TOPOLOGY OF REAL MULTI-AFFINE HYPERSURFACES AND A HOMOLOGICAL STABILITY PROPERTY

SAUGATA BASU AND DANIEL PERRUCCI

Abstract. Let R be a real closed field. We prove that the number of semi-algebraically connected components of a real hypersurface in R^n defined by a multi-affine polynomial of degree d is bounded by 2^{d-1}. This bound is sharp and is independent of n (as opposed to the classical bound of $d(2d-1)^{n-1}$ on the Betti numbers of hypersurfaces defined by arbitrary polynomials of degree d in R^n due to Petrovskiĭ and Oleĭnik, Thom and Milnor). Moreover, we show there exists $c > 1$, such that given a sequence $(B_n)_{n>0}$ where B_n is a closed ball in R^n of positive radius, there exist hypersurfaces $(V_n \subset \mathrm{R}^n)_{n>0}$ defined by symmetric multi-affine polynomials of degree 4, such that $\sum_{i \leqslant 5} b_i(V_n \cap B_n) > c^n$, where $b_i(\cdot)$ denotes the i-th Betti number with rational coefficients. Finally, as an application of the main result of the paper we verify a representational stability conjecture due to Basu and Riener on the cohomology modules of symmetric real algebraic sets for a new and much larger class of symmetric real algebraic sets than known before.

1. Introduction

We fix a real closed field R. For any closed semi-algebraic set $S \subset \mathrm{R}^n$, we denote by $b_i(S)$ the dimension of the i-th homology group $\mathbf{H}_i(S)$ with rational coefficients (the i-th Betti number of S)[1]. We will denote

$$b(S) = \sum_{i \geqslant 0} b_i(S).$$

In particular, $b_0(S)$ equals the number of semi-algebraically connected components of S.

The problem of proving upper bounds on the Betti numbers of a real algebraic variety $V \subset \mathrm{R}^n$ in terms of the degrees of polynomials defining V is a very well-studied problem in real algebraic geometry. Before stating the classical result in this direction it is useful to first introduce some notation that we are also going to use later in the paper.

Notation 1.1. For $\mathcal{P}$ a finite subset of $\mathrm{R}[X_1, \ldots, X_n]$, B a semi-algebraic subset of R^n, we denote by $\mathrm{Z}(\mathcal{P}, B)$ the set of common zeros of $\mathcal{P}$ in B. If $\mathcal{P} = \{P\}$, will denote $\mathrm{Z}(\mathcal{P}, B)$ by $\mathrm{Z}(P, B)$. If ϕ is a quantifier-free formula in the first order theory of the reals (i.e. a Boolean combination of atoms of the form $P \geqslant 0, P \in$

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[1] Since we only consider homology and cohomology groups of semi-algebraic sets with rational coefficients we have $\mathbf{H}_i(S) \cong \mathbf{H}^i(S)$ for any closed semi-algebraic set S by the universal coefficients theorem [25, page 243].

$\mathrm{R}[X_1,\ldots,X_n]$), then we will denote by $\mathcal{R}(\phi,B)$ the semi-algebraic subset of B defined by ϕ.

An upper bound on the sum of the Betti numbers of a real algebraic set in R^n in terms of the degrees of its defining polynomials was proved by Petrovskiĭ and Oleĭnik [24], Thom [26], and Milnor [22]. However, the proof of this result actually proves an a priori stronger result, namely a bound on the Betti numbers of the intersection of the real algebraic set with any closed Euclidean ball in R^n (see for example the proof of Proposition 7.28 in [6]). In order to make explicit this distinction we introduce the following notation.

Definition 1.1. Let $\mathbf{B} = (B_n)_{n>0}$ be a sequence of closed semi-algebraic subsets of R^n, and $\mathbf{F} = (\mathcal{F}_n)_{n>0}$, a sequence where $\mathcal{F}_n \subset \mathrm{R}[X_1,\ldots,X_n]$ for each $n>0$. We define for each $p \geqslant 0$,

$$\begin{aligned}\beta_{\mathbf{F},\mathbf{B},p}(n) &= \max_{P\in\mathcal{F}_n}\left(\sum_{i\leqslant p} b_i(\mathrm{Z}(P,B_n))\right),\\ \beta_{\mathbf{F},p}(n) &= \beta_{\mathbf{F},(\mathrm{R}^n)_{n>0},p}(n),\end{aligned}$$

and also define,

$$\begin{aligned}\beta_{\mathbf{F},\mathbf{B}}(n) &= \beta_{\mathbf{F},\mathbf{B},n}(n),\\ \beta_{\mathbf{F}}(n) &= \beta_{\mathbf{F},n}(n).\end{aligned}$$

We next observe that under certain conditions on $\mathbf{B}$ and $\mathbf{F}$, $\beta_{\mathbf{F},\mathbf{B},p}(n)$ (respectively, $\beta_{\mathbf{F},\mathbf{B}}(n)$) is an upper bound on $\beta_{\mathbf{F},p}(n)$ (respectively, $\beta_{\mathbf{F}}(n)$).

Following the notation in Definition 1.1:

Proposition 1.1. *Suppose that for each $n>0$, B_n is a closed and bounded convex semi-algebraic set having dimension n, and $\mathcal{F}_n$ is closed under translations $X \mapsto X - x, x \in \mathrm{R}^n$, and scalings $X \mapsto \lambda\cdot X, \lambda\in\mathrm{R}$. Then,*

$$\begin{aligned}\beta_{\mathbf{F},p}(n) &\leqslant \beta_{\mathbf{F},\mathbf{B},p}(n), \text{ for } p\geqslant 0, \text{ and}\\ \beta_{\mathbf{F}}(n) &\leqslant \beta_{\mathbf{F},\mathbf{B}}(n).\end{aligned}$$

Proof. Since $\mathcal{F}_n$ is stable under translations and $\dim B_n = n$, one can assume that B_n contains the origin in its interior. As B_n is convex, this implies that for $\lambda>0$, $\lambda\cdot B_n$ is an increasing family of semi-algebraic sets (increasing with λ), $\mathrm{R}^n = \bigcup_{\lambda>0}\lambda\cdot B_n$, and each $\lambda\cdot B_n$ is a closed and bounded semi-algebraic set. It follows from the conic structure theorem at infinity of semi-algebraic sets (see for instance [6, Proposition 5.49]) that there exists $\lambda_0>0$, such that $\mathrm{Z}(P,\lambda_0\cdot B_n)$ is a semi-algebraic deformation retract of $\mathrm{Z}(P,\mathrm{R}^n)$.

