On the minimum of a polynomial function on a basic closed semialgebraic set and applications

Gabriela Jeronimo[◇,♮,*], Daniel Perrucci[◇,♭,*], Elias Tsigaridas[♯,†]

◇ Departamento de Matemática, FCEN, Universidad de Buenos Aires, Argentina
♮ IMAS, CONICET–UBA, Argentina
♭ CONICET, Argentina
♯ POLSYS Project, INRIA Paris-Rocquencourt, UPMC, Univ Paris 06, France

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Abstract

We give an explicit upper bound for the algebraic degree and an explicit lower bound for the absolute value of the minimum of a polynomial function on a compact connected component of a basic closed semialgebraic set when this minimum is not zero. We also present extensions of these results to non-compact situations. As an application, we obtain a lower bound for the separation of two disjoint connected components of basic closed semialgebraic sets, when at least one of them is compact.

1 Introduction

Let $T \subset \mathbb{R}^n$ be a basic closed semialgebraic set defined by polynomials with integer coefficients and let C be a connected component of T. In this work, we consider the following problem: given a polynomial $g \in \mathbb{Z}[x_1, \dots, x_n]$ that attains a non-zero minimum value over C, find bounds $\delta > 0$ and $b > 0$ such that the minimum is an algebraic number of degree at most δ and its absolute value is greater than or equal to b. We look for explicit bounds δ and b in terms of the number of variables, the number of polynomials defining T, and upper bounds for the degrees and coefficient size of these polynomials and g. Such explicit bounds are of fundamental importance in the complexity analysis of symbolic and numerical methods for optimization and polynomial system solving (see, for instance, [1]).

Generally, in the case of polynomial optimization with polynomial constraints, the solutions to the problem are algebraic numbers; hence, it is natural to study the degree of the minimal

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polynomial that defines them ([14]). A standard technique to handle optimization problems with inequality constraints is to use the Karush-Kuhn-Tucker conditions (see [16, Chapter 12]). In [14], this approach is followed to obtain a general formula for the algebraic degree of the minimizers of a polynomial over a closed semialgebraic subset of $\mathbb{R}^n$ defined by at most n generic polynomials. Moreover, when the polynomials are not generic but the Karush-Kuhn-Tucker system is still zero-dimensional, by means of deformation techniques, the authors prove that the same formula is an upper bound for this algebraic degree.

In this paper, we prove bounds for the algebraic degree and the absolute value of the minimum for an arbitrary family of polynomial constraints, under the assumption that the set of minimizers over the considered connected component is compact (see Theorem 14). These bounds will be easily deduced from the case of compact connected components. Since the system which gives the critical points for g on T may not satisfy certain required hypothesis or may provide us with an infinite set of possible minimizers, we use deformation techniques; more precisely, we follow the approach in [11] relying on [2, Chapter 13]. The deformation enables us to deal with 'nice' systems which, in the limit, define a finite set of minimizing points. A careful analysis of the perturbed systems combined with resultant-based estimations using results from [19] leads us to the explicit bounds (see [3], [5], [9] and [10] for similar applications of these techniques). Our key result is the following:

Theorem 1 *Let $T = \{x \in \mathbb{R}^n \mid f_1(x) = \cdots = f_l(x) = 0, f_{l+1}(x) \geq 0, \ldots, f_m(x) \geq 0\}$ be defined by polynomials $f_1, \ldots, f_m \in \mathbb{Z}[x_1, \ldots, x_n]$ with $n \geq 2$, degrees bounded by an even integer d and coefficients of absolute value at most H, and let C be a compact connected component of T. Let $g \in \mathbb{Z}[x_1, \ldots, x_n]$ be a polynomial of degree at most d and coefficients of absolute value bounded by H. Then, the minimum value that g takes over C is a real algebraic number of degree at most*

$$\max_{0 \leq s \leq \min\{m,n\}} \binom{n}{s} d^s (d-1)^{n-s} \leq 2^{n-1} d^n$$

and, if it is not zero, its absolute value is greater than or equal to

$$(2^{4-\frac{n}{2}} \tilde{H} d^n)^{-n2^n d^n},$$

where $\tilde{H} = \max\{H, 2n + 2m\}$.

Even though the compactness assumptions are in general hard to check, they hold naturally in some applications; this is the case for instance in certain problems from game theory ([8]) and from mathematical programming (see Section 4).

The lower bound for the absolute value for the minimum in Theorem 1 can be applied, for instance, to make more explicit the upper bound for the degrees in Schmüdgen's Positivstellensatz from [17, Theorem 3], which is stated in terms of the minimum of a polynomial on a compact set.

Our bound for the algebraic degree is similar to that in [14, Theorem 2.2] and, in this sense, our result can be seen as an extension of those in [14]. In fact, Theorem 1 also enables us to obtain an explicit upper bound of the same kind for the algebraic degree of the minimum under the weaker assumption that this minimum is attained (see Theorem 15).

An application of Theorem 14, which is in fact the original motivation of this work, is an explicit lower bound for the separation between disjoint connected components of basic closed semialgebraic sets. Bounds of this kind can be applied to estimate the running time of numeric algorithms dealing with polynomial equations and inequalities (see, for instance, [12], [20]). They can also be applied in robotics, more precisely, in the motion planning problem: the connected components of the configuration space represent the forbidden area, that is, where the robot should not go, and the separation bound gives the number of bits needed in the worst case to compute a point on the path that avoids the obstacles (see [4]).

The problem of estimating the separation for isolated points has already been studied both in the complex and real settings (see, for instance, [4], [5], [9]). Our result, which includes positive dimensional situations, is the following:

Theorem 2 *Let $T_1 = \{x \in \mathbb{R}^n \mid f_1(x) = \cdots = f_{l_1}(x) = 0, f_{l_1+1}(x) \geq 0, \dots, f_{m_1}(x) \geq 0\}$, $T_2 = \{x \in \mathbb{R}^n \mid g_1(x) = \cdots = g_{l_2}(x) = 0, g_{l_2+1}(x) \geq 0, \dots, g_{m_2}(x) \geq 0\}$ be defined by polynomials $f_1, \dots, f_{m_1}, g_1, \dots, g_{m_2} \in \mathbb{Z}[x_1, \dots, x_n]$ with degrees bounded by an even integer d and coefficients of absolute value at most H. Let C_1 be a compact connected component of T_1 and C_2 a connected component of T_2. Then, if $C_1 \cap C_2 = \emptyset$, the distance between C_1 and C_2 is at least*

$$(2^{4-n}\tilde{H}d^{2n})^{-n2^{2n}d^{2n}},$$

where $\tilde{H} = \max\{H, 4n + 2m_1 + 2m_2\}$.

The paper is organized as follows: Section 2 is devoted to proving the bounds for the minimum. First, we introduce the deformation techniques we use and prove some geometric properties of this deformation which, in particular, enable us to give a characterization of minimizers as solutions to a polynomial system. Then, we prove Theorem 1 and, finally, we extend this theorem to the non-compact case (Theorems 14 and 15). In Section 3, we prove Theorem 2 and present an easy example to show that the double exponential nature of our bounds is unavoidable. Section 4 presents further applications of our bounds.

