Algebraic K-theory of ample groupoids

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Motivation and tools

We study isomorphism conjectures such as Farrell-Jones, Baum-Connes or discretisation. These claim that one can compute the K-theory of group(oid) algebras in terms of *easier* things. A possible approach for Baum-Connes is the Meyer-Nest [4] one:

Baum-Connes via localisation: approximations $P(A) \to A$ in KK^G for any G-algebra A by *compactly induced objects* induces an assembly map

$$K^{\mathrm{top}}(P(A) \rtimes G) \to K^{\mathrm{top}}(A \rtimes G).$$

$P(A) \in \langle \mathrm{ind}_H^G(B) \rangle$, the localising subcategory generated by algebras induced by compact subgroups. **Green's Imprimitivity:** *easier* left hand-side

$$K^{\mathrm{top}}(\mathrm{ind}_H^G(B) \rtimes G) \simeq K^{\mathrm{top}}(H \rtimes B).$$

- Cortiñas-Thom in the algebraic setup [2]: There exist a universal M_∞-stable, homotopy invariant, excisive functor $j\colon \mathrm{Alg}_\ell \to kk$ into a triangulated category. Furthermore, it represents Weibel's homotopy K-theory: $kk(l, A) = KH(A)$.

Obstruction: it is not known whether the functor $j\colon \mathrm{Alg}_\ell \to kk$ preserves any kind of colimits, which is needed for the Meyer-Nest approach.

Analogue to [1], a functor over Alg_ℓ is ***homological*** if is M_∞-stable, homotopy invariant, excisive, **additive** and **filtered colimit** preserving.

Theorem (Arnone, Kranz, Mukherjee, N.): There exists a presentable stable ∞-category E with a homological functor $j\colon \mathrm{Alg}_\ell \to E$ such that

$$j^*\colon \mathrm{Fun}^{\mathrm{colim}}(E, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{hom}}(\mathrm{Alg}_\ell, \mathcal{C})$$

for any stable cocomplete ∞-category $\mathcal{C}$. Furthermore, it also represents homotopy K-theory: $E(\ell, A) \simeq KH(A)$.

Goal: study isomorphism conjectures in the algebraic setup for ***ample groupoids*** $\rightsquigarrow$ Application: discrete groups.

Steinberg algebra

Fix $\mathcal{G}$ an ample groupoid and ℓ a commutative unital ring.

- If Z is a space $\rightsquigarrow \mathcal{C}_c(Z, \ell) := \mathrm{span}_\ell\langle \chi_U : U \subseteq Z \text{ compact open}\rangle$.
- ***Steinberg algebra*** $\mathcal{A}_\ell(\mathcal{G}) := \mathcal{C}_c(\mathcal{G}, \ell)$ with convolution product
- If $\mathcal{G}^{(0)} = * \rightsquigarrow \mathcal{A}_\ell(\mathcal{G}) = \ell[G]$ the group algebra.

A ***$\mathcal{G}$-algebra*** B is an $\mathcal{A}_\ell(\mathcal{G})$-module together with an $\mathcal{A}_\ell(\mathcal{G})$-linear map

$$B \otimes_{\mathcal{C}_c(\mathcal{G}^0, \ell)} B \to B.$$

Examples: if $\tau\colon Z \to \mathcal{G}^{(0)}$ is a Hausdorff étale $\mathcal{G}$-space

- $\mathcal{C}_c(Z, \ell)$ with pointwise multiplication;
- $M_Z := \mathcal{C}_c(Z \,{}_\tau\!\times_\tau Z, \ell)$ with *matrix-like* multiplication.

We restrict ourselves to **Hausdorff** ample groupoids.

Algebraic $E^{\mathcal{G}}$-theory for Hausdorff ample groupoids

Joint work with Arnone, Kranz, Mukherjee. A functor over $\mathrm{Alg}_{\mathcal{G}}$ is:

- ***$\mathcal{G}$-stable*** if inverts *corner inclusions* $M_Z \hookrightarrow M_{Z \sqcup W}$.
- ***$\mathcal{G}$-homological*** if is $\mathcal{G}$-stable, $\mathcal{G}$-homotopy invariant, $\mathcal{G}$-excisive, additive and filtered colimit preserving.