Now let $P_0 = P(\lambda_0\cdot X_1,\ldots,\lambda_0\cdot X_n)\in\mathcal{F}_n$. Then, $\mathrm{Z}(P_0,B_n)$ is semi-algebraically homeomorphic to $\mathrm{Z}(P,\lambda_0\cdot B_n)$. Hence,

$$\begin{aligned}b_p(\mathrm{Z}(P_0,B_n)) &= b_p(\mathrm{Z}(P,\mathrm{R}^n)), \text{ for } p\geqslant 0,\\ b(\mathrm{Z}(P_0,B_n)) &= b(\mathrm{Z}(P,\mathrm{R}^n)).\end{aligned}$$

This proves both inequalities in the proposition. □

Remark 1.1. The inequalities in Proposition 1.1 can be strict. Take for example,

$$\mathbf{F} = (\mathrm{R}[X_1,\ldots,X_n]_{\leqslant 2})_{n>0}$$

(where $\mathrm{R}[X_1, \ldots, X_n]_{\leqslant d}$ denotes the subset polynomials of degree at most d), and

$$\mathbf{B} = ([-1,1]^n)_{n>0}.$$

Then for $n \geqslant 2$,

$$\begin{aligned}\beta_{\mathbf{F},0}(n) &= 2,\\ \beta_{\mathbf{F},\mathbf{B},0}(n) &\geqslant 2^n.\end{aligned}$$

The first equation is obvious. For the second inequality, consider

$$P_n = \sum_{i=1}^{n} X_i^2 - n.$$

Then, $\mathrm{Z}(P_n, B_n) = \{-1,1\}^n$, and thus $b_0(\mathrm{Z}(P_n, B_n)) = 2^n$.

The theorem of Petrovskiĭ and Oleĭnik [24], Thom [26], and Milnor [22] can now be restated as follows.

Theorem 1 (Petrovskiĭ and Oleĭnik [24], Thom [26], and Milnor [22])**.** *For each $n > 0$, let B_n be a closed Euclidean ball in R^n of positive radius, and*

$$\mathbf{F}_d = (\mathrm{R}[X_1, \ldots, X_n]_{\leqslant d})_{n>0}.$$

Then,

$$\beta_{\mathbf{F}_d,\mathbf{B}}(n) \leqslant d(2d-1)^{n-1}.$$

Using Proposition 1.1 one immediately obtains from Theorem 1 the following corollary.

Corollary 1.

$$\beta_{\mathbf{F}_d}(n) \leqslant d(2d-1)^{n-1}. \tag{1}$$

Note that the upper bound in (1) grows exponentially in n for d fixed. Another point to note is that the proofs of the upper bounds on the sum of the Betti numbers in (1) ultimately rely on bounding the number of critical points of certain Morse functions. As such it does not give any additional information on a specific Betti number (say the zero-th Betti number). In fact the problem of proving bounds on individual Betti numbers which are better than the bounds on the sum of all Betti numbers is of great interest in real algebraic geometry. One of the main results in this paper (Theorem 2 below) furnishes such a bound (on the zero-th Betti number) for a special class of real algebraic hypersurfaces in R^n that we define below.

1.1. **Multi-affine polynomials.** We consider real algebraic varieties in R^n defined by polynomials of a special shape.

Definition 1.2. We call $P \in \mathrm{R}[X_1, \ldots, X_n]$ a *multi-affine* polynomial if for every $i, 1 \leqslant i \leqslant n$, $\deg_{X_i} P \leqslant 1$. We denote the subset of multi-affine polnomials in $\mathrm{R}[X_1, \ldots, X_n]_{\leqslant d}$ by $\mathcal{A}_{d,n}$, and the sequence $(\mathcal{A}_{d,n})_{n>0}$ by $\mathbf{A}_d$.

Real multi-affine polynomials occur in several applications. For example, multi-affine polynomials appear in computational complexity theory, since every element of the coordinate ring,

$$\mathrm{R}[\mathcal{B}_n] = \mathrm{R}[X_1, \ldots, X_n]/(X_1(X_1-1), \ldots, X_n(X_n-1)),$$

of the Boolean hypercube, $\mathcal{B}_n = \{0,1\}^n$, can be represented by a multi-affine polynomial. The degree of the unique multi-affine polynomial representing a Boolean function $f : \mathcal{B}_n \to \{0, 1\}$ is called the degree of f, and is used as a measure of complexity of f [23].

The multi-affine polynomial

$$P_{\mathcal{M}} = \sum_I X^I \in \mathrm{R}[X_1, \ldots, X_n],$$

where for $I \subset [1, n]$, X^I denotes the monomial $\prod_{i \in I} X_i$, and where I varies over the bases of a matroid $\mathcal{M}$, is called the *basis generating polynomial of* $\mathcal{M}$. Its properties (such as real stability) play an important role in the study of matroids, for instance in the spectacular recent works by Anari et al. [4] and Brändén and Huh [13]. However, the topology of the real hypersurfaces they define has not been studied much to the best of our knowledge. In this paper we prove quantitative results on certain topological invariants (Betti numbers) of real hypersurfaces defined by multi-affine polynomials of any fixed degree.

Finally, elementary symmetric polynomials, as well as linear combinations of them, furnish examples of multi-affine polynomials. This last class of polynomials, which are multi-affine as well as symmetric, will appear again later in the paper.

2. Main Results

We now state the main results proved in this paper in the following three subsections.

2.1. **Bound on the zero-th Betti number.** Our first result is a bound on the number of semi-algebraically connected components of hypersurfaces in R^n defined by multi-affine polynomials which is independent of n. More precisely, we prove the following theorem.

Theorem 2.

$$\beta_{\mathbf{A}_d,0}(n) \leqslant 2^{d-1}.$$

Example 2.1 (Sharpness)**.** The bound in Theorem 2 is sharp: for $d, n \in \mathbb{N}$, let $P = X_1 \ldots X_d - 1 \in \mathcal{A}_{d,n}$. Then, $b_0(\mathrm{Z}(P, \mathrm{R}^n)) = 2^{d-1}$.

Remark 2.1. Unlike the proof of Corollary 1 above, we will prove Theorem 2 directly without first proving a bounded version.

2.2. **Varieties defined by more than one polynomials and higher Betti numbers.** A remarkable property of the bound in Theorem 2 is that it is independent of n (unlike the bound in (1)). However, there are two restrictive features of the bound in Theorem 2 that are worth pointing out.

(a) The bound applies only to varieties defined by a single multi-affine polynomial. Note that the usual trick in real algebraic geometry of reducing the number of polynomials defining a variety to one by taking a sum of squares does not work well with the class of multi-affine polynomials. The square of a multi-affine polynomial is no longer necessarily multi-affine.

(b) The bound in Theorem 2 applies only to the zero-th Betti number (as opposed to the sum of all the Betti numbers).

It is natural to ask whether one could improve Theorem 2 by removing the restrictions (a) and (b). We show that this is not possible if we want to have an upper bound that is independent of n (in the case of restriction (b) our result only applies to the bounded version – see Theorem 3).