2 The minimum of a polynomial function

Let $f_1, \dots, f_m, g \in \mathbb{Z}[x_1, \dots, x_n]$ with $n \geq 2$, d an even positive integer such that $\deg(f_1), \dots, \deg(f_m) \leq d$, and $d_0 = \deg(g) \leq d$. Let $H \in \mathbb{N}$ be an upper bound on the absolute values of all the coefficients of $f_1, \dots, f_m$ and $H_0 \in \mathbb{N}$, $H_0 \leq H$, an upper bound on the absolute values of the coefficients of g. Let $T = \{x \in \mathbb{R}^n \mid f_1(x) = \cdots = f_l(x) = 0, f_{l+1}(x) \geq 0, \dots, f_m(x) \geq 0\}$ and let C be a compact connected component of T.

2.1 The deformation

Here we introduce some notations that we will use throughout this section. Let

- $A \in \mathbb{Z}^{(m+1)\times(n+1)}$, $A = (a_{ij})_{0\leq i\leq m,\ 0\leq j\leq n}$ be a matrix such that each of its submatrices has maximal rank and $a_{ij} > 0$ for every i, j.
- For every $1 \leq i \leq m$, $\tilde{f}_i(x) = \sum_{j=1}^n a_{ij}x_j^d + a_{i0}$, $F_i^+(t, x) = f_i(x) + t\tilde{f}_i(x)$ and $F_i^-(t, x) = f_i(x) - t\tilde{f}_i(x)$.
- $\tilde{g}(x) = \sum_{j=1}^n a_{0j}x_j^d + a_{00}$ and $G(t, x) = g(x) + t\tilde{g}(x)$.

- For every $S \subset \{1, \dots, m\}$ and $\sigma \in \{+, -\}^S$,

$$\hat{W}_{S,\sigma} = \{(t,x) \in \mathbb{A} \times \mathbb{A}^n \mid F_i^{\sigma_i}(t,x) = 0 \text{ for every } i \in S\},$$

$$\hat{Z}_{S,\sigma} = \{(t,x) \in \mathbb{A} \times \mathbb{A}^n \mid (t,x) \in \hat{W}_{S,\sigma} \text{ and } \{\nabla_x F_i^{\sigma_i}(t,x), i \in S\} \text{ is linearly dependent}\},$$

  and

$$\hat{V}_{S,\sigma} = \{(t,x,\lambda) \in \mathbb{A} \times \mathbb{A}^n \times \mathbb{P}^{|S|} \mid (t,x) \in \hat{W}_{S,\sigma} \text{ and }$$
$$\lambda_0 \nabla_x G(t,x) = \sum_{i \in S} \lambda_i \nabla_x F_i^{\sigma_i}(t,x)\},$$

  where $\mathbb{A}$ and $\mathbb{P}$ denote the affine and projective spaces over the complex numbers respectively. We consider the decomposition of $\hat{W}_{S,\sigma}$ as $\hat{W}_{S,\sigma} = W_{S,\sigma}^{(0)} \cup W_{S,\sigma}^{(1)} \cup W_{S,\sigma}$, where

  - $W_{S,\sigma}^{(0)}$ is the union of the irreducible components of $\hat{W}_{S,\sigma}$ included in $t = 0$,
  - $W_{S,\sigma}^{(1)}$ is the union of the irreducible components of $\hat{W}_{S,\sigma}$ included in $t = t_0$ for some $t_0 \in \mathbb{C} - \{0\}$,
  - $W_{S,\sigma}$ is the union of the remaining irreducible components of $\hat{W}_{S,\sigma}$,

  and the analogous decompositions of $\hat{Z}_{S,\sigma}$ and $\hat{V}_{S,\sigma}$ as $\hat{Z}_{S,\sigma} = Z_{S,\sigma}^{(0)} \cup Z_{S,\sigma}^{(1)} \cup Z_{S,\sigma}$ and $\hat{V}_{S,\sigma} = V_{S,\sigma}^{(0)} \cup V_{S,\sigma}^{(1)} \cup V_{S,\sigma}$ respectively.

- For a group of variables y, Π_y will indicate the projection to the coordinates y.

We start by constructing a matrix A satisfying the conditions required above and bounding its entries.

Lemma 3 *There exists a matrix $A \in \mathbb{Z}^{(m+1)\times(n+1)}$, $A = (a_{ij})_{0 \le i \le m,\ 0 \le j \le n}$, such that each of its submatrices has maximal rank and $0 < a_{ij} \le 2(n+m)$ for every i, j.*

Proof. Let p be a prime number such that $n + m + 2 \le p \le 2n + 2m + 1$, which exists by Bertrand's postulate.

Consider the Hilbert matrix $A_1 = (\frac{1}{i+j+1})_{0 \le i \le m,\ 0 \le j \le n}$, which is a particular case of a Cauchy matrix; therefore, every submatrix of A_1 has maximal rank. Let $A_2 = (n + m + 1)!A_1$; then, $A_2 \in \mathbb{Z}^{(m+1)\times(n+1)}$ and the positive prime factors of every entry of A_2 are prime numbers lower than or equal to $n + m + 1$. Looking at the formula for the determinant of Cauchy matrices, one can see that the determinant of every square submatrix of A_2 is an integer (different from 0) such that all its prime factors are lower than or equal to $n + m + 1$.

Finally, take A as the matrix obtained by replacing every entry of A_2 by its remainder in the division by p, which is never equal to 0. Then it is clear that A has the required properties. $\square$

Before proceeding, we will state two basic facts about the varieties previously defined. We postpone the proof of these results to Section 2.3.

Lemma 4 *Let $S \subset \{1, \dots, m\}$ and $\sigma \in \{+, -\}^S$. If $|S| > n$, the variety $W_{S,\sigma}$ is empty.*

Lemma 5 *For every $S \subset \{1, \dots, m\}$ and $\sigma \in \{+, -\}^S$, the variety $Z_{S,\sigma}$ is empty.*

2.2 Geometric properties

For every $t \geq 0$, let

$$T_t = \{x \in \mathbb{R}^n \mid F_1^+(t,x) \geq 0, \dots, F_l^+(t,x) \geq 0, \quad F_{l+1}^+(t,x) \geq 0, \dots, F_m^+(t,x) \geq 0,$$
$$F_1^-(t,x) \leq 0, \dots, F_l^-(t,x) \leq 0\}.$$

As $\tilde{f}_i(x) > 0$ for every $1 \leq i \leq m$ and $x \in \mathbb{R}^n$, it is clear that:

- If $0 \leq t_1 \leq t_2$, then $T_{t_1} \subset T_{t_2}$,
- $T_0 = T$.

Since T is a closed set, its connected components are closed. Then, since C is a compact connected component of T, there exists $\mu > 0$ such that $\mathrm{dist}(C, C') \geq 2\mu$ for every connected component C' of T, $C' \neq C$. Let us denote

$$C_\mu = \{x \in \mathbb{R}^n \mid \mathrm{dist}(x, C) < \mu\}.$$

Lemma 6 *There exists $\varepsilon > 0$ such that for every $0 \leq t \leq \varepsilon$, the connected component of T_t containing C is included in C_μ.*

Proof. Assume the statement does not hold. Let $(t_k)_{k\in\mathbb{N}}$ be a decreasing sequence of positive numbers converging to 0 such that, if C'_k is the connected component of T_{t_k} containing C, then $C'_k \not\subset C_\mu$.

