Weak universality of the KPZ equation

M. Hairer, joint with J. Quastel

University of Warwick

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KPZ equation

Introduced by Kardar, Parisi and Zhang in 1986.

Stochastic partial differential equation:

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi \qquad\qquad (d = 1)$$

Here ξ is space-time white noise: Gaussian generalised random field with $\mathbf{E}\xi(s,x)\xi(t,y) = \delta(t-s)\delta(y-x)$.

Model for propagation of nearly flat interfaces.

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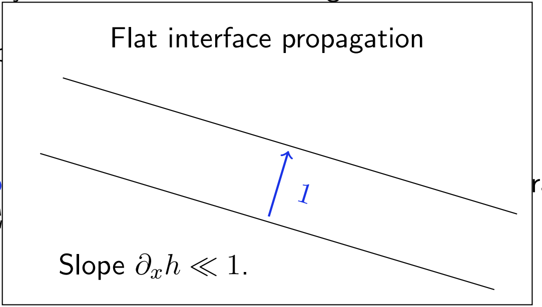

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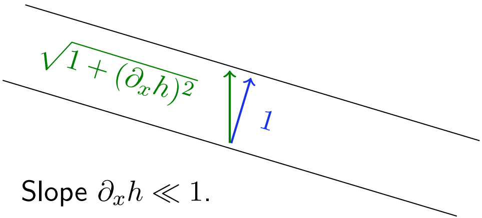
Flat interface propagation



Slope $\partial_x h \ll 1$.

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Strong Universality conjecture

At large scales, the fluctuations of every $1+1$-dimensional model $\tilde{h}$ of interface propagation exhibits the same fluctuations as the KPZ equation. These fluctuations are self-similar with exponents $1-2-3$:

$$\lim_{\lambda\to\infty} \lambda^{-1}\tilde{h}(\lambda^2 x, \lambda^3 t) - \tilde{C}_\lambda t = c_1 \lim_{\lambda\to\infty} \lambda^{-1} h(c_2\lambda^2 x, \lambda^3 t) - C_\lambda t \;.$$

Spectacular recent progress: Amir, Borodin, Corwin, Quastel, Sasamoto, Spohn, etc. Relies on considering models that are "exactly solvable". Partial characterisation of limiting "KPZ fixed point": experimental evidence.

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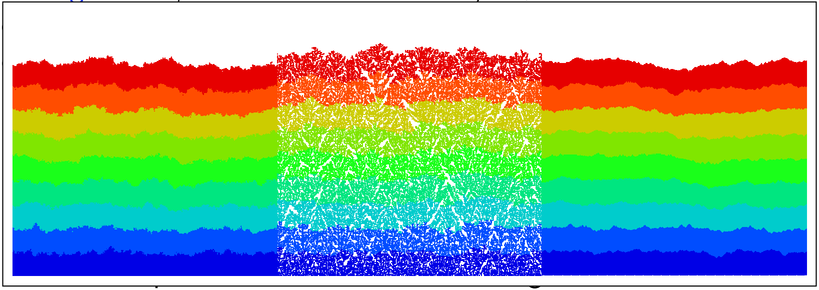
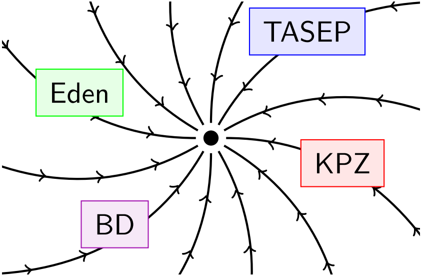


"exactly solvable". Partial characterisation of limiting "KPZ fixed point": experimental evidence.

Heuristic picture

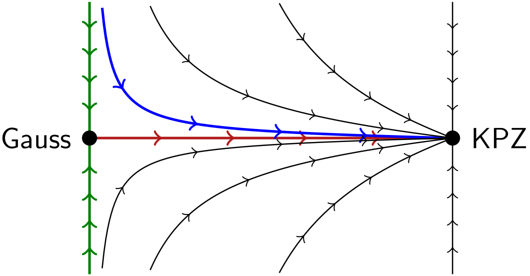
Schematic evolution in “space of models” under rescaling (modulo height shifts):



KPZ equation just one model among many...

All interface fluctuation models

Universality for symmetric interface fluctuation models: exponents $1-2-4$, Gaussian limit. Picture for all interface models:



KPZ equation: red line.

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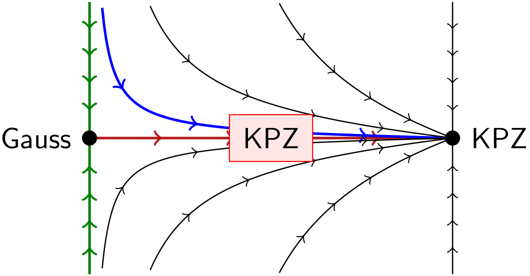

KPZ equation: red line.