We first address (a). We construct below a sequence of examples each involving three multi-affine polynomials in $\mathrm{R}[X_1,\ldots,X_n]$ of degree at most 4, such that the number of connected components of the real variety they define grows with n. In order to construct these polynomials we need to introduce some notation.

Notation 2.1. For $n \in \mathbb{N}$ and $\ell \in \mathbb{Z}$ with $\ell \geqslant -1$, we denote by $\sigma_{\ell,n} \in \mathrm{R}[X_1,\ldots,X_n]$ the ℓ-th elementary symmetric polynomial in $X_1,\ldots,X_n$ defined as follows:

- $\sigma_{-1,n} = 0$,
- $\sigma_{0,n} = 1$,
- $\sigma_{\ell,n} = \sum_{1 \leqslant i_1 < \cdots < i_\ell \leqslant n} X_{i_1} \ldots X_{i_\ell}$ for $1 \leqslant \ell \leqslant n$,
- $\sigma_{\ell,n} = 0$ for $\ell > n$.

It is clear that for $n \in \mathbb{N}, n \geqslant 1$ and $\ell \in \mathbb{Z}$ with $\ell \geqslant -1$, $\sigma_{\ell,n}$ is multi-affine. Also, for $0 \leqslant \ell \leqslant n$,

$$\sigma_{\ell,n} = X_n \sigma_{\ell-1,n-1} + \sigma_{\ell,n-1}.$$

Notation 2.2. For $\ell, n \in \mathbb{N}$, we denote by $N_{\ell,n}$ the ℓ-th power sum polynomial,

$$N_{\ell,n} = X_1^\ell + \cdots + X_n^\ell.$$

When the value of n is clear from the context, we will simply write σ_ℓ to denote $\sigma_{\ell,n}$ and N_ℓ to denote $N_{\ell,n}$.

Example 2.2. Consider any fixed value of $k \in \mathbb{N}$. For $n \geqslant k$, consider $P_1, P_2, P_3 \in \mathrm{R}[X_1,\ldots,X_n]$ of degree bounded by $d = 4$:

$$\begin{aligned} P_1(X) &= \sigma_1(X) - k, \\ P_2(X) &= \sigma_2(X) - \tfrac{1}{2}k(k-1), \\ P_3(X) &= (4k-6)\sigma_3(X) - 4\sigma_4(X) - \tfrac{1}{2}k(k-1)^2(k-2). \end{aligned}$$

Using the Newton identities

$$\begin{aligned} N_1 &= \sigma_1, \\ N_2 &= N_1\sigma_1 - 2\sigma_2, \\ N_3 &= N_2\sigma_1 - N_1\sigma_2 + 3\sigma_3, \\ N_4 &= N_3\sigma_1 - N_2\sigma_2 + N_1\sigma_3 - 4\sigma_4, \end{aligned}$$

for $x \in Z(\{P_1, P_2, P_3\}, \mathrm{R}^n)$ we have

$$N_1(x) = k,$$

$$N_2(x) = k,$$

and

$$\begin{aligned} \sum_{1 \leqslant i \leqslant n} x_i^2 (x_i - 1)^2 &= N_4(x) - 2N_3(x) + N_2(x) \\ &= (4k-6)\sigma_3(x) - 4\sigma_4(x) - \frac{1}{2}k(k-1)^2(k-2) \\ &= 0. \end{aligned}$$

This implies that $\mathrm{Z}(\{P_1, P_2, P_3\}, \mathrm{R}^n)$ is a finite set with $\binom{n}{k}$ points: each point is an element of $\{0,1\}^n$ with exactly k coordinates equal to 1 and $n-k$ coordinates equal to 0. Therefore

$$b_0(\mathrm{Z}(\{P_1, P_2, P_3\}, \mathrm{R}^n)) = \binom{n}{k}$$

cannot be bounded only in terms of $d = 4$ (independently from n).

Example 2.2 shows that it is impossible to obtain a bound on the number of semi-algebraically connected components of a real variety in R^n defined by three multi-affine polynomials which is independent of n. We do not know if such a bound exists for a real variety defined by two multi-affine polynomials of degree at most d.

We now address (b). We first introduce a notation.

Notation 2.3. For $0 \leqslant d \leqslant n$ we denote

$$\Sigma_{d,n} = \left\{ P \in \mathrm{R}[X_1, \ldots, X_n]_{\leqslant d} \ \middle|\ P = \sum_{0 \leqslant i \leqslant d} a_i \sigma_{i,n}, a_i \in \mathrm{R}, 0 \leqslant i \leqslant d \right\}.$$

Moreover, we denote $\mathbf{\Sigma}_d = (\Sigma_{d,n})_{n>0}$.

It is natural to wonder whether one can obtain a bound on $\beta_{\mathbf{A}_d}(n)$ that is independent of n. We prove the following theorem which rules out a bound independent of n for the intersection of these hypersurfaces with bounded closed balls.

Theorem 3. *There exists a constant $c > 1$ having the following property. Let $\mathbf{B} = (B_n)_{n>0}$, where each B_n is a symmetric (i.e. stable under the standard action of $\mathfrak{S}_n$ on R^n), convex, closed and bounded semi-algebraic subset of R^n containing the origin in its interior. Then for $n > 1$,*

$$\beta_{\mathbf{\Sigma}_4, \mathbf{B}, 5}(n) > c^n.$$

In particular, since for each $d, n > 0$, $\Sigma_{d,n} \subset \mathcal{A}_{d,n}$, we also have for $n > 1$,

$$\beta_{\mathbf{A}_4, \mathbf{B}, 5}(n) > c^n.$$

Remark 2.2. The proof of Theorem 3 uses two different ingredients and will be given in Section 3.2. First, it uses representation theory of the symmetric group. Second, it uses a certain spectral sequence argument originally used by Agrachev [3, 2], and later by other authors [10, 1, 20] for proving *upper bounds* on the Betti numbers of semi-algebraic sets defined by quadratic inequalities (in the non-symmetric situation). We use it in this paper for proving *lower bounds* on the maximum Betti numbers occurring in a family symmetric real varieties (i.e. for proving existence of symmetric real varieties with large Betti numbers). This technique of proof might be of independent interest for proving lower bounds on the maximum possible Betti number of real varieties defined by other families of (symmetric) polynomials than those we consider in this paper.

Remark 2.3. Also, note that proving existence of real varieties with maximum possible Betti numbers is a well studied problem in real algebraic geometry (see [17, 12]). Theorem 3 is distinguished from these results because of several reasons.

(a) The results in the papers cited above are about real projective or more generally toric varieties, while we study real affine varieties in this paper.