Since C'_k is connected, contains C and intersects the set $\{x \in \mathbb{R}^n \mid \mathrm{dist}(x, C) \geq \mu\}$, there is a point $r_k \in C'_k$ with $\mathrm{dist}(r_k, C) = \mu$. Since $(r_k)_{k\in\mathbb{N}}$ is a sequence contained in the compact set $\{x \in \mathbb{R}^n \mid \mathrm{dist}(x, C) = \mu\}$, it has a subsequence which converges to a point r such that $\mathrm{dist}(r, C) = \mu$. Without loss of generality, we may assume this subsequence to be the original one.

On the other hand, since $r_k \in C'_k \subset T_{t_k}$, we have that, for every $1 \leq i \leq m$,

$$F_i^+(t_k, r_k) \geq 0, \quad \text{and so, } F_i^+(0, r) = \lim_{k\to\infty} F_i^+(t_k, r_k) \geq 0,$$

and, for every $1 \leq i \leq l$,

$$F_i^-(t_k, r_k) \leq 0, \quad \text{and so, } F_i^-(0, r) = \lim_{k\to\infty} F_i^-(t_k, r_k) \leq 0.$$

This implies that $r \in T$, leading to a contradiction, since there is no point in T whose distance to C equals μ. □

The following proposition shows that in order to obtain minimizers for the polynomial function g on the compact connected component C it is enough to consider polynomial systems with at most as many equations as variables.

Proposition 7 *There exist $z \in C$, $S \subset \{1, \dots, m\}$ with $0 \leq |S| \leq n$, and $\sigma \in \{+, -\}^S$ with $\sigma_i = +$ for $l + 1 \leq i \leq m$, such that $(0, z) \in \Pi_{(t,x)}(V_{S,\sigma})$ and $g(z) = \min\{g(x) \mid x \in C\}$.*

Proof. Let $\varepsilon > 0$ be such that:

- for every $0 \le t \le \varepsilon$, the connected component of T_t containing C is included in C_μ,
- for every $S \subset \{1, \dots, m\}, \sigma \in \{+,-\}^S$ and $t_0 \in \Pi_t(W^{(1)}_{S,\sigma}) \cup \Pi_t(Z^{(1)}_{S,\sigma}) \cup \Pi_t(V^{(1)}_{S,\sigma})$, $\varepsilon < |t_0|$.

Let $(t_k)_{k\in\mathbb{N}}$ be a decreasing sequence of positive numbers converging to 0 with $t_1 \le \varepsilon$. Consider the connected component C'_k of T_{t_k} which contains C (note that C'_k is a compact set) and let z_k be a point in C'_k at which the function $G(t_k, \cdot)$ attains its minimum value over C'_k. Since the sequence $(z_k)_{k\in\mathbb{N}}$ is bounded, it has a convergent subsequence; without loss of generality, we may assume this subsequence to be the original one. Let $z = \lim_{k\to\infty} z_k$. Proceeding as in the proof of Lemma 6, we have that $z \in C$.

In order to see that $g(z) = \min\{g(x) \mid x \in C\}$, note that for every $x \in C \subset C'_k$, we have that $G(t_k, z_k) \le G(t_k, x)$ for every k; therefore,

$$g(z) = G(0, z) = \lim_{k\to\infty} G(t_k, z_k) \le \lim_{k\to\infty} G(t_k, x) = G(0, x) = g(x).$$

Now, for every k and every $x \in \mathbb{R}^n$, at most one of the polynomials $F_i^+(t_k, x)$ and $F_i^-(t_k, x)$ may vanish, since $\tilde{f}_i(x) > 0$. For every $k \in \mathbb{N}$, let

$$S_k = \{i \in \{1, \dots, l\} \mid F_i^+(t_k, z_k) = 0 \text{ or } F_i^-(t_k, z_k) = 0\} \cup \{i \in \{l+1, \dots, m\} \mid F_i^+(t_k, z_k) = 0\}.$$

Without loss of generality, we may assume that S_k is the same set S for every $k \in \mathbb{N}$; moreover, we may assume that, for each $i \in S$, it is always the same polynomial $F_i^+(t_k, z_k)$ or $F_i^-(t_k, z_k)$ the one which vanishes, thus defining a function $\sigma \in \{+,-\}^S$.

Since $(t_k, z_k) \in \hat{W}_{S,\sigma}$, $t_k \notin \Pi_t(W^{(0)}_{S,\sigma} \cup W^{(1)}_{S,\sigma})$ and $W_{S,\sigma} = \emptyset$ if $|S| > n$ (Lemma 4), we have that $|S| \le n$. In addition, since $t_k \notin \Pi_t(Z^{(0)}_{S,\sigma} \cup Z^{(1)}_{S,\sigma})$ and $Z_{S,\sigma} = \emptyset$ (Lemma 5), it follows that $(t_k, z_k) \notin \hat{Z}_{S,\sigma}$; therefore, $\{\nabla_x F_i^{\sigma_i}(t_k, z_k), i \in S\}$ is a linearly independent set for every $k \in \mathbb{N}$. Finally, since the function $G(t_k, \cdot)$ attains a local minimum at the point z_k when restricted to the set $\{x \in \mathbb{R}^n \mid F_i^{\sigma_i}(t_k, x) = 0 \text{ for every } i \in S\}$, by the Lagrange Multiplier Theorem, there exists $(\lambda_{i,k})_{i\in S}$ such that

$$\nabla_x G(t_k, z_k) = \sum_{i\in S} \lambda_{i,k} \nabla_x F_i^{\sigma_i}(t_k, z_k).$$

Therefore, $(t_k, z_k, (1, (\lambda_{i,k})_{i\in S})) \in \hat{V}_{S,\sigma}$; but since $t_k \notin \Pi_t(V^{(0)}_{S,\sigma} \cup V^{(1)}_{S,\sigma})$, we conclude that $(t_k, z_k, (1, (\lambda_{i,k})_{i\in S})) \in V_{S,\sigma}$. Without loss of generality, we may assume that $(1, (\lambda_{i,k})_{i\in S})_{k\in\mathbb{N}}$ converges to a point $(\lambda_0, (\lambda_{i,0})_{i\in S})) \in \mathbb{P}^{|S|}$; then $(0, z, (\lambda_0, (\lambda_{i,0})_{i\in S})) \in V_{S,\sigma}$ and, therefore, $(0, z) \in \Pi_{(t,x)}(V_{S,\sigma})$ as we wanted to prove. □

2.3 Obtaining the bounds

In this section we prove Lemmas 4 and 5 and we do the estimates to obtain the bounds we are looking for.