Weak Universality conjecture

Conjecture: the KPZ equation is the only model on the "red line".

Conjecture: Let $\tilde{h}_\varepsilon$ be any "natural" one-parameter family of asymmetric interface models with ε denoting the strength of the asymmetry such that propagation speed $\approx \sqrt{\varepsilon}$.

As $\varepsilon \to 0$, there is a choice of $C_\varepsilon \sim \varepsilon^{-1}$ such that $\sqrt{\varepsilon}\tilde{h}_\varepsilon(\varepsilon^{-1}x, \varepsilon^{-2}t) - C_\varepsilon t$ converges to solutions h to the KPZ equation.

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Jara-Gonçalves (2010): accumulation points satisfy weak version of KPZ for generalisations of WASEP.

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Difficulty

Problem: KPZ equation is ill-posed:

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi \ ,$$

Solution behaves like Brownian motion for fixed times: nowhere differentiable!

Trick: Write $Z = e^h$ (Hopf-Cole) and formally derive

$$\partial_t Z = \partial_x^2 Z + Z\xi \ ,$$

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Recent progress

Theory of rough paths / regularity structures / paraproducts gives direct meaning to nonlinearity in a robust way:

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$\mathscr{F}$: Constant C in $\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi_\varepsilon - C$.

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Strategy: find $M_\varepsilon \in \mathfrak{R}$ such that $M_\varepsilon \Psi(\xi_\varepsilon)$ converges.

Universality result for KPZ

Consider the model

$$\partial_t h_\varepsilon = \partial_x^2 h_\varepsilon + \sqrt{\varepsilon} P(\partial_x h_\varepsilon) + \eta \,,$$

with P an even polynomial, η a Gaussian field with compactly supported correlations $\varrho(t,x)$ s.t. $\int \varrho = 1$.

Theorem (H., Quastel, 2014) As $\varepsilon \to 0$, there is a choice of $C_\varepsilon \sim \varepsilon^{-1}$ such that $\sqrt{\varepsilon} h(\varepsilon^{-1} x, \varepsilon^{-2} t) - C_\varepsilon t$ converges to solutions to (KPZ)$_\lambda$ with λ depending in a non-trivial way on all coefficients of P.

Remark: Convergence to KPZ with $\lambda \neq 0$ even if $P(u) = u^4$!!

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Nonlinearity $\lambda(\partial_x h)^2$

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Write $\tilde h_\varepsilon(x,t) = \sqrt{\varepsilon} h(\varepsilon^{-1}x, \varepsilon^{-2}t) - C_\varepsilon t$. Satisfies

$$\partial_t \tilde h_\varepsilon = \partial_x^2 h_\varepsilon + \varepsilon(\partial_x \tilde h_\varepsilon)^4 + \xi_\varepsilon - C_\varepsilon\ ,$$

with ξ_ε an ε-approximation to white noise.

Fact: Derivatives of microscopic model do not converge to 0 as $\varepsilon \to 0$: no small gradients! Heuristic: gradients have $\mathcal{O}(1)$ fluctuations but are small on average over large scales... General formula:

$$\lambda = \frac{1}{2}\int P''(u)\,\mu(du)\ , \qquad C_\varepsilon = \frac{1}{\varepsilon}\int P(u)\,\mu(du) + \mathcal{O}(1)\ ,$$

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Main step in proof

Rewrite general equation in integral form as

$$H = \mathcal{P}\big(\mathcal{E}(\mathscr{D}H)^4 + a(\mathscr{D}H)^2 + \Xi\big)\;,$$

with $\mathcal{E}$ an abstract integration operator of order 1.

Find two-parameter lift of noise $\eta \mapsto \Psi_{\alpha,c}(\eta)$ so that $h = \mathcal{R}H$ solves

$$\begin{aligned}\partial_t h &= \partial_x^2 h + \alpha H_4(\partial_x h, c) + aH_2(\partial_x h, c) + \eta \\ &= \partial_x^2 h + \alpha(\partial_x h)^4 + (a - 6\alpha c)(\partial_x h)^2 + (3\alpha c^2 - ac) + \eta\;.\end{aligned}$$

Show that $\Psi_{\varepsilon,1/\varepsilon}(\xi_\varepsilon)$ converges to same limit as $\Psi_{0,1/\varepsilon}(\xi_\varepsilon)$!
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Some open questions:

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2. Convergence of particle models. (Are "weak solutions" unique??)
3. Convergence on whole space instead of circle (cf. Labbé).
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