordinate system) then, the result follows. In fact, since $u_0$ is homogeneous, we only have one of the following:

(1) $u_0 = 0$ in $\mathbb{R}$.
(2) $u_0 < 0$ in $\{x_1 > 0\}$ and $u_0 < 0$ in $\{x_1 < 0\}$.
(3) $u_0 < 0$ in $\{x_1 > 0\}$ and $u_0 = 0$ in $\{x_1 < 0\}$.
(4) $u_0 > 0$ in $\{x_1 > 0\}$ and $u_0 > 0$ in $\{x_1 < 0\}$.
(5) $u_0 > 0$ in $\{x_1 > 0\}$ and $u_0 < 0$ in $\{x_1 < 0\}$.
(6) $u_0 > 0$ in $\{x_1 > 0\}$ and $u_0 = 0$ in $\{x_1 < 0\}$.

Actually, (1), (2), (3) and (4) are not possible. In fact, if (1), (2) or (3) hold, we have that $u_0 \leq 0$ in $\mathbb{R}$ and then, $\chi_0$ is constant equal to $0$ or $M$ in $\mathbb{R}$. If (4) holds, we have that $\chi_0 = M$ in $\mathbb{R}$. Thus, in any of these cases we either have $\delta(x_0) = 0$ or $\delta(x_0) = 6M$, a contradiction.

Therefore, under the hypotheses of this theorem only (5) and (6) are possible.

There holds that $u_0$ is harmonic where positive and where negative, $u_0$ uniformly Lipschitz in $\mathbb{R}$ and $u_0(0) = 0$. Therefore, if we have (5) or (6), there exists $\alpha > 0$ such that $u_0 = \alpha x_1$

in $\{x_1 > 0\}$. If we have (5), by the same argument there exists $\gamma > 0$ such that $u_0 = \gamma x_1$ in $\{x_1 < 0\}$. In any of the cases (5) or (6), there holds (3.5) with $\alpha > 0$ and $\gamma \geq 0$. Thus, the theorem is true if $u_0$ depends on 1 variable.

Now, as in the proof of Theorem 3.1, we will show by a dimension reduction argument, that $u_0$ depends only on 1 variable. Following the arguments in that proof, assuming $u_0$ depends on $k$ variables with $k \geq 2$, we may suppose that we are in $\mathbb{R}^k$ and therefore, it is enough to show that there exists in $\mathbb{R}^k$ a point $0 \neq \bar{x} \in \partial\{u_0 > 0\}$ such that $\mathcal{W}_{(\bar{x},0)}(0^+) = 3M$.

More precisely, we will see that there exists a point $0 \neq \bar{x} \in \partial_{red}\{\chi_0 > 0\} \subset \partial\{u_0 > 0\}$, where $\partial_{red}$ denotes reduced boundary. Recall that

$$\mathcal{W}_{(\bar{x},0)}(0^+) = \int_{-4}^{-1} \int_{\mathbb{R}^k} 2\chi_{00}(x)\, G(x, -t)\, dx\, dt,$$

for $\chi_{00}(x) = \lim_{j\to\infty} \chi_0(\bar{x} + \mu_j x)$ with $\mu_j \to 0$. If $\bar{x} \in \partial_{red}\{\chi_0 > 0\}$, then $\chi_{00}(x) = M\chi_{\{\langle x,\nu\rangle > 0\}}$ for $\nu$ the unit interior normal to $\{\chi_0 > 0\}$ at $\bar{x}$ in the measure theoretic sense. Therefore, $\mathcal{W}_{(\bar{x},0)}(0^+) = 3M$ and, from here the proof follows as that of Theorem 3.1.