Notation 8 *For $p \in \mathbb{Q}[x_1, \dots, x_n]$ and $e \in \mathbb{N}$, $e \ge \deg p$, $(p)^0_e$ will denote the polynomial $x_0^e p(x_1/x_0, \dots, x_n/x_0) \in \mathbb{Q}[x_0, \dots, x_n]$ which is obtained by homogenizing p up to degree e.*

- *For every $1 \le i \le m$,*

$$\overline{F_i^+}(t_0, t, x_0, x) = t_0\,(f_i)^0_d(x_0, x) + t\,(\tilde{f}_i)^0_d(x_0, x) = t_0\,(f_i)^0_d(x_0, x) + t\left(\sum_{j=0}^{n} a_{ij} x_j^d\right),$$

$$\overline{F_i^-}(t_0,t,x_0,x) = t_0\,(f_i)^0_d(x_0,x) - t\,(\tilde{f}_i)^0_d(x_0,x) = t_0\,(f_i)^0_d(x_0,x) - t\left(\sum_{j=0}^{n} a_{ij}x_j^d\right).$$

- *For $S \subset \{1,\dots,m\}$ and $\sigma \in \{+,-\}^S$, for every $1 \le j \le n$,*

$$\overline{G}_{S,\sigma,j}(t_0,t,x_0,x,\lambda_0,\lambda) = t_0\left(\lambda_0\frac{\partial g}{\partial x_j} - \sum_{i\in S}\lambda_i\frac{\partial f_i}{\partial x_j}\right)^0_{d-1} + t\left(\lambda_0\frac{\partial \tilde{g}}{\partial x_j} - \sum_{i\in S}\lambda_i\sigma_i\frac{\partial \tilde{f}_i}{\partial x_j}\right)^0_{d-1} =$$

$$= t_0\left(\lambda_0\Big(\frac{\partial g}{\partial x_j}\Big)^0_{d-1} - \sum_{i\in S}\lambda_i\Big(\frac{\partial f_i}{\partial x_j}\Big)^0_{d-1}\right) + t\;dx_j^{d-1}\left(\lambda_0 a_{0j} - \sum_{i\in S}\lambda_i\sigma_i a_{ij}\right).$$

Proof of Lemma 4. Consider the polynomials $\overline{F_i^{\sigma_i}}$ for every $i \in S$. These polynomials are bi-homogeneous in the sets of variables (t_0,t), (x_0,x); therefore, they define a variety $\overline{\hat{W}}_{S,\sigma}$ in $\mathbb{P}^1 \times \mathbb{P}^n$ (which contains $\hat{W}_{S,\sigma}$ when embedded in $\mathbb{P}^n$). Now, the fiber $\Pi^{-1}_{(t_0,t)}(0,1)$ with respect to the projection $\Pi_{(t_0,t)} : \overline{\hat{W}}_{S,\sigma} \to \mathbb{P}^1$ is given by the set of common zeroes of the polynomials $\sum_{j=0}^n a_{ij}x_j^d$ for $i \in S$. But this system has no solution in $\mathbb{P}^n$, since, by the assumption on A and the fact that $|S| > n$, the matrix $(a_{ij})_{i\in S, 0\le j\le n}$ has maximal rank $n+1$. We conclude that $\Pi_{(t_0,t)}(\overline{\hat{W}}_{S,\sigma})$ is not equal to $\mathbb{P}^1$. Since $\mathbb{P}^n$ is a complete variety, $\Pi_{(t_0,t)}(\overline{\hat{W}}_{S,\sigma})$ is closed and hence, it is a finite set. Therefore, $W_{S,\sigma} = \emptyset$. $\square$

Proof of Lemma 5. Consider the variety $\mathcal{Z}_{S,\sigma}$ defined in $\mathbb{P}^1 \times \mathbb{P}^n \times \mathbb{P}^{|S|-1}$ by the polynomials $\overline{F_i^{\sigma_i}}$, $i \in S$, and each of the n components of the vector $\sum_{i\in S}\lambda_i\nabla_x\overline{F_i^{\sigma_i}}$. Note that the projection to $\mathbb{P}^1 \times \mathbb{P}^n$ of $\mathcal{Z}_{S,\sigma}$ contains $\hat{Z}_{S,\sigma}$ (when embedded in $\mathbb{P}^1 \times \mathbb{P}^n$). Consider the projection $\Pi_{(t_0,t)} : \mathcal{Z}_{S,\sigma} \to \mathbb{P}^1$. We will show that the fiber $\Pi^{-1}_{(t_0,t)}(0,1)$ is empty or, equivalently, that the system

$$\begin{cases} \displaystyle\sum_{j=0}^{n} a_{ij}x_j^d &= 0 \qquad i \in S \\ \displaystyle dx_j^{d-1}\sum_{i\in S}\sigma_i a_{ij}\lambda_i &= 0 \qquad j = 1,\dots,n. \end{cases}$$

has no solution in $\mathbb{P}^n \times \mathbb{P}^{|S|-1}$. Assume, on the contrary, that (x_0,x,λ) is a solution and let $k = |\{j \in \{1,\dots,n\} \mid x_j = 0\}|$. When specializing x in the second set of equations, we get a linear equation system for λ consisting of $n-k$ linearly independent equations in $|S|$ unknowns which has a non-trivial solution; hence $|S| \ge n+1-k$. This implies that the first $|S|$ equations do not have a common solution in $\mathbb{P}^n$ with k vanishing coordinates.

We conclude that $\Pi_{(t_0,t)}(\mathcal{Z}_{S,\sigma})$ is not equal to $\mathbb{P}^1$. Since $\mathbb{P}^n \times \mathbb{P}^{|S|-1}$ is complete, as in the proof of the previous lemma, it follows that $Z_{S,\sigma} = \emptyset$. $\square$

Now we use the previous constructions to derive our bounds. We will define univariate polynomials $Q_{S,\sigma}(U)$ having the minimum that g takes over the compact connected components of T as roots and we will obtain our bounds by means of these polynomials. Let

$$P(U,x_0,x) = Ux_0^{d_0} - (g)^0_{d_0}(x_0,x).$$

For $S \subset \{1, \dots, m\}$ with $|S| \leq n$ and $\sigma \in \{+, -\}^S$, let

$$R_{S,\sigma}(t_0, t, U) = \mathrm{Res}_{(x_0,x),(\lambda_0,\lambda)}(P; \overline{F_i^{\sigma_i}}, i \in S; \overline{G}_{S,\sigma,j}, 1 \leq j \leq n) \in \mathbb{Z}[t_0, t, U],$$

where $\mathrm{Res}_{(x_0,x),(\lambda_0,\lambda)}$ denotes the bihomogeneous resultant associated to the bi-degrees of the polynomials involved: $(d_0, 0), (d, 0)$ repeated s times, and $(d-1, 1)$ repeated n times (see [6, Ch. 13, Sec. 2] for the definition and basic properties of multihomogeneous resultants).

Lemma 9 *The polynomial $R_{S,\sigma}(t_0, t, U)$ is not identically zero.*

Proof. Let $\mathcal{S}$ be the polynomial system

$$\begin{cases} \overline{F_i^{\sigma_i}}(t_0, t, x_0, x) = 0 & i \in S, \\ \overline{G}_{S,\sigma,j}(t_0, t, x_0, x, \lambda_0, \lambda) = 0 & 1 \leq j \leq n. \end{cases}$$

By specializing $(t_0, t) = (0, 1)$ in the polynomials of the system $\mathcal{S}$, we get the following polynomial system of equations:

$$\mathcal{S}_\infty = \begin{cases} \displaystyle\sum_{j=0}^{n} a_{ij} x_j^d = 0 & i \in S, \\ d x_j^{d-1} \left(a_{0j}\lambda_0 - \displaystyle\sum_{i \in S} \sigma_i a_{ij} \lambda_i \right) = 0 & 1 \leq j \leq n. \end{cases}$$

We will show that $\mathcal{S}_\infty$ has finitely many solutions in $\mathbb{P}^n \times \mathbb{P}^s$, none of them lying in the hyperplane $\{x_0 = 0\}$. As a consequence of this fact, it follows that the only roots of $R_{S,\sigma}(0, 1, U)$ are the finitely many values $g(x)$ where $(1, x, \lambda_0, \lambda)$ is a solution to $\mathcal{S}_\infty$; therefore $R_{S,\sigma}(t_0, t, U)$ is not identically zero.