(b) The asymptotics in the above cited papers are for fixed n, with the degree of the polynomial tending to infinity. In this paper, we consider the degree to be fixed and let n be large.
(c) Finally, it is not clear if the method of "combinatorial patchworking" used in [17, 12] can be used to construct real *symmetric* varieties having large Betti numbers.

2.3. **Stability conjecture.** We now describe a connection between the results stated above with the study of the cohomology groups of symmetric semi-algebraic sets as modules over the symmetric group.

2.3.1. *Some background.* The symmetric group $\mathfrak{S}_n$ acts on R^n by permuting coordinates. We say that a semi-algebraic subset $S \subset \mathrm{R}^n$ is symmetric if it is stable under this action. The action of $\mathfrak{S}_n$ on a closed symmetric semi-algebraic set $S \subset \mathrm{R}^n$ induces an action on the cohomology $\mathbf{H}^*(S)$, giving $\mathbf{H}^*(S)$ the structure of *a finite dimensional* $\mathfrak{S}_n$*-module.*

Remark 2.4. Note that if $\mathbf{H}$ is a finite-dimensional $\mathfrak{S}_n$ module (over $\mathbb{Q}$), $\mathrm{Hom}(\mathbf{H}, \mathbb{Q})$ has a canonically defined induced $\mathfrak{S}_n$-module structure, and is isomorphic to $\mathbf{H}$ as an $\mathfrak{S}_n$-module. [2]

Also, using the universal coefficient theorem, we have that for any closed semi-algebraic set $S \subset \mathrm{R}^n$, $\mathbf{H}^i(S) \cong \mathrm{Hom}(\mathbf{H}_i(S), \mathbb{Q})$. If S is additionally symmetric, then we have that $\mathbf{H}^i(S) \cong_{\mathfrak{S}_n} \mathbf{H}_i(S)$.

General facts from group representation theory then tell us that the $\mathfrak{S}_n$-module $\mathbf{H}^*(S)$ admits a canonically defined *isotypic decomposition* as a direct sum of sub-$\mathfrak{S}_n$-submodules, each of which is a multiple of a certain irreducible $\mathfrak{S}_n$-module. The irreducible $\mathfrak{S}_n$-modules are well studied, and they are in bijection with the finite set of partitions of the number n – the module corresponding to the partition $\lambda \vdash n$ will be denoted by $\mathbb{S}^\lambda$ in what follows, and is called the *Specht*-module corresponding to λ (see the book [19] for the precise definitions of these objects). We will use the following notation.

Notation 2.4. For any finite dimensional $\mathfrak{S}_n$-module $\mathbf{H}$, [3] and $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$, we will denote by $\mathbf{H}_\lambda$ the isotypic component corresponding to the Specht module $\mathbb{S}^\lambda$ in $\mathbf{H}$. Thus, the isotypic decomposition of $\mathbf{H}$ is the direct sum decomposition

$$\mathbf{H} \cong_{\mathfrak{S}_n} \bigoplus_{\lambda \vdash n} \mathbf{H}_\lambda,$$

and each $\mathbf{H}_\lambda \cong_{\mathfrak{S}_n} m_\lambda \mathbb{S}^\lambda$, where $m_\lambda \geqslant 0$. We will denote $\mathrm{mult}_\lambda(\mathbf{H}) = m_\lambda$.

Thus the isotypic decomposition of $\mathbf{H}^*(S)$ gives a canonically defined direct sum decomposition (direct sum in the category of $\mathfrak{S}_n$-modules)

$$\mathbf{H}^i(S) \cong_{\mathfrak{S}_n} \bigoplus_{\lambda \vdash n} m_{i,\lambda}(S) \mathbb{S}^\lambda, \tag{2}$$

where

$$m_{i,\lambda}(S) = \mathrm{mult}_\lambda(\mathbf{H}^i(S)).$$

[2]This is a consequence of the fact that the group $\mathfrak{S}_n$ is ambivalent; every element is conjugate to its inverse.

[3]The choice of $\mathbf{H}$ to denote the representation is deliberate since all the $\mathfrak{S}_n$-modules considered in this paper will be of the form $\mathbf{H}^*(V)$ or $\mathbf{H}_*(V)$ for some symmetric real algebraic set V.

The dimension of the Specht module $\mathbb{S}^\lambda$, has a simple expression (see for example [19, Theorem 2.3.21]

$$\dim \mathbb{S}^\lambda = \frac{n!}{\prod_{i,j} h_{i,j}(\lambda)}, \tag{3}$$

where

$$h_{i,j}(\lambda) = \lambda_i + \lambda'_j - i - j + 1,$$

and λ' is the transpose of λ. [4] Note that these dimensions can be exponentially big even for relatively simple partitions (say the partition $(n/2, n/2)$ for even n). For a symmetric semi-algebraic set $S \subset \mathrm{R}^n$, knowing the multiplicities $m_{i,\lambda}(S), \lambda \vdash n$, allows one to compute the dimension of $\mathbf{H}^i(S)$, and thus the i-th Betti number of S (using Eqn. (3)).

The partition $(n) \vdash n$ having length one plays a special role. The corresponding Specht-module $\mathbb{S}^{(n)}$ is the one dimensional *trivial representation* of $\mathfrak{S}_n$, and the isotypic component of $\mathbf{H}^i(S)$ corresponding to the partition (n) is thus isomorphic to the fixed part $\mathbf{H}^i(S)^{\mathfrak{S}_n}$ of $\mathbf{H}^i(S)$, which in turn is isomorphic to $\mathbf{H}^i(S/\mathfrak{S}_n)$ (see [7] for details and subtleties regarding these isomorphisms). We will use this last fact later in the paper (in the proof of Proposition 3.3).

The decomposition of the cohomology modules of a closed semi-algebraic set $S \subset \mathrm{R}^n$ defined by symmetric polynomials having degrees at most d into isotypic components was studied in [7] and [11] where several results are proved. One important result is a severe restriction on the partitions that are allowed to appear in the isotypic decomposition of the cohomology – which cuts down the possibilities for the allowed partitions *from exponential to polynomial* (for fixed d). The following theorem is a slightly simplified version of Theorem 4 in [11] and will be used in the proof of our new stability result (Theorem 5 below).

Theorem 4. [11] *Let $d \geqslant 2$, and $V \subset \mathrm{R}^n$ be a real variety defined by symmetric polynomials of degree bounded by d. Then, for all $\lambda \vdash n$, if $m_{i,\lambda}(V) > 0$, then*

$$\mathrm{length}(\lambda) < i + 2d - 1.$$

Independent of the above results, the phenomenon of representational and homological stability (see for example [15]) is an active topic of research in algebraic topology. One basic phenomenon of (homological) stability that motivates this study is the fact that for any fixed p, and any compact and oriented manifold X, $b_p(C_n(X))$, where $C_n(X)$ is the ordered n-th configuration space of X, is eventually given by a polynomial in n. The space $C_n(X)$ admits an $\mathfrak{S}_n$ action which induces an $\mathfrak{S}_n$-module structure on $\mathbf{H}_p(C_n(X))$. The homological stability is then a consequence of the stability of the multiplicities of certain Specht modules in $\mathbf{H}_p(C_n(X))$ for large n. All the above can be put in a much broader context of the category of FI-modules. However, we do not need this generality for the application that we discuss below.