So, let $E = \{\chi_0 > 0\}$. Then, $|E| > 0$ and $|E^c| > 0$. If not, either $\chi_0 = 0$ a.e. or $\chi_0 = M$ a.e., contradicting the fact that $\delta(x_0) = 3M$. Thus, there exists $R_0 > 0$ such that $|E \cap B_{R_0}| > 0$ and $|E^c \cap B_{R_0}| > 0$, where we denote $B_{R_0} = B_{R_0}(0)$. We claim that $0 < Per(\partial E; B_{R_0}) < \infty$. In fact, the perimeter is finite in $B_{R_0}$ since $\chi_0 = M\chi_E$ and $\chi_0 \in BV(B_{R_0})$. Now, by the isoperimetric inequality,

$$Per(\partial E; B_{R_0}) \geq C_k \min\left\{|E \cap B_{R_0}|, |E^c \cap B_{R_0}|\right\}^{(k-1)/k} > 0.$$

On the other hand,

$$Per(\partial E; B_{R_0}) = \mathcal{H}^{k-1}(B_{R_0} \cap \partial_{red}E).$$

Therefore, there exists $0 \neq \bar{x} \in \partial_{red}\{\chi_0 > 0\}$ as claimed, and the theorem is proved.  $\square$

In the remainder of the section we will let $u^{\varepsilon_j}$ be stationary solutions to $P_{\varepsilon_j}(f_{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ with $\|f_{\varepsilon_j}\|_{L^\infty(\Omega)} \leq C$, $\|u^{\varepsilon_j}\|_{L^\infty(\Omega)} \leq C'$ and $\varepsilon_j \to 0$ and we will let $u = \lim u^{\varepsilon_j}$ uniformly on compact subsets of $\Omega$, and $\chi = \lim B_{\varepsilon_j}(u^{\varepsilon_j}) *$–weakly in $L^\infty(\Omega)$. For points $x \in \Omega \cap \partial\{u > 0\}$ we will consider $\delta(x)$ as defined in (3.1).

We prove

**Lemma 3.1.** *Let $x_0 \in \Omega \cap \partial\{u > 0\}$. Then,*

$$\limsup_{\substack{x \to x_0 \\ x \in \Omega \cap \partial\{u>0\}}} \delta(x) \leq \delta(x_0).$$

*Proof.* Let $\sigma > 0$ such that $B_{2\sigma}(x_0) \subset\subset \Omega$ and $\varphi \in C_0^\infty(B_\sigma(0))$, with $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ in $B_{\sigma/2}(0)$. For $\bar{x} \in B_\sigma(x_0) \cap \partial\{u > 0\}$ we denote $\psi^{\bar{x}}(x) = \varphi(x - \bar{x})$ and define

$$\mathcal{W}_{(\bar{x},0)}(r) = \mathcal{W}_{(\bar{x},0)}(r, u, \psi^{\bar{x}}, \chi) = \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} \left(\left|\nabla(\psi^{\bar{x}}u)\right|^2 + 2(\psi^{\bar{x}})^2\chi + \frac{(\psi^{\bar{x}}u)^2}{2t}\right) G(x - \bar{x}, -t)dx\, dt.$$

Now fix $\eta > 0$. Then, by Theorem 2.2 in [7], there exists $r_0 = r_0(\eta)$ such that

(3.6)     $$\delta(\bar{x}) = \mathcal{W}_{(\bar{x},0)}(0^+) \leq \mathcal{W}_{(\bar{x},0)}(r) + \frac{\eta}{2} \qquad \text{if } r \leq r_0,$$

where $r_0$ can be taken independent of the point $\bar{x} \in B_\sigma(x_0)$ if the constants in [7] are suitably chosen.

On the other hand, there exists $\theta = \theta(r, \eta) \leq \sigma$ such that

(3.7) $$\mathcal{W}_{(\bar{x},0)}(r) \leq \mathcal{W}_{(x_0,0)}(r) + \frac{\eta}{2} \qquad \text{if } \bar{x} \in B_\theta(x_0).$$

In fact, since in $\mathbb{R}^N \times [-4r^2, -r^2]$,

$$\left| |\nabla(\psi^{\bar{x}} u)|^2 + 2(\psi^{\bar{x}})^2 \chi + \frac{(\psi^{\bar{x}} u)^2}{2t} \right| \leq C_r,$$

$$\left| \left( |\nabla(\psi^{\bar{x}} u)|^2 + 2(\psi^{\bar{x}})^2 \chi + \frac{(\psi^{\bar{x}} u)^2}{2t} \right) - \left( |\nabla(\psi^{x_0} u)|^2 + 2(\psi^{x_0})^2 \chi + \frac{(\psi^{x_0} u)^2}{2t} \right) \right| \leq C_r |\bar{x} - x_0|,$$

with $C_r$ independent of $\bar{x}$, we get

$$|\mathcal{W}_{(\bar{x},0)}(r) - \mathcal{W}_{(x_0,0)}(r)| \leq$$

$$\leq \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} C_r \left| G(x - \bar{x}, -t) - G(x - x_0, -t) \right| dx\, dt +$$

$$+ \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} C_r |\bar{x} - x_0| G(x - x_0, -t) dx\, dt \leq$$

$$\leq C_r' |\bar{x} - x_0|,$$

which implies (3.7).