First, note that if $(x_0, x, \lambda_0, \lambda)$ is a solution to $\mathcal{S}_\infty$, the last n equations of this system imply that, for every $1 \leq j \leq n$, either $x_j = 0$ or $a_{0j}\lambda_0 - \sum_{i \in S} \sigma_i a_{ij} \lambda_i = 0$. Let us show that, for every $J \subset \{1, \dots, n\}$, the system $\mathcal{S}_\infty$ has only finitely many solutions such that $x_j = 0$ if and only if $j \in J$. For a fixed J, these solutions are the solutions to

$$\mathcal{S}_\infty^{(1,J)} = \left\{ \sum_{j \notin J} a_{ij} x_j^d = 0 \quad i \in S \right. \quad \text{and} \quad \mathcal{S}_\infty^{(2,J)} = \left\{ a_{0j}\lambda_0 - \sum_{i \in S} \sigma_i a_{ij} \lambda_i = 0 \quad j \notin J. \right.$$

Taking into account that any submatrix of (a_{ij}) has maximal rank, we have that:

- If $|J| > n - s$, the system $\mathcal{S}_\infty^{(1,J)}$ implies that $x_j = 0$ for every $j \notin J$, contradicting the definition of J.

- If $|J| < n - s$, then $\mathcal{S}_\infty^{(2,J)}$ has a unique solutions $(\lambda_0, \lambda) = 0$, since it consists of at least as many equations as unknowns; then, $\mathcal{S}_\infty$ has no solutions in $\mathbb{P}^n \times \mathbb{P}^s$ corresponding to J.

- If $|J| = n - s$, $\mathcal{S}_\infty^{(2,J)}$ has a unique solution in $\mathbb{P}^s$. On the other hand, $\mathcal{S}_\infty^{(1,J)}$ has no solutions with $x_0 = 0$ and exactly d^s solutions with $x_0 = 1$.

□

Write $R_{S,\sigma}(t_0,t,U) = t^{e_{S,\sigma}}\tilde{R}_{S,\sigma}(t_0,t,U)$ with $e_{S,\sigma} \in \mathbb{N}_0$ and $\tilde{R}_{S,\sigma}(t_0,t,U)$ not a multiple of t. Note that $R_{S,\sigma}(1,t,g(x))$ vanishes on $\Pi_{(t,x)}(\hat{V}_{S,\sigma})$ and so, $\tilde{R}_{S,\sigma}(1,t,g(x))$ vanishes on $\Pi_{(t,x)}(V_{S,\sigma})$. Let

$$Q_{S,\sigma}(U) = \tilde{R}_{S,\sigma}(1,0,U).$$

Proposition 10 *The polynomial $Q_{S,\sigma}(U) \in \mathbb{Z}[U]$ is not identically zero. The degree of $Q_{S,\sigma}(U)$ is at most*

$$\binom{n}{s} d^s (d-1)^{n-s},$$

where $s = |S|$, and its coefficients have an absolute value lower than

$$M_{S,\sigma} = (2H_0)^{M_1}(2\widetilde{H})^{sM_2+nM_3} d^{nM_3} N_1^{M_1}\, N_2^{sM_2}\, N_3^{nM_3} \binom{M_1+N_1-1}{N_1-1}\binom{M_2+N_2-1}{N_2-1}^s \binom{M_3+N_3-1}{N_3-1}^n,$$

where

- $\widetilde{H} = \max\{H, 2n+2m\}$,
- $M_1 = \binom{n}{s} d^s(d-1)^{n-s}$, $M_2 = \binom{n}{s} d_0 d^{s-1}(d-1)^{n-s}$, $M_3 = \binom{n-1}{s} d_0 d^s (d-1)^{n-s-1}$,
- $N_1 = \binom{d_0+n}{n}$, $N_2 = \binom{d+n}{n}$, $N_3 = \binom{d-1+n}{n}(s+1)$.

Proof. Since $\tilde{R}_{S,\sigma}(t_0,t,U)$ is homogeneous in the variables (t_0,t) and it is not a multiple of t, it follows that $Q_{S,\sigma}(U)$ is not identically zero.

The degree of the polynomials f_i is bounded by d and their coefficients are of absolute value at most H. The corresponding quantities for g are $d_0 \le d$ and H_0. By abuse of notation, let A be an upper bound for the absolute values of the elements of the matrix A. From Lemma 3, we may assume $A \le 2(n+m)$.

We deduce that $P \in (\mathbb{Z}[U])[x_0,x]$ is a polynomial of degree d_0 and its coefficients are linear polynomials in U with coefficients of magnitude at most H_0. Also, $\overline{F_i^{\pm}}(t_0,t,x_0,x) \in (\mathbb{Z}[t_0,t])[x_0,x]$ are polynomials of degree d and their coefficients are linear forms in (t_0,t) with coefficients of magnitude at most $\widetilde{H}$. Finally, $\overline{G}_{S,\sigma,j}(t_0,t,x_0,x,\lambda_0,\lambda) \in (\mathbb{Z}[t_0,t])[x_0,x,\lambda_0,\lambda]$ are bihomogeneous polynomials in $((x_0,x),(\lambda_0,\lambda))$ of degree $d-1$ in the variables (x_0,x) and linear in the variables (λ_0,λ), and their coefficients are linear forms in (t_0,t) with coefficients of magnitude at most $d\widetilde{H}$.

We compute the resultant $R_{S,\sigma}$ that eliminates $((x_0,x),(\lambda_0,\lambda))$, which is a polynomial in $(\mathbb{Z}[U])[t_0,t]$. Recall that the bihomogeneous resultant $\mathrm{Res}_{(x_0,x),(\lambda,\lambda_0)}$ of a bihomogeneous system of $n+s+1$ polynomials consisting of a polynomial of bidegree $(d_0,0)$, s polynomials of bidegree $(d,0)$ and n polynomials of bidegree $(d-1,1)$ is a multihomogeneous polynomial of degree

$$M_1 = \mathrm{Bez}((d,0),s;(d-1,1),n) = \binom{n}{s} d^s(d-1)^{n-s}$$

in the coefficients of the polynomial of bidegree $(d_0,0)$, of degree

$$M_2 = \mathrm{Bez}((d_0,0),1;(d,0),s-1;(d-1,1),n) = \binom{n}{s} d_0 d^{s-1}(d-1)^{n-s}$$

in the coefficients of each of the s polynomials of bidegree $(d, 0)$, and of degree

$$M_3 = \mathrm{Bez}((d_0, 0), 1; (d, 0), s; (d-1, 1), n-1) = \binom{n-1}{s} d_0 d^s (d-1)^{n-s-1}$$

in the coefficients of each of the n polynomials of bidegree $(d-1, 1)$. Here $\mathrm{Bez}(\boldsymbol{d}_1, s_1; \ldots; \boldsymbol{d}_r, s_r)$ denotes the Bézout number of a bihomogeneous system formed by s_i polynomials of bi-degree $\boldsymbol{d}_i = (d_{i,1}, d_{i,2})$ for $1 \leq i \leq r$ (see [18, Chapter IV, Sec. 2]).