Inspired by the representational stability phenomenon, the following conjecture was made in [7] about the *growth rate* of the multiplicities of the Specht modules in the cohomology modules of certain natural sequences of symmetric semi-algebraic sets. We state this conjecture below. But in order to do so we first need to introduce some definitions.

[4] Since $h_{i,j}(\lambda)$ in the above formula is equal to the length of the hook with corner in the box (i,j) in the Young diagram of λ, the formula (3) is often called the *hook length formula*.

We let

$$\Lambda_n = \mathrm{R}[X_1, \ldots, X_n]^{\mathfrak{S}_n}$$

denote the graded ring of invariant polynomials, with natural graded homomorphisms $\Lambda_{n+m} \to \Lambda_n$ obtained by setting $X_{n+m}, \ldots, X_{n+1}$ to 0. We denote by

$$\Lambda = \operatorname{proj\,lim} \Lambda_n$$

(where the limit is taken in the category of graded rings), and denote by

$$\phi_n : \Lambda \to \Lambda_n$$

the graded homomorphisms induced by the limit (see [21, pages 18-19]).

By a standard abuse of notation, after dropping n from the subscript, we will consider the symmetric polynomials σ_ℓ, N_ℓ (see Notation 2.1 and Notation 2.2) as elements of the ring Λ.

More precisely, for every $\ell, n \geqslant 0$, we have

$$\begin{aligned} \phi_n(\sigma_\ell) &= \sigma_{\ell,n}, \\ \phi_n(N_\ell) &= N_{\ell,n}. \end{aligned}$$

Now, suppose $I = (f_1, \ldots, f_k)$ is a finitely generated ideal of Λ. Then, I defines in a natural way symmetric real algebraic sets

$$V_n(I) = \mathrm{Zer}(\phi_n(f_1), \ldots, \phi_n(f_k)) \subset \mathrm{R}^n, n > 0.$$

For any fixed partition $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash d$, we denote for $n \geqslant \lambda_1 + d$

$$\{\lambda\}_n = (n - d, \lambda_1, \ldots, \lambda_\ell).$$

(Note that the above definition of the sequence of partitions

$$(\{\lambda = (\lambda_1, \ldots) \vdash d\}_n)_{n \geqslant \lambda_1 + d}$$

is standard in the asymptotic study of representations of $\mathfrak{S}_n$ as $n \to \infty$ (see for example [16, Eqn. (6.3.1)]).)

We are now in a position to state the the conjecture made in [7].

Conjecture 1. [7] *For any fixed $p \geqslant 0$, $m_{p,\{\lambda\}_n}(V_n(I))$ (see (2) for definition) is eventually given by a polynomial in n.*

The evidence in favor of Conjecture 1 is a little sparse. It was verified in the following very special case in [7].

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash d$, and $I \subset \Lambda$ the ideal generated by the symmetric function $N_4 - 2N_3 + N_2 \in \Lambda$. In this case, for which for each $n > 0$, the corresponding real algebraic set $V_n(I)$ equals $\mathcal{B}_n = \{0,1\}^n$. We have for all large enough n (see [7, Remark 5.3]),

$$m_{i,\{\lambda\}_n}(V_n(I)) = \begin{cases} n - 2\lambda_1 + 1, & \text{if } i = 0 \text{ and length}(\lambda) \leqslant 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Notice that the right hand side of Eqn. (4) is a polynomial in n for any fixed λ.

In this paper we verify Conjecture 1 for an infinite class of ideals. Instead of considering just the ideal generated by a very particular linear combination of *Newton symmetric functions* as above, we are able to handle all principal ideals in Λ which are generated by *arbitrary* linear combinations of the *elementary symmetric functions*.

We prove the following theorem.

Theorem 5. *Let $f = \sum_{i=0}^{d} a_i\sigma_i \in \Lambda$ be a linear combination of the elementary symmetric functions $\sigma_i \in \Lambda, 0 \leqslant i \leqslant d$, and let $I = (f)$. Then, for any partition λ and n large enough, $m_{0,\{\lambda\}_n}(V_n(I))$ equals 0 if $\mathrm{length}(\lambda) > 0$, and stabilizes to a (possibly non-zero) constant if $\mathrm{length}(\lambda) = 0$ (i.e when λ is the empty partition).*

Remark 2.5. Theorem 5 verifies Conjecture 1 for ideals generated by one linear combination of elementary symmetric functions, with $p = 0$. Note that in comparison to the special case of Conjecture 1 proved in [7], the family of ideals that we are able to handle (while still being principal) is considerably larger. It should also be noted that Theorem 5 proves a strong form of Conjecture 1 for the principal ideals that we consider in this paper – in that the multiplicities of the Specht modules corresponding to $\{\lambda\}_n$ actually stabilize to a constant (not just a polynomial in n). In general such a strong version of Conjecture 1 cannot hold as exhibited in Eqn. (4).

Remark 2.6. Note that the limit

$$\lim_{n\to\infty} m_{0,\{\lambda\}_n}(V_n(I))$$

which exists by Theorem 5 can be strictly bigger than 1. For instance, we will show at the end of Section 3.3 that if $f = \sigma_2 - 1, g = \sigma_3 - \sigma_1$, $I = (f), J = (g), \lambda = ()$, then

$$\begin{aligned}\lim_{n\to\infty} m_{0,\{\lambda\}_n}(V_n(I)) &= 2,\\ \lim_{n\to\infty} m_{0,\{\lambda\}_n}(V_n(J)) &= 3.\end{aligned}$$

The rest of the paper is devoted to the proofs of the theorems stated above.

3. Proofs of the main results

Even though theorems that we have stated in the previous section were formulated over an arbitrary real closed field R, using a standard application of the Tarski-Seidenberg transfer principle (see for example [6, Theorem 2.80]) it suffices to prove them for $\mathrm{R} = \mathbb{R}$. In the rest of the paper we will assume $\mathrm{R} = \mathbb{R}$ so that we are free to use certain basic results (such as existence of Leray spectral sequence, proper base change theorem etc.) without having to formulate these over arbitrary real closed fields.