Finally, from (3.6) and (3.7) we get, for $r \leq r_0(\eta)$,

$$\limsup_{\substack{x \to x_0 \\ x \in \Omega \cap \partial\{u > 0\}}} \delta(x) \leq \mathcal{W}_{(x_0,0)}(r) + \eta,$$

and the result follows by letting $r \to 0$ first, and then $\eta \to 0$. $\qquad \square$

**Lemma 3.2.** *Let $x_0 \in \Omega \cap \partial\{u > 0\}$ be such that $x_0 \notin \partial_*\{\chi > 0\}$, where we denote $\partial_*\{\chi > 0\}$ the set of points $x \in \mathbb{R}^N$ such that*

$$\limsup_{r \to 0} \frac{|B_r(x) \cap \{\chi > 0\}|}{|B_r(x)|} > 0, \quad \limsup_{r \to 0} \frac{|B_r(x) \cap \{\chi > 0\}^c|}{|B_r(x)|} > 0.$$

*Then, $\delta(x_0) = 0$ or $\delta(x_0) = 6M$.*

*Proof.* Let $\lambda_n \to 0$ be such that there exist $u_0 = \lim_{n \to \infty} u_{\lambda_n}$ uniformly on compact sets of $\mathbb{R}^N$ and $\chi_0 = \lim_{n \to \infty} \chi^{\lambda_n}$ $*$–weakly in $L_{loc}^\infty(\mathbb{R}^N)$ (here $u_{\lambda_n}(x) = \frac{1}{\lambda_n} u(x_0 + \lambda_n x)$ and $\chi^{\lambda_n}(x) = \chi(x_0 + \lambda_n x)$).

If

$$\limsup_{r \to 0} \frac{|B_r(x_0) \cap \{\chi > 0\}|}{|B_r(x_0)|} = 0,$$

then for any $R > 0$,

$$\lim_{n \to \infty} \frac{|B_R(0) \cap \{\chi^{\lambda_n} > 0\}|}{|B_R(0)|} = 0$$

and therefore, $\chi_0 \equiv 0$. Recalling (3.2) we obtain that $\delta(x_0) = 0$.

Now, if

$$\limsup_{r \to 0} \frac{|B_r(x_0) \cap \{\chi > 0\}^c|}{|B_r(x_0)|} = 0,$$

we argue similarly and deduce that $\chi_0 \equiv M$, which implies that $\delta(x_0) = 6M$ and proves the lemma. $\qquad \square$

As a consequence of the previous lemmas we get

**Proposition 3.1.** *Let $x_0 \in \Omega \cap \partial\{u > 0\}$. Then, $\delta(x_0) \in [3M, 6M]$.*

*Proof.* We first recall that the arguments in the proof of Theorem 3.1 imply that $\delta(x_0) \in [0, 6M]$.

Next let $\rho > 0$ such that $B_\rho(x_0) \subset\subset \Omega$. Since $x_0 \in \Omega \cap \partial\{u > 0\}$, there exists $\hat{x} \in B_{\rho/2}(x_0)$ such that $u(\hat{x}) > 0$. Let us take $0 < \mu \leq \rho/2$ such that there exists $\bar{x} \in \partial B_\mu(\hat{x}) \cap \partial\{u > 0\}$ with $B_\mu(\hat{x}) \subset \{u > 0\}$.

Since $u > 0$ in $B_\mu(\hat{x})$, there holds that $\chi \equiv M$ in $B_\mu(\hat{x})$.

Now let $\lambda_n \to 0$ be such that there exist $u_0(x) = \lim_{n\to\infty} \frac{1}{\lambda_n} u(\bar{x} + \lambda_n x)$ uniformly on compact sets of $\mathbb{R}^N$ and $\chi_0(x) = \lim_{n\to\infty} \chi(\bar{x} + \lambda_n x) *$–weakly in $L^\infty_{loc}(\mathbb{R}^N)$. Then $\chi_0 \equiv M$ in $\{\langle x, \nu \rangle > 0\}$ with $\nu = \frac{\hat{x} - \bar{x}}{|\hat{x} - \bar{x}|}$.

