Topological groupoids and algebraic K-theory of Steinberg Algebras

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Topological, étale and ample groupoids

- A ***topological groupoid*** is a groupoid object in **Top**.

$$\begin{tikzcd} \mathcal{G}^{(1)} \underset{\mathbf{d}\ \mathbf{r}}{\times} \mathcal{G}^{(1)} \arrow[r, "\mathbf{m}"] & \mathcal{G}^{(1)} \arrow[r, bend left, "\mathbf{d}"] \arrow[r, bend right, "\mathbf{r}"'] \arrow[loop below, "(-)^{-1}"] & \mathcal{G}^{(0)} \arrow[l, "\mathbf{i}"'] \end{tikzcd}$$

A topological groupoid $\mathcal{G}$ is:

- ***étale*** if $\mathbf{d}\colon \mathcal{G}^{(1)} \to \mathcal{G}^{(0)}$ is a local homeomorphism and $\mathcal{G}^{(0)}$ is LCH.
- ***ample*** if $\mathcal{G}$ is étale and $\mathcal{G}^{(0)}$ is totally disconnected.

ExAmples:

- topological spaces $\rightsquigarrow \mathcal{G}^{(1)} = \mathcal{G}^{(0)} = X$ (ample if X is LCH and td),
- discrete groups $\rightsquigarrow \mathcal{G}^{(0)} = *$ (ample),
- translation groupoids $G \ltimes X$ when $G \curvearrowright X$ (ample if X is LCH and td),
- discrete groupoids $\mathcal{G}$

$$\mathcal{G} = \bigsqcup_{[x]\in\mathcal{G}^{(0)}/\mathcal{G}} \mathcal{G}\cdot x.$$

Local description of proper Hausdorff groupoids

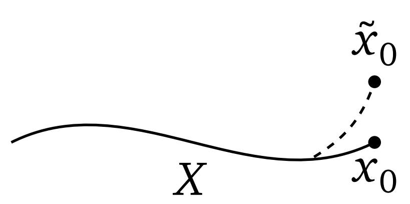
- An étale groupoid $\mathcal{G}$ is ***proper*** if $(\mathbf{d},\mathbf{r})\colon \mathcal{G}^{(1)} \to \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ is proper.
- If $x \in \mathcal{G}^{(0)}$ the ***stabilizer group*** is $\mathcal{G}_x^x := \mathbf{d}^{-1}(x) \cap \mathbf{r}^{-1}(x)$.

Theorem (N. [1]): If $\mathcal{G}$ is étale (ample) proper and Hausdorff then for every $x \in \mathcal{G}^{(0)}$ there exists an (compact) open set $x \in U_x \subseteq \mathcal{G}^{(0)}$ such that

$$\mathcal{G}_x^x \ltimes U_x \cong \mathcal{G}|_{U_x}^{U_x}.$$

Corollary: If $G \curvearrowright X$ properly over a LCH space X, then for every $x \in U \subseteq X$ there exists $x \in U_x \subseteq U$ such that U_x is $\mathrm{stab}_G(x)$-invariant.

There are counterexamples for non-Hausdorff or non-proper groupoids



Two-headed snake

Farrell-Jones conjecture

Idea: compute K-theory of group ring in terms of simpler things.

General version in terms of homotopy colimits over the ***orbit category***:

$$\mathrm{Or}(G,\mathcal{F}) = \{\text{discrete spaces } G/H,\ H\in\mathcal{F} \text{ with } G\text{-equivariant maps}\}.$$

FJ conjecture: The assembly map is a weak equivalence:

$$\operatorname*{hocolim}_{\mathrm{Or}(G,\mathcal{VCYC})} \mathbb{K}_R \xrightarrow{\simeq} K(RG),$$

where $\mathcal{VCYC}$ is the family of virtually cyclic subgroups.

Applications of Farrell-Jones to topology?

FJ conjecture for Whitehead group: If G is torsion free, $\mathrm{Wh}(G) = 0$.

By means of surgery theory:

Conjecture (Borel '53): any homotopy equivalence between closed aspherical topological manifolds is homotopic to a homeomorphism.