3.1. **Proof of Theorem 2.** The idea of the proof is as follows. Let

$$P \in \mathrm{R}[X_1, \dots, X_n]$$

be a multi-affine polynomial of degree $d \in \mathbb{N}$. Suppose

$$P(X_1, \dots, X_n) = X_n Q(X_1, \dots, X_{n-1}) + R(X_1, \dots, X_{n-1})$$

with Q, R multi-affine, $Q \neq 0$ and $\deg Q = d - 1$. The main point of the proof is to show that there is no semi-algebraic connected component of $Z(P, \mathrm{R}^n)$ included in $Z(Q, \mathrm{R}^n)$. Once this is done, $b_0(Z(P, \mathrm{R}^n))$ is bounded by the number of semi-algebraic connected components of the set $Z(P, \mathrm{R}^n) \cap (\mathrm{R}^n \backslash Z(Q, \mathrm{R}^n))$. Finally, we bound this last number using Theorem 6, which we prove first and might be of independent interest.

Theorem 6. *Let $P \in \mathrm{R}[X_1, \dots, X_n]$ be a multi-affine polynomial of degree $d \in \mathbb{N}$. The number of semi-algebraically connected components of $\mathrm{R}^n \backslash \mathrm{Z}(P, \mathrm{R}^n)$ is bounded by 2^d.*

Proof. The proof is by induction on d. The result is clear for $d = 0$ and $d = 1$. Suppose now $d \geqslant 2$ and

$$P(X_1, \ldots, X_n) = X_n Q(X_1, \ldots, X_{n-1}) + R(X_1, \ldots, X_{n-1})$$

with Q, R multi-affine. Without loss of generality we suppose $Q \neq 0$ and $\deg Q = d - 1$. Since every semi-algebraically connected component of $\mathrm{R}^n \backslash \mathrm{Z}(P, \mathrm{R}^n)$ intersects $\mathrm{R}^n \backslash \mathrm{Z}(Q, \mathrm{R}^n)$, the number of semi-algebraically connected components of $\mathrm{R}^n \backslash \mathrm{Z}(P, \mathrm{R}^n)$ is bounded by the number of semi-algebraically connected components of

$$(\mathrm{R}^n \backslash \mathrm{Z}(P, \mathrm{R}^n)) \cap (\mathrm{R}^n \backslash \mathrm{Z}(Q, \mathrm{R}^n))) =$$
$$\left\{ (x_1, \ldots x_n) \in \mathrm{R}^n \mid Q(x_1, \ldots, x_{n-1}) \neq 0,\ x_n \neq \frac{-R(x_1, \ldots, x_{n-1})}{Q(x_1, \ldots, x_{n-1})} \right\},$$

which is twice the number of semi-algebraically connected components of $\mathrm{R}^{n-1} \backslash Z(Q, \mathrm{R}^{n-1})$ or, equivalently, of $\mathrm{R}^n \backslash Z(Q, \mathrm{R}^n)$. We conclude using the inductive hypothesis. $\square$

Proof of Theorem 2. We denote by $e_1, \ldots, e_n$ the elements of the standard basis of R^n, and by $\langle e_i \rangle$ the span of e_i. For $x = (x_1, \ldots, x_n) \in \mathrm{R}^n$ we denote $\bar{x} = (x_1, \ldots, x_{n-1}) \in \mathrm{R}^{n-1}$ and $\tilde{x} = (x_1, \ldots, x_{n-2}) \in \mathrm{R}^{n-2}$.

We consider first the case of P reducible in $\mathrm{R}[X_1, \ldots, X_n]$. Suppose without loss of generality $P = P_1 P_2$ with P_1 and P_2 non-constant multi-affine polynomials, $P_1 \in \mathrm{R}[X_1, \ldots, X_m]$ and $P_2 \in \mathrm{R}[X_{m+1}, \ldots, X_n]$. Then

$$\mathrm{Z}(P, \mathrm{R}^n) = (\mathrm{Z}(P_1, \mathrm{R}^m) \times \mathrm{R}^{n-m}) \cup (\mathrm{R}^m \times \mathrm{Z}(P_1, \mathrm{R}^{n-m}))$$

which is semi-algebraically connected and therefore, $b_0(Z(P, \mathrm{R}^n)) = 1$.

Now we consider the case of P irreducible in $\mathrm{R}[X_1, \ldots, X_n]$. Suppose

$$P(X_1, \ldots, X_n) = X_n Q(X_1, \ldots, X_{n-1}) + R(X_1, \ldots, X_{n-1})$$

with Q, R multi-affine. Without loss of generality suppose $Q \neq 0$ and $\deg Q = d - 1$. We will prove that there is no connected component of $Z(P, \mathrm{R}^n)$ included in $Z(Q, \mathrm{R}^n)$. If $n = 1$ then $Z(Q, \mathrm{R}^n) = \varnothing$ and we are done. From now on we consider $n \geqslant 2$. Suppose that C is a connected component of $Z(P, \mathrm{R}^n)$ included in $Z(Q, \mathrm{R}^n)$.

For any $z \in C$, $P(z) = Q(\bar{z}) = 0$, then $R(\bar{z}) = 0$ and therefore $z + \langle e_n \rangle \subset C$.

Take now a fixed $z \in C$ and consider $\varnothing \neq I \subset \{1, \ldots n\}$ of maximum cardinality such that

$$z + \langle e_i \mid i \in I \rangle \subset C.$$

Reasoning as above we have $n \in I$, and since $P \not\equiv 0$ we have $\#I \leqslant n - 1$. Without loss of generality suppose $n - 1 \notin I$. Take

$$Q(X_1, \ldots, X_{n-1}) = X_{n-1} S(X_1, \ldots, X_{n-2}) + T(X_1, \ldots, X_{n-2}),$$
$$R(X_1, \ldots, X_{n-1}) = X_{n-1} U(X_1, \ldots, X_{n-2}) + V(X_1, \ldots, X_{n-2}),$$

with S, T, U, V multi-affine. We consider the following cases:

- For every $y \in z + \langle e_i \mid i \in I \rangle$, $S(\tilde{y}) = U(\tilde{y}) = 0$:

  Since $P(y) = Q(\bar{y}) = 0$, then $R(\bar{y}) = 0$. Since in addition $S(\tilde{y}) = U(\tilde{y}) = 0$, then $T(\tilde{y}) = V(\tilde{y}) = 0$. For $w \in y + \langle e_{n-1} \rangle$, $Q(\bar{w}) = R(\bar{w}) = 0$ and then $P(w) = 0$. This proves

  $$z + \langle e_i \mid i \in I \cup \{n-1\} \rangle \subset C,$$

  which contradicts the maximality of I.