Recalling again (3.2) we obtain that $\delta(\bar{x}) \geq 3M$. Since $\bar{x} \in B_\rho(x_0) \cap \partial\{u > 0\}$, where $\rho > 0$ can be chosen arbitrarily small, we deduce from Lemma 3.1 that $\delta(x_0) \geq 3M$ and this completes the proof. $\qquad\square$

**Proposition 3.2.** *There holds that $\delta(x_0) = 3M$ or $\delta(x_0) = 6M$ for $\mathcal{H}^{N-1}$-almost every $x_0 \in \Omega \cap \partial\{u > 0\}$.*

*Proof.* We first observe that the bounds in the proof of Lemma 3.1 in [8] imply that $\chi \in BV_{loc}(\Omega)$. In particular, $\chi = M\chi_{\{\chi > 0\}}$ and thus, $\{\chi > 0\}$ is a set of locally finite perimeter.

Let $x_0 \in \Omega \cap \partial\{u > 0\}$.

Assume $x_0 \in \partial_{red}\{\chi > 0\} \subset \partial_*\{\chi > 0\}$. Then, if $\lambda_n \to 0$, we have $\chi_0(x) = \lim_{n\to\infty} \chi(x_0 + \lambda_n x) = M\chi_{\{\langle x, \nu \rangle > 0\}}$ for $\nu$ the unit interior normal to $\{\chi > 0\}$ at $x_0$ in the measure theoretic sense and therefore, $\delta(x_0) = 3M$.

Now assume $x_0 \notin \partial_*\{\chi > 0\}$. Then, it follows from Proposition 3.1 and Lemma 3.2 that $\delta(x_0) = 6M$.

Finally, we obtain the desired result observing that, by Lemma 1 in [5], Section 5.8, there holds that $\mathcal{H}^{N-1}(\partial_*\{\chi > 0\} \setminus \partial_{red}\{\chi > 0\}) = 0$. $\qquad\square$

**Proposition 3.3.** *Assume $N = 2$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$. Then, $\delta(x_0) = 3M$ or $\delta(x_0) = 6M$.*

*Proof.* Let $\lambda_n \to 0$ be such that there exist $u_0(x) = \lim_{n\to\infty} \frac{1}{\lambda_n} u(x_0 + \lambda_n x)$ uniformly on compact sets of $\mathbb{R}^N$ and $\chi_0(x) = \lim_{n\to\infty} \chi(x_0 + \lambda_n x) *$–weakly in $L^\infty_{loc}(\mathbb{R}^N)$.

If $u_0 \leq 0$, then $\chi_0 \equiv 0$ or $\chi_0 \equiv M$, and thus Proposition 3.1 implies that $\delta(x_0) = 6M$.

If $u_0 > 0$ somewhere, we consider $\mathcal{A}$ a connected component of $\{u_0 > 0\}$. Then, from the homogeneity of $u_0$ we get that, in some system of coordinates, either $\mathcal{A} \subset \{x_1 > 0\}$ or else $\{x_1 > 0\} \subset \mathcal{A}$. In the first case, Lemma A1 in [3] implies that $u_0(x) = \alpha x_1^+ + o(|x|)$ in $\{x_1 > 0\}$, with $\alpha \geq 0$ and then the homogeneity of $u_0$ yields

$$u_0(x) = \alpha x_1^+ \quad \text{in } \{x_1 > 0\} \quad \text{and} \quad \alpha > 0.$$

Now, with a similar analysis in $\{x_1 < 0\}$ we conclude that

$$(3.8) \qquad\qquad u_0(x) = \alpha x_1^+ + \bar{\alpha} x_1^- \quad \alpha > 0, \; \bar{\alpha} \in \mathbb{R}.$$

The case in which $\{x_1 > 0\} \subset \mathcal{A}$ gives, with the same arguments, that again (3.8) holds and therefore, $\delta(x_0) = 3M$ or $\delta(x_0) = 6M$. $\qquad\square$

**Remark 3.1.** In [8] we obtained results on the regularity of the boundary of $\{u > 0\}$, for $u = \lim u^{\varepsilon_j}$, with $u^{\varepsilon_j}$ stationary solutions to $P_{\varepsilon_j}(f_{\varepsilon_j})$. In particular we dealt with the cases of energy minimizers (Theorem 10.2 in [8]) and travelling waves of a combustion model (Theorem 10.1 in [8], see also Theorem 4.2 in Section 4).