It follows that $R_{S,\sigma}$ is a sum of terms of the form

$$\rho\, \alpha \prod_{i \in S} \beta_i \prod_{1 \leq j \leq n} \gamma_j, \tag{1}$$

where $\rho \in \mathbb{Z}$ is a coefficient of the bihomogeneous resultant $\mathrm{Res}_{(x_0,x),(\lambda,\lambda_0)}$, α denotes a monomial in the coefficients of P of total degree M_1, β_i denotes a monomial in the coefficients of $\overline{F_i^{\pm}}$ of total degree M_2 for every $i \in S$, and γ_j denotes a monomial in the coefficients of $\overline{G}_{S,\sigma,j}$ of total degree M_3 for every $1 \leq j \leq n$. In particular, the degree of $R_{S,\sigma}$ in the variable U is at most M_1.

Note that the polynomial $Q_{S,\sigma}(U)$ is the coefficient of $R_{S,\sigma} \in (\mathbb{Z}[U])[t_0, t]$ corresponding to the smallest power of t. Therefore,

$$\deg Q_{S,\sigma}(U) \leq \deg_U R_{S,\sigma}(U, t_0, t) \leq M_1 = \binom{n}{s} d^s (d-1)^{n-s}.$$

In order to estimate the magnitude of its coefficients, we may set $t_0 = 1$ in $R_{S,\sigma}$ and, by abuse of notation, write $R_{S,\sigma}$ for the specialized polynomial. For every k, we will compute an upper bound for the magnitude of the coefficients of the polynomial in $\mathbb{Z}[U]$ that appears as coefficient of t^k in $R_{S,\sigma}$.

First, we apply [19, Theorem 1.1] to bound the coefficients $\rho \in \mathbb{Z}$ of the resultant $\mathrm{Res}_{(x_0,x),(\lambda_0,\lambda)}$. We obtain:

$$|\rho| \leq N_1^{M_1}\, (N_2^{M_2})^s\, (N_3^{M_3})^n, \tag{2}$$

where $N_1 = \binom{d_0+n}{n}$ and $N_2 = \binom{d+n}{n}$ are the cardinalities of the supports of generic homogeneous polynomials of degrees d_0 and d respectively, and $N_3 = \binom{d-1+n}{n}(s+1)$ is the cardinality of the support of a generic bihomogeneous polynomial of bidegree $(d-1, 1)$ in $(x_0, x), (\lambda_0, \lambda)$.

Note that $\alpha \in \mathbb{Z}[U]$ is a polynomial in U with integer coefficients and degree bounded by M_1, and the absolute value of the coefficient of the power U^j in α^{M_1} is at most

$$\binom{M_1}{j} H_0^{M_1-j} < (2H_0)^{M_1}. \tag{3}$$

On the other hand, the product $\prod_{i \in S} \beta_i \prod_{1 \leq j \leq n} \gamma_j \in \mathbb{Z}[t]$ is a polynomial in the variable t with integer coefficients, which is a product of sM_2 linear factors that are coefficients of the $\overline{F_i^{\pm}}$ and nM_3 linear factors that are coefficients of the $\overline{G}_{S,\sigma,j}$. Thus, for a fixed k, using the upper bounds on the coefficients of the polynomials $\overline{F_i^{\pm}}$ and $\overline{G}_{S,\sigma,j}$, it follows that the coefficient of t^k in this product is at most

$$\binom{sM_2 + nM_3}{k} \widetilde{H}^{sM_2} (d\widetilde{H})^{nM_3} < (2\widetilde{H})^{sM_2+nM_3} d^{nM_3}. \tag{4}$$

Finally, taking into account the multihomogeneous structure of the resultant, it follows that $R_{S,\sigma}$ is a sum of at most

$$\binom{M_1+N_1-1}{N_1-1}\binom{M_2+N_2-1}{N_2-1}^s\binom{M_3+N_3-1}{N_3-1}^n \tag{5}$$

terms of the form (1).

Combining the upper bounds (2), (3), (4) and (5) we obtain the stated upper bound for the absolute value of the coefficients of $Q_{S,\sigma}(U)$. $\square$

To prove Theorem 1, we will use the bounds for the degree and the absolute value of the coefficients of the polynomials $Q_{S,\sigma}(U)$ just obtained and the following bound on the roots of a univariate polynomial in terms of its coefficients:

Proposition 11 *([13, Proposition 2.5.9]) Let $P(x)=\sum_{j=0}^n c_jx^j\in\mathbb{C}[x]$, $c_n\neq 0$. If $z\in\mathbb{C}$ and $P(z)=0$, then $|z|\le 1+\max_{0\le j\le n-1}|\frac{c_j}{c_n}|$.*

The following corollary follows easily from the previous proposition:

Corollary 12 *Let $P(x)=\sum_{j=0}^n c_jx^j\in\mathbb{Z}[x]$ be a non-zero polynomial and $M\in\mathbb{Z}$ such that $|c_j|<M$ for every $0\le j\le n$. If $z\in\mathbb{C}-\{0\}$ and $P(z)=0$, then $|z|\ge M^{-1}$.*

We can prove now the main result of the paper:

Proof of Theorem 1. By Proposition 7, the polynomial g attains its minimum value over C at a point $z_0\in C$ such that $(0,z_0)\in\Pi_{(t,x)}(V_{S,\sigma})$ for certain $S\subset\{1,\dots,m\}$ with $0\le|S|\le n$, and $\sigma\in\{+,-\}^S$ with $\sigma_i=+$ for $l+1\le i\le m$. Now, for every $(0,z)\in\Pi_{(t,x)}(V_{S,\sigma})$, we have that $Q_{S,\sigma}(g(z))=0$.

Then, if $s=|S|$, we have that $0\le s\le\min\{m,n\}$ and Proposition 10 implies that $g(z_0)$ is an algebraic number of degree at most

$$\binom{n}{s}d^s(d-1)^{n-s}\le 2^{n-1}d^n.$$

Furthermore, if $g(z_0)\neq 0$, by Corollary 12, its absolute value is greater than or equal to $M_{S,\sigma}^{-1}$.

We keep the notation in Proposition 10. In order to get the stated bound for the minimum, we use the following facts:

- $N_1,N_2\le\frac{3}{2}d^n$,
- $N_3\le\frac{9}{4}d^n$: for $n=2$ and $n=3$ the bound holds easily, for $n\ge 4$,

$$N_3\le(n+1)\prod_{i=1}^n\frac{d-1+i}{i}\le(n+1)d\Big(\frac{\sum_{i=2}^n\frac{d-1+i}{i}}{n-1}\Big)^{n-1}\le(n+1)d\Big((d-1)\frac{\log(n)}{n-1}+1\Big)^{n-1}\le$$

$$\le(n+1)d\Big((d-1)0.47+1\Big)^{n-1}\le(n+1)0.74^{n-1}d^n\le\frac{9}{4}d^n.$$

- $\binom{M_i+N_i-1}{N_i-1}\le 2^{M_i+N_i}$ for $1\le i\le 3$.