- There exists $y \in z + \langle e_i \mid i \in I \rangle$ such that $S(\tilde{y}) \neq 0$ and $U(\tilde{y}) = 0$:
  Since $P(y) = Q(\bar{y}) = 0$, then $R(\bar{y}) = 0$. Since in addition $U(\tilde{y}) = 0$, then $V(\tilde{y}) = 0$. For $w \in (\bar{y}, 0) + \langle e_{n-1} \rangle$, $R(\bar{w}) = 0$ and then $P(w) = 0 \cdot Q(\bar{w}) + R(\bar{w}) = 0$. This proves $(\bar{y}, 0) + \langle e_{n-1} \rangle \subset C$. Nevertheless, if we take $w \in (\bar{y}, 0) + \langle e_{n-1} \rangle$ with $w_{n-1} \neq -T(\tilde{y})/S(\tilde{y})$ we have $Q(w) \neq 0$ which is impossible.
- There exists $y \in z + \langle e_i \mid i \in I \rangle$ such that $U(\tilde{y}) \neq 0$:
  Take $\varepsilon > 0$ such that U does not vanish on $B(\tilde{y}, \varepsilon)$. Since $P(y) = Q(\bar{y}) = 0$, then $R(\bar{y}) = 0$, and $y_{n-1} = -V(\tilde{y})/U(\tilde{y})$.
  For $w \in B(\tilde{y}, \varepsilon)$, $P(w, -V(w)/U(w), 0) = 0$ and therefore
  $$\{(w, -V(w)/U(w), 0) \mid w \in B(\tilde{y}, \varepsilon)\} \subset C.$$
  This implies that for $w \in B(\tilde{y}, \varepsilon)$, $Q(w, -V(w)/U(w)) = 0$ and then
  $$U(w)T(w) = V(w)S(w).$$
  Since this equality holds in the non-empty open set $B(\tilde{y}, \varepsilon)$, we have
  $$UT = VS \in R[X_1, \ldots, X_{n-2}].$$
  Take $U = U_1 E$ and $V = V_1 E$ with $E = \gcd(U, V) \in \mathrm{R}[X_1, \ldots, X_{n-2}]$, then $S = U_1 F$ and $T = V_1 F$ for some $F \in \mathrm{R}[X_1, \ldots, X_{n-2}]$, and
  $$P = X_n(X_{n-1}U_1F + V_1F) + X_{n-1}U_1E + V_1E = (X_nF + E)(X_{n-1}U_1 + V_1).$$
  This contradicts the assumption of P being irreducible.

After considering all possible cases, we conclude that there is no semi-algebraic connected component of $Z(P, \mathrm{R}^n)$ included in $Z(Q, \mathrm{R}^n)$. This implies that $b_0(Z(P, \mathrm{R}^n))$ is bounded by the number of semi-algebraic connected components of the set

$$Z(P, \mathrm{R}^n) \cap (\mathrm{R}^n \backslash Z(Q, \mathrm{R}^n)) =$$
$$\left\{ (x_1, \ldots x_n) \in \mathrm{R}^n \mid Q(x_1, \ldots, x_{n-1}) \neq 0, x_n = \frac{-R(x_1, \ldots, x_{n-1})}{Q(x_1, \ldots, x_{n-1})} \right\},$$

which equals the number of semi-algebraically connected components of the set $(\mathrm{R}^{n-1} \backslash Z(Q, \mathrm{R}^{n-1}))$. This number is bounded by 2^{d-1} by Theorem 6. □

3.2. **Proof of Theorem 3.** For every $n > 0$, want to produce a symmetric multi-affine polynomial $P \in \mathrm{R}[X_1, \ldots, X_n]$ of small degree (in fact we will take the degree to be 4) having large Betti number (growing super-polynomially with n). As mentioned earlier the usual trick of taking sum of squares does not work well with multi-affine polynomials. For example, the sequence of polynomials

$$P_n = \sum_{i=1}^{n} X_i^2 (X_i - 1)^2$$

has the property that each polynomial is symmetric, of degree 4, having sum of Betti numbers equal to 2^n (and so growing exponentially with n), but P_n is not multi-affine.

Therefore, we take an indirect approach. Suppose that B is symmetric, closed and bounded semi-algebraic set containing $\mathcal{B}_n$.

We leverage the fact that the polynomials P_1, P_2, P_3 in Example 2.2, being linear combinations of elementary symmetric polynomials, are each symmetric and multi-affine. Moreover,

$$\mathbf{H}^0(\mathrm{Z}(\{P_1, P_2, P_3\}, B))$$

as a $\mathfrak{S}_n$-module is easy to understand and has a Specht module occurring in it of large dimension.

We prove (Proposition 3.2 below) using a spectral sequence argument that each Specht module that appears in $\mathbf{H}^0(\mathrm{Z}(\{P_1, P_2, P_3\}, B))$ must appear in at least one of the cohomology modules $\mathbf{H}^0(\mathrm{Z}(P, B)), \ldots, \mathbf{H}^5(\mathrm{Z}(P, B))$ for some P in the linear span of P_1, P_2, P_3.

Proposition 3.2 follows from a more general result (Proposition 3.1 below). Proposition 3.1 relates the vanishing of the multiplicities of a Specht module in the low dimensional (up to dimension $2p-1$ for some $p > 0$) cohomology modules of the hypersurfaces defined by symmetric polynomials in any linear subspace of symmetric polynomials, to the vanishing of the same Specht module in the *zero-th* cohomology of the *intersections* of at most p of such hypersurfaces.

The key idea here is that if a finite group acts on the stalks of a constructible sheaf and the isotypic component corresponding to a certain irreducible representation is zero at all stalks, then the isotypic component of that irreducible occurs with zero multiplicity in the cohomology of that sheaf (see Claim 3.5 in the proof of Proposition 3.1 below).

Proposition 3.1. *Let $\lambda \vdash n, \lambda \neq (n)$, $p > 0$, $L \subset \mathrm{R}[X_1, \ldots, X_n]^{\mathfrak{S}_n}$ a linear subspace of the vector space of symmetric polynomials, and $B \subset \mathrm{R}^n$ a symmetric, convex, closed and bounded semi-algebraic set.*

Suppose that for all $P \in L$ and $0 \leqslant i \leqslant 2p-1$,

$$m_{i,\lambda}(\mathrm{Z}(P, B)) = 0 \tag{5}$$

(cf. Eqn. (2)).

Then, for all $q, 1 \leqslant q \leqslant p$, and $P_1, \ldots, P_q \in L$,

$$m_{0,\lambda}(\mathrm{Z}(\{P_1, \ldots, P_q\}, B)) = 0.$$

We will use the following lemma in the proof of Proposition 3.1.

Lemma 3.1. *Suppose that $V_1, \ldots, V_m$ be symmetric closed semi-algebraic subsets of R^n. For $J \subset [1, m]$ denote*

$$V^J = \bigcup_{j \in J} V_j, V_J = \bigcap_{j \in J} V_j.$$

Then for $i \geqslant 0$ and $\lambda \vdash n$,

$$m_{i,\lambda}(V_{[1,m]}) \leqslant \sum_{j=1}^{n-i} \sum_{J \subset [1,m], \mathrm{card}(J)=j} m_{i+j-1,\lambda}(V^J). \tag{6}$$

Proof. The proof uses Schur's lemma and an $\mathfrak{S}_n$-equivariant version of the proof of a similar inequality in the non-symmetric case in [6, Proposition 7.33 (b)]. We first observe that claim is obviously true when $m = 1$.