The results in [8] imply, for the first of these applications, that $\delta(x_0) = 3M$ for every $x_0 \in \Omega \cap \partial\{u > 0\}$, when $N = 2$ or $N = 3$, and similarly for the second one, when $N = 2$. Moreover, in both cases, $\delta(x_0) = 3M$ for $\mathcal{H}^{N-1}$-almost every $x_0 \in \Omega \cap \partial\{u > 0\}$, in any dimension.

**Remark 3.2.** The reciprocal results of Theorems 3.1 and 3.2 are also true.

In fact, assume that a blow up limit $u_0$ at $x_0 \in \Omega \cap \partial\{u > 0\}$ has the form $u_0(x) = \alpha|x_1|$ with $\alpha \geq 0$. Then $\chi_0 \equiv 0$ or $\chi_0 \equiv M$ and therefore, $\delta(x_0) = 0$ or $\delta(x_0) = 6M$. Recalling Proposition 3.1, we deduce that $\delta(x_0) = 6M$ so the reciprocal of Theorem 3.1 holds.

Now assume that a blow up limit $u_0$ at $x_0 \in \Omega \cap \partial\{u > 0\}$ has the form $u_0(x) = \alpha x_1^+ - \gamma x_1^-$ with $\alpha > 0$ and $\gamma \geq 0$. Then $\chi_0 = M\chi_{\{x_1 > 0\}}$ or $\chi_0 \equiv M$ and therefore, $\delta(x_0) = 3M$ or $\delta(x_0) = 6M$. But $\delta(x_0) = 6M$ would give, by Theorem 3.1, that $u_0(x) = \tilde{\alpha}|\langle x, \nu\rangle|$ for some direction $\nu$ and some $\tilde{\alpha} \geq 0$, a contradiction. Therefore $\delta(x_0) = 3M$ so that the reciprocal of Theorem 3.2 also holds.

## 4. APPLICATION: REGULARITY OF THE FREE BOUNDARY

In this section we present applications of the results in Section 3. They deal with the regularity of the boundary of $\{u > 0\}$ ($u = \lim u^{\varepsilon_j}$) in the stationary case including, in particular, regularity results for travelling waves of a combustion model.

First, we consider a family $u^{\varepsilon_j}$ of stationary solutions to $P_{\varepsilon_j}(f_{\varepsilon_j})$ such that $u^{\varepsilon_j}$ and $f_{\varepsilon_j}$ are uniformly bounded in $L^\infty$ norm. In [8] we proved that $u^{\varepsilon_j}$ are locally uniformly bounded in $Lip$ norm. So that, the results of the present paper apply to this family. Let $u = \lim u^{\varepsilon_j}$ uniformly on compact subsets as $\varepsilon_j \to 0$. In [8] we proved that $u$ is a solution to

$$\Delta u = f\chi_{\{u \neq 0\}} \quad \text{in } \{u > 0\} \cup \{u \leq 0\}^\circ$$

where $f = \lim f_{\varepsilon_j}$ $*$-weakly in $L^\infty$.

Moreover, in [8] we proved that, under suitable assumptions, $\partial\{u > 0\}$ is smooth and $u$ is a classical solution to the following free boundary problem

$$(E(f)) \qquad \begin{aligned} \Delta u &= f\chi_{\{u \neq 0\}} && \text{in } \{u > 0\} \cup \{u \leq 0\}^\circ, \\ |\nabla u^+|^2 - |\nabla u^-|^2 &= 2M && \text{on } \partial\{u > 0\}. \end{aligned}$$

The purpose of this section is to state some theorems on the regularity of the free boundary $\partial\{u > 0\}$ that are proved in [8] for which the results in this paper are an essential tool.