Theorem: There is a presentable stable ∞-category $E^{\mathcal{G}}$ with a $\mathcal{G}$-homological functor $j_{\mathcal{G}}\colon \mathrm{Alg}_{\mathcal{G}} \to E^{\mathcal{G}}$ s.t. for any stable cocomplete $\mathcal{C}$:

$$j_{\mathcal{G}}^*\colon \mathrm{Fun}^{\mathrm{colim}}(E^{\mathcal{G}}, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathcal{G}\text{-hom}}(\mathrm{Alg}_{\mathcal{G}}, \mathcal{C})$$

Moreover, we have crossed product functor and an ind-res adjunction:

$$\begin{tikzcd} E^{\mathcal{H}} \arrow[r, bend left, "\mathrm{ind}_{\mathcal{H}}^{\mathcal{G}}"] \arrow[r, phantom, "\perp"] & E^{\mathcal{G}} \arrow[l, bend left, "\mathrm{res}_{\mathcal{H}}^{\mathcal{G}}"] \arrow[r, "-\rtimes\mathcal{G}"] & E \end{tikzcd}$$

Farrell-Jones conjecture via localisations

Joint w/ Arnone, Kranz, Mukherjee: algebraic Meyer-Nest approach.

Theorem (Green's imprimitivity): There is an E-equivalence

$$A \rtimes \mathcal{H} \hookrightarrow \mathrm{ind}_{\mathcal{H}}^{\mathcal{G}}(A) \rtimes \mathcal{G}.$$

if $\mathcal{H}$ is an open subgroupoid of a Hausdorff ample groupoid $\mathcal{G}$.

- There are approximations $P(A) \to A$ by *properly* induced algebras.
- ***Meyer-Nest assembly map***: $KH(P(A) \rtimes \mathcal{G}) \to KH(A \rtimes \mathcal{G})$.
- If $A = \mathcal{C}_c(\mathcal{G}^{(0)}, \ell)$ the right hand-side is $KH(\mathcal{A}_\ell(\mathcal{G}))$.

Goal: identify this map with FJ assembly map (Cortiñas-Li '26).

Discretisation conjecture for trees

Idea [3]: assign to $\mathcal{G}$ a discrete groupoid $\mathcal{G}_d$ and a *proper* correspondence

$$\Omega_{\mathcal{G}}\colon \mathcal{G}_d \to \mathcal{G}.$$

Goal: show that $\Omega_{\mathcal{G}}$ induces isomorphisms in: homology, K^{top}, KH, etc.

Current case study: group G acting on an ***oriented tree*** (X^1, X^0)

- $\mathcal{G} = G \ltimes \overline{X^1}$, for $\overline{X^1} = X^1 \cup \{\infty\}$ a t.d. space and $\mathcal{G}_d = G \ltimes X^0$.
- Known for K-theory of groupoid C^*-algebras [5].

Theorem (N.): $\Omega_{G \ltimes \overline{X^1}}$ induces isomorphism in homology (as in [3]). Moreover, if G fixes a vertex $\Omega_{G \ltimes \overline{X^1}}$ induces isomorphism in KH.

References

[1] Bunke, U. *E-theory of X-C^*-algebras and functor formalisms.* arXiv:2605.19863

[2] Cortiñas, G. and Thom, A. *Bivariant algebraic K-theory.* J. Reine Angew. Math. 610, 2007.

[3] Li, X. and Miller, A. *Discretisation and independent resolutions of ample groupoids.* arXiv:2606.21761

[4] Meyer, R. and Nest, R., *The Baum-Connes conjecture via localisation of categories.* Topology, 2006.

[5] Pimsner, M. *KK-groups of crossed products by groups acting on trees.* Inventiones mathematicae 1986.