Steinberg algebra

- If Z is a topological space, we define the R-module:

$$C_c(Z,R) := \langle \chi_U : U \subseteq Z \text{ compact open}\rangle \subseteq R^Z.$$

- $\mathcal{G}$ *ample* $\rightsquigarrow$ ***Steinberg algebra*** [2] $\mathcal{A}_R(\mathcal{G}) = C_c(\mathcal{G}^{(1)},R)$ with product:

$$(\psi * \varphi)(g) := \sum_{\mathbf{d}(g)=\mathbf{d}(h)} \psi(gh^{-1})\varphi(h).$$

- System of *local units*: $\{\chi_U : U \subseteq \mathcal{G}^{(0)} \text{ compact open}\}$.

Groupoid homology: $H_*(\mathcal{G}) = \mathrm{Tor}_*^{\mathcal{A}_{\mathbb{Z}}(\mathcal{G})}(C_c(\mathcal{G}^{(0)},\mathbb{Z}), C_c(\mathcal{G}^{(0)},\mathbb{Z}))$.

ExAmples:

- X LCH and td $\rightsquigarrow \mathcal{A}_R(X) = C_c(X,R)$, where R is discrete.
- G discrete group $\rightsquigarrow \mathcal{A}_R(G) = RG$, the group algebra.
- $\mathcal{G}_X$ full equivalence relation in $X \times X \rightsquigarrow \mathcal{A}_R(\mathcal{G}_X) = M_X(R)$.
- $\mathcal{G}$ discrete groupoid

$$\mathcal{A}_R(\mathcal{G}) = \bigoplus_{[x]\in\mathcal{G}^{(0)}/\mathcal{G}} M_{\mathcal{G}\cdot x}\left(R[\mathcal{G}_x^x]\right).$$

Modules over Steinberg algebra as sheaves

A ***$\mathcal{G}$-space*** is a space Z together with an action of an étale groupoid:

$$\rho\colon Z \to \mathcal{G}^{(0)} \qquad Z\,{}_\rho\!\times_{\mathbf{d}} \mathcal{G}^{(1)} \xrightarrow{\bullet} Z$$

A ***sheaf of R-modules*** is a $\mathcal{G}$-space with ρ etale and its fibers in mod_R.

Theorem (Steinberg [3]) There is an equivalence of categories

$$\mathrm{Mod}_{\mathcal{A}_R(\mathcal{G})} \simeq \mathrm{Sh}(\mathcal{G}, \mathrm{mod}_R).$$

Idea: use Steinberg's equivalence to describe $\mathrm{Proj}_{\mathcal{A}_R(\mathcal{G})}$.

- F.g. projective modules are idempotents $\chi_U \mathcal{A}_R(\mathcal{G})^n \to \chi_U \mathcal{A}_R(\mathcal{G})^n$.
- What sheaf corresponds to $\chi_U \mathcal{A}_R(\mathcal{G})^n$?
- $\mathcal{G}$ étale, free (i.e. $(\mathbf{d},\mathbf{r})$ is injective) and proper $\rightsquigarrow \mathcal{G} \simeq_{\mathrm{morita}} \mathcal{G}^{(0)}/\mathcal{G}$.

Theorem (N. [1]): If $\mathcal{G}$ is ample, free and proper then

$$H_0(\mathcal{G}) \cong K_0(\mathcal{A}_{\mathbb{Z}}(\mathcal{G})).$$

Isomorphism conjectures for groupoids

Cortiñas-Li ('26): orbit category $\mathrm{Or}(\mathcal{G},\mathcal{F})$ and assembly map for $K(\mathcal{A}_R(\mathcal{G}))$.

which family of subgroupoids is analogue to virtually cyclic subgroups?

Goal: reduce FJ for groupoids to the discrete group case.

Discretization idea: $\mathcal{G}$ *nice* ample groupoid $\rightsquigarrow \mathcal{G}_d$ discrete groupoid s.t.

$$K_*(\mathcal{A}_R(\mathcal{G})) \underset{\textit{conj.}}{\cong} K_*(\mathcal{A}_R(\mathcal{G}_d)) \cong \bigoplus_{[x]\in\mathcal{G}_d^{(0)}/\mathcal{G}_d} K_*(R[(\mathcal{G}_d)_x^x]).$$

Examples of *nice* groupoids: spaces, groups acting on lattices, graphs.

References

[1] Nico, V. (2026). *Álgebras de Steinberg e invariantes homológicos.* Diploma thesis. Universidad de Buenos Aires.

[2] Steinberg, B. (2010). *A groupoid approach to discrete inverse semigroup algebras.* Advances in Mathematics, 223(2), 689-727.

[3] Steinberg, B. (2014). *Modules over étale groupoid algebras as sheaves.* Journal of the Australian Mathematical Society, 97(3), 418-429.