In fact, assume $u$ is defined in $B_\sigma(x_0)$ with $x_0 \in \partial\{u > 0\}$. Let $\chi = \lim B_{\varepsilon_j}(u^{\varepsilon_j})$ $*$-weakly in $L^\infty(B_\sigma(x_0))$ and consider $\delta(x_0)$ as in (3.1).

In the next theorem, we assume that, in $B_\sigma(x_0)$, $u^+$ is uniformly nondegenerate. This property holds in many applications (see, for instance, Theorem 4.2 below). By uniform nondegeneracy we mean that there exists $c > 0$ such that

$$u^+(x) \geq c \operatorname{dist}(x, \{u \leq 0\}).$$

As a first application, we have the following result,

**Theorem 4.1** (Theorem 9.7 in [8])**.** *There holds that $\delta(x_0) = 3M$ if and only if the free boundary is $C^{1,\alpha}$ in a neighborhood of $x_0$. This implies that $u$ is a classical solution to the free boundary problem $E(f)$ in a neighborhood of $x_0$.*

Theorem 3.2 is a key tool in the proof of Theorem 4.1.

We point out that there are examples in [6] that show that the free boundary condition in $E(f)$ may not hold at any free boundary point. In fact, $u^+$ may degenerate or the density of $\{u \leq 0\}$ may be zero at a boundary point. Thus, some extra assumption is needed if one wants to show that $u$ is a solution to $E(f)$.

The results in the present paper are also used in [8] to obtain the following regularity result for travelling waves of a combustion model. In fact, we have

**Theorem 4.2** (Theorem 10.1 in [8]). *Let $x = (x_1, y) \in \Omega = \mathbb{R} \times \Sigma$, with $\Sigma \subset \mathbb{R}^{N-1}$ a smooth bounded domain, let $a$ be a continuous positive function on $\overline{\Sigma}$ and let $0 < \widetilde{\sigma} < 1$ be given.*
*Consider travelling wave solutions to the following combustion model*

$$\Delta v^\varepsilon - a(y)v_t^\varepsilon = \beta_\varepsilon(v^\varepsilon),$$

*where $\beta_\varepsilon$ is as before with $\beta'(0) > 0$. This is, $v^\varepsilon(x,t) = u^\varepsilon(x_1 + c^\varepsilon t, y)$, with $u^\varepsilon$ solutions to*

$$\Delta u^\varepsilon - c^\varepsilon a(y)u_{x_1}^\varepsilon = \beta_\varepsilon(u^\varepsilon) \qquad in \ \Omega,$$

$$u^\varepsilon(-\infty, y) = (1 - \widetilde{\sigma})^{-1}, \qquad u^\varepsilon(+\infty, y) = 0 \qquad in \ \Sigma,$$

$$\frac{\partial u^\varepsilon}{\partial \eta} = 0 \qquad on \ \mathbb{R} \times \partial\Sigma,$$

*for some suitable $c^\varepsilon$.*
*Let $u = \lim u^{\varepsilon_j}$ ($\varepsilon_j \to 0$). Then, there is a subset $\mathcal{R}$ of the free boundary $\Omega \cap \partial\{u > 0\}$ which is locally a $C^{1,\alpha}$ surface and $u$ is a classical solution to the free boundary problem $E(f)$ in a neighborhood of $\mathcal{R}$ ($f = ca(y)u_{x_1}$ with $c = \lim c^{\varepsilon_j}$). Moreover, $\mathcal{R}$ is open and dense in $\Omega \cap \partial\{u > 0\}$ and the remainder of the free boundary has $(N-1)-$dimensional Hausdorff measure zero.*
*In dimension 2 we have $\mathcal{R} = \Omega \cap \partial\{u > 0\}$.*
*In addition, in any dimension, if $a \in C_{loc}^{k,\alpha}$ (resp. analytic) then, $\mathcal{R} \in C_{loc}^{k+2,\alpha}$ (resp. analytic).*

We remark that this travelling wave problem was first studied in [1], where the authors obtained existence of $(u^\varepsilon, c^\varepsilon)$, strict monotonicity in the $x_1$ direction, uniform Lipschitz estimates and uniform nondegeneracy of the family $u^\varepsilon$, as well as uniform estimates of the velocities $c^\varepsilon$.

The proof of Theorem 4.2 relies on the fact that the density of the zero set is positive at every free boundary point. We obtain this density property by a contradiction argument that strongly uses Theorem 3.1.

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