Then we have

$$M_{S,\sigma} \leq 2^{2(M_1+sM_2+nM_3)+(\log_2(3)-1)(M_1+sM_2)+2(\log_2(3)-1)nM_3+N_1+sN_2+nN_3} \cdot$$

$$\cdot\ \tilde{H}^{M_1+sM_2+nM_3} d^{n(M_3+M_1+sM_2+nM_3)}.$$

Since $M_1 + sM_2 + nM_3 \leq (n+1)\binom{n}{s}d^n \leq (n+1)2^{n-1}d^n$ and $M_3 \leq 2^{n-2}d^n$, we have

$$M_{S,\sigma} \leq 2^{(((3\log_2(3)+1)n+2\log_2(3)+2)2^{n-2}+\frac{3}{2}(n+1)+\frac{9}{4}n)d^n}\, \tilde{H}^{(n+1)2^{n-1}d^n}\, d^{(2n^2+3n)2^{n-2}d^n}.$$

Taking into account that $\tilde{H} \geq 4$ and $d \geq 2$, we obtain

$$M_{S,\sigma} \leq 2^{((-2n^2+3\log_2(3)n+2\log_2(3)+6)2^{n-2}+\frac{15}{4}n+\frac{3}{2})d^n}\, \tilde{H}^{n2^n d^n}\, d^{n^2 2^n d^n},$$

and the bound follows since, for $n \geq 2$,

$$(-2n^2 + 3\log_2(3)n + 2\log_2(3) + 6)2^{n-2} + \frac{15}{4}n + \frac{3}{2} \leq \left(4 - \frac{n}{2}\right)n2^n.$$

□

Remark 13 *The algebraic degrees of the coordinates of a minimizer are also bounded by* $\max_{0\leq s\leq \min\{m,n\}} \binom{n}{s}d^s(d-1)^{n-s} \leq 2^{n-1}d^n$. *This can be seen simply by replacing the polynomial* g *by a coordinate* x_i *in the previous construction, namely, taking* $P(U, x_0, x) = Ux_0 - x_i$.

2.4 Non-compact situations

In applications (see Sections 3 and 4), sometimes the minimization of a polynomial needs to be done over a component which is not necessarily compact, but with a compact set of minimizers. The result in Theorem 1 can be extended to this situation:

Theorem 14 *Let* $T = \{x \in \mathbb{R}^n \mid f_1(x) = \cdots = f_l(x) = 0, f_{l+1}(x) \geq 0, \ldots, f_m(x) \geq 0\}$ *be defined by polynomials* $f_1, \ldots, f_m \in \mathbb{Z}[x_1, \ldots, x_n]$ *with* $n \geq 2$, *degrees bounded by an even integer* d *and coefficients of absolute value at most* H, *and let* C *be a connected component of* T. *Let* $g \in \mathbb{Z}[x_1, \ldots, x_n]$ *be a polynomial of degree at most* d *and coefficients of absolute value bounded by* H *that takes a minimum value* $g_{\min,C}$ *over* C. *Assume that the set*

$$C_{\min} = \{z \in C \mid g(z) = g_{\min,C}\}$$

is compact. Then, $g_{\min,C}$ *is an algebraic number of degree at most*

$$\max_{0\leq s\leq \min\{m,n\}} \binom{n}{s} d^s(d-1)^{n-s} \leq 2^{n-1}d^n$$

and, if it is not zero, its absolute value is greater than or equal to

$$(2^{4-\frac{n}{2}}\tilde{H}d^n)^{-n2^n d^n},$$

where $\tilde{H} = \max\{H, 2n + 2m\}$.

Proof. Take $M \in \mathbb{R}_{>0}$ so that $C_{\min}$ is contained in the open ball $B(0, M)$, and let C' be a connected component of the set

$$T' = \{x \in \mathbb{R}^n \mid f_1(x) = \cdots = f_l(x) = 0, f_{l+1}(x) \geq 0, \ldots, f_m(x) \geq 0, M^2 - \sum_{i=1}^{n} x_i^2 \geq 0\}$$

that contains a connected component of $C_{\min}$. Note that C' is compact, since T' is bounded.

By Proposition 7, there exist $z_0 \in C'$, $S \subset \{1, \ldots, m+1\}$ with $|S| \leq n$, and $\sigma \in \{+, -\}^S$ such that $(0, z_0) \in \Pi_{(t,x)}(V_{S,\sigma})$ (for the corresponding variety $V_{S,\sigma}$ associated to the equations of T') such that $g(z_0) = g_{\min,C}$.

Since $M^2 - \sum_{i=1}^{n}(z_{0,i})^2 \neq 0$, we have that $S \subset \{1, \ldots, m\}$ (see the proof of Proposition 7). Therefore, to define the deformation varieties we may consider an initial matrix $A \in \mathbb{Z}^{(m+2)\times(n+1)}$ obtained by taking an $(m+1) \times (n+1)$ matrix as in Lemma 3 and adding as a last row any $n+1$ vector with positive entries and such that every submatrix of A has maximal rank. In this way, all the entries of the first $m+1$ rows of A, which are those relevant to our estimations, are natural numbers less than or equal to $2(n+m)$. Now the result follows by Proposition 10, proceeding as in the proof of Theorem 1. □

Under the weaker assumption that the minimium is attained, we can deduce from Theorem 1 a bound for the algebraic degree of the minimum.

Theorem 15 *Let $T = \{x \in \mathbb{R}^n \mid f_1(x) = \cdots = f_l(x) = 0, f_{l+1}(x) \geq 0, \ldots, f_m(x) \geq 0\}$ be defined by polynomials $f_1, \ldots, f_m \in \mathbb{Z}[x_1, \ldots, x_n]$ with $n \geq 2$, degrees bounded by an even integer d and coefficients of absolute value at most H, and let C be a connected component of T. Let $g \in \mathbb{Z}[x_1, \ldots, x_n]$ be a polynomial of degree at most d and coefficients of absolute value bounded by H that takes a minimum value $g_{\min,C}$ over C. Then, $g_{\min,C}$ is an algebraic number of degree at most*

$$\max_{0 \leq s \leq \min\{m+1,n\}} \binom{n}{s} d^s (d-1)^{n-s} \leq 2^{n-1} d^n.$$

Proof. Take $M \in \mathbb{N}$ so that $g_{\min,C}$ is attained in the open ball $B(0, M)$, and let C' be a connected component of the set

$$T' = \{x \in \mathbb{R}^n \mid f_1(x) = \cdots = f_l(x) = 0, f_{l+1}(x) \geq 0, \ldots, f_m(x) \geq 0, M^2 - \sum_{i=1}^{n} x_i^2 \geq 0\}$$

that contains a point of C where $g_{\min,C}$ is attained. Since C' is compact, the result follows by applying the bound for the algebraic degree in Theorem 1. □

3 Bounds for the separation between disjoint connected components of basic closed semialgebraic sets

In this section we apply our previous results to the case when g is the square of the Euclidean distance in order to obtain bounds for the separation between two disjoint (and at least one compact) connected components of semialgebraic sets defined by non-strict inequalities. In particular, this gives a separation bound for two connected components of a closed semialgebraic set provided that one of them is compact.

Proof of Theorem 2. We have that $C_1 \times C_2$ is a connected component of the set $T_1 \times T_2 = \{(x,y) \in \mathbb{R}^{2n} \mid f_1(x) = \cdots = f_{l_1}(x) = 0, f_{l_1+1}(x) \geq 0, \dots, f_{m_1}(x) \geq 0, g_1(y) = \cdots = g_{l_2}(y) = 0, g_{l_2+1}(y) \geq 0, \dots, g_{m_2}(y) \geq 0\}$, and if $D(x,y) = \sum_{i=1}^{n}(x_i - y_i)^2$, then the minimum value that D takes over $C_1 \times C_2$ equals $\operatorname{dist}^2(C_1, C_2) > 0$. In addition, the set$\{(x,y) \in C_1 \times C_2 \mid \operatorname{dist}(x,y) = \operatorname{dist}(C_1, C_2)\}$ is bounded and, therefore, compact. Then, the result follows from Theorem 14. □

Example 16 *Consider $d, H, n \in \mathbb{N}$ with even d and $f_1, \dots, f_n \in \mathbb{Z}[x_1, \dots, x_n]$ defined by*

$$f_1(x) = Hx_1 - 1, \qquad f_i(x) = x_i - x_{i-1}^d \ \text{ for } 2 \leq i \leq n-1, \qquad f_n(x) = x_n^2 - x_{n-1}^d.$$

The set $\{x \in \mathbb{R}^n \mid f_1(x) = \cdots = f_n(x) = 0\}$ equals $\{p, q\}$ with

$$p = (H^{-1}, H^{-d}, \dots, H^{-d^{n-2}}, H^{-\frac{1}{2}d^{n-1}}), \qquad q = (H^{-1}, H^{-d}, \dots, H^{-d^{n-2}}, -H^{-\frac{1}{2}d^{n-1}})$$

and the distance between p and q is $2H^{-\frac{1}{2}d^{n-1}}$. This shows that the double exponential nature of our bound is unavoidable even in the case of different connected components of a single closed semialgebraic set.

4 Further applications

In this section, we apply Theorems 1, 14 and 15 to some standard optimization problems (c.f. [14, Section 3]).

Example 17 (Unconstrained optimization) *Let $n \geq 2$ and $g \in \mathbb{Z}[x_1, \dots, x_n]$ a polynomial of degree bounded by an even integer d and coefficients of absolute value at most H. Then:*

- *If the minimum of g is attained, by Theorem 15, it is an algebraic number of degree at most $nd(d-1)^{n-1}$.*
- *If the set where the minimum of g is attained is non-empty and compact, by Theorem 14, the minimum is an algebraic number of degree at most $(d-1)^n$ and, if it is not zero, its absolute value is greater than or equal to $(2^{4-\frac{n}{2}}\tilde{H}d^n)^{-n2^nd^n}$, where $\tilde{H} = \max\{H, 2n\}$.*

Example 18 (Quadratically constrained quadratic programming (QCQP)) *Let $n \geq 2$. The standard form of QCQP is the following:*

$$\begin{array}{ll} \min\limits_{x \in \mathbb{R}^n} & \frac{1}{2}\, x^T A_0\, x + b_0^T x + c_0 \\ \text{s.t.} & \frac{1}{2}\, x^T A_i\, x + b_i^T x + c_i \leq 0, \quad 1 \leq i \leq m, \end{array}$$

with symmetric $A_i \in \mathbb{R}^{n \times n}$, $b_i \in \mathbb{R}^n$ and $c_i \in \mathbb{R}$ for $0 \leq i \leq m$. Consider a feasible instance of QCQP defined by integer data and let H be an upper bound for the absolute value of the entries of all A_i, b_i and c_i. Our results enable us to obtain the following bounds:

- *If the minimum is attained, by Theorem 15, it is an algebraic number of degree at most $\max_{0 \leq s \leq \min\{m+1,n\}} \binom{n}{s} 2^s$.*
- *If the feasible set is compact, or, more generally, if the set where the mimimum is attained is non-empty and compact, by Theorem 1 or 14 respectively, the minimum is an algebraic number of degree at most $\max_{0 \leq s \leq \min\{m,n\}} \binom{n}{s} 2^s$ and, if it is not zero, its absolute value is greater than or equal to $(2^{4+\frac{n}{2}}\tilde{H})^{-n2^{2n}}$, where $\tilde{H} = \max\{H, 2n+2m\}$.*

Note that the feasible set is compact if, for instance, A_i is positive definite for at least one i, $1 \leq i \leq m$. On the other hand, the set where the minimum is attained is non-empty and compact if, for instance, A_0 is positive definite.

Example 19 (Semidefinite programming (SDP)) *Let $n \geq 2$. The standard form of SDP is the following:*

$$\begin{array}{ll} \min\limits_{X \in S_n} & A_0 \cdot X \\ \text{s.t.} & A_i \cdot X = b_i, \quad 1 \leq i \leq m, \\ & X \succeq 0, \end{array}$$

with symmetric $A_i \in \mathbb{R}^{n\times n}$ for $0 \leq i \leq m$, and $b_i \in \mathbb{R}$ for $1 \leq i \leq m$. Recall that S_n is the set of symmetric matrices of size n with real coefficients, $A \cdot X = \operatorname{tr}(AX)$ and the notation $X \succeq 0$ means that X is positive semidefinite.

This problem can be seen as a polynomial minimization problem in $\frac{n(n+1)}{2}$ real variables which represent the entries of the matrix X, since the condition $X \succeq 0$ can be described by the non-negativity of the $2^n - 1$ principal minors of X.

Consider feasible instances defined by integer data. The problem of computing the algebraic degree in SDP in the generic case has been studied in [15] and in [7], where precise formulas in terms of combinatorial numbers and Pfaffians are given. Our results enable us to obtain a simple upper bound for the algebraic degree and a lower bound for the absolute value of the minimum:

- *Since the number of polynomial constraints is always greater than or equal to the number of variables, the bounds in Theorems 1, 14 and 15 for the algebraic degree are the same. Therefore, if the minimum is attained, by Theorem 15, it is an algebraic number of degree at most*

$$\max_{0 \leq s \leq \frac{n(n+1)}{2}} \binom{\frac{n(n+1)}{2}}{s} \hat{n}^s (\hat{n}-1)^{n-s},$$

  where $\hat{n}$ is the smallest even integer greater than or equal to n.

- *If the feasible set is compact, or, more generally, if the set where the mimimum is attained is non-empty and compact, and the minimum is not zero, by Theorem 1 or 14 respectively, its absolute value is greater than or equal to*

$$(2^{4-\frac{n(n+1)}{4}} \tilde{H} \hat{n}^{\frac{n(n+1)}{2}})^{-\frac{n(n+1)}{2}(2\hat{n})^{\frac{n(n+1)}{2}}},$$

  where $\hat{n}$ is the smallest even integer greater than or equal to n and $\tilde{H} = \max\{2H, n(n+1) + 2m + 2^{n+1} - 2\}$ with H an upper bound for the absolute value of the entries of all A_i and b_i (the value of $\tilde{H}$ is obtained taking into account that 2^n is an upper bound for the absolute value of the coefficients of the principal minors of X).

Note that the feasible set is compact if, for instance, A_i is positive definite for at least one i, $1 \leq i \leq m$. On the other hand, the set where the minimum is attained is non-empty and compact if, for instance, A_0 is positive definite, or, more generally, if the dual problem is strictly feasible.

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