The claim is now proved by induction on m. Assume that the induction hypothesis holds for all $m-1$ closed, symmetric semi-algebraic subsets of R^n, and for all $i \geqslant 0$ and $\lambda \neq (n)$.

It follows from the standard Mayer-Vietoris sequence that there is an exact sequence where each map is $\mathfrak{S}_n$-equivariant.

$$\cdots \to \mathbf{H}^i(V_{[1,m-1]}) \oplus \mathbf{H}^i(V_m) \to \mathbf{H}^i(V_{[1,m]}) \to \mathbf{H}^{i+1}(V_{[1,m-1]} \cup V_m) \to \cdots$$

Using Schur's lemma and restricting to the isotypic component corresponding to $\mathbb{S}^\lambda$ we obtain an exact sequence

$$\cdots \to \mathbf{H}^i(V_{[1,m-1]})_\lambda \oplus \mathbf{H}^i(V_m)_\lambda \to \mathbf{H}^i(V_{[1,m]})_\lambda \to \mathbf{H}^{i+1}(V_{[1,m-1]} \cup V_m)_\lambda \to \cdots$$

from which it follows that

$$m_{i,\lambda}(V_{[1,m]}) \leqslant m_{i,\lambda}(V_{[1,m-1]}) + m_{i,\lambda}(V_m) + m_{i+1,\lambda}(V_{[1,m-1]} \cup V_m). \tag{7}$$

Applying the induction hypothesis to the closed symmetric semi-algebraic sets $V_1, \ldots, V_{m-1}$, we deduce that

$$m_{i,\lambda}(V_{[1,m-1]}) \leqslant \sum_{j=1}^{n-i} \sum_{J\subset[1,m-1],\mathrm{card}(J)=j} m_{i+j-1,\lambda}(V^J). \tag{8}$$

Next, applying the induction hypothesis to the closed symmetric semi-algebraic sets, $V_1 \cup V_m, \ldots, V_{m-1} \cup V_m$ we obtain

$$m_{i+1,\lambda}(V_{[1,m-1]} \cup V_m) \leqslant \sum_{j=1}^{n-i-1} \sum_{J\subset[1,m-1],\mathrm{card}(J)=j} m_{i+j,\lambda}(V^{J\cup\{m\}}). \tag{9}$$

We obtain from inequalities (7), (8), and (9) that

$$m_{i,\lambda}(V_{[1,m]}) \leqslant \sum_{j=1}^{n-i} \sum_{J\subset[1,m],\mathrm{card}(J)=j} m_{i+j-1,\lambda}(V^J),$$

which finishes the induction. □

Proof of Proposition 3.1. We first prove a series of claims (Claims 3.1-3.5 below). In these claims we will use the following notation. Let $\underline{P} = (P_1, \ldots, P_q) \in L^q$ for some $q \geqslant 1$, and we denote

$$\Omega = \{\omega = (\omega_1, \ldots, \omega_q) \in \mathbf{S}^{q-1} \mid \omega_1 \geqslant 0, \ldots, \omega_q \geqslant 0\},$$

where $\mathbf{S}^{q-1}$ denotes the unit sphere in R^q.

Following a technique introduced by Agrachev [3, 2], for $\omega \in \Omega$ we denote

$$\omega\underline{P} = \omega_1 P_1 + \cdots + \omega_q P_q,$$

and denote

$$S(\underline{P}, B) = \{(\omega, x) \in \Omega \times B \mid \omega\underline{P}(x) \leqslant 0\}.$$

We denote by $\pi_1 : S(\underline{P}, B) \to \Omega$ and $\pi_2 : S(\underline{P}, B) \to B$ the restrictions to $S(\underline{P}, B)$ of the projection maps $\Omega \times B \to \Omega$ and $\Omega \times B \to B$ respectively.

Claim 3.1.

$$\pi_2(S(\underline{P}, B)) = \{x \in B \mid \bigvee_{j=1}^{q} (P_j(x) \leqslant 0)\}.$$

Proof of Claim 3.1. Suppose that $P_j(x) \leqslant 0$, with $1 \leqslant j \leqslant q$. Let

$$\omega^{(j)} = (\delta_{1,j}, \ldots, \delta_{q,j}) \in \Omega.$$

Then clearly $\omega^{(j)}\underline{P}(x) \leqslant 0$ and hence $(\omega^{(j)}, x) \in S(\underline{P}, B)$, proving that $x \in \pi_2(S(\underline{P}, B))$.

Conversely, if $x \in \pi_2(S(\underline{P}, B))$, then there exists $\omega \in \Omega$, such that $\omega\underline{P}(x) \leqslant 0$. If $P_j(x) > 0$ for every $j, 1 \leqslant j \leqslant q$, then $\omega\underline{P}(x) > 0$, since ω has at least one coordinate not equal to 0 and hence strictly positive. This is a contradiction. So there exists $j, 1 \leqslant j \leqslant q$, such that $P_j(x) \leqslant 0$.

This completes the proof of the claim. $\square$

Claim 3.2. *The map π_2 induces an isomorphism of $\mathfrak{S}_n$-modules*

$$\mathbf{H}_*(S(\underline{P}, B)) \to \mathbf{H}_*(\pi_2(S(\underline{P}, B))).$$

Proof of Claim 3.2. The map π_2 is clearly $\mathfrak{S}_n$-equivariant. For $x \in \pi_2(S(\underline{P}, B))$, the fiber $\pi_2^{-1}(x)$ is a non-empty intersection of the sphere $\mathbf{S}^{q-1}$ with the polyhedral cone defined by the linear inequalities,

$$\omega_1 \geqslant 0, \ldots, \omega_q \geqslant 0, \omega_1 P_1(x) + \cdots \omega_q P_q(s) \leqslant 0,$$

and hence is contractible. This implies that the induced map $\pi_{2,*} : \mathbf{H}_*(S(\underline{P}, B)) \to \mathbf{H}_*(\pi_2(S(\underline{P}, B)))$ is an isomorphism by the Vietoris-Begle theorem [25, page 344]. $\square$

Claim 3.3. *Eqn. (5) implies that for all $P \in L$, $0 \leqslant i \leqslant 2p-1$,*

$$m_{i,\lambda}(\mathcal{R}(P \leqslant 0, B)) = 0.$$

Proof of Claim 3.3. The Mayer-Vietoris exact sequence in homology yields the following exact sequence relating the homology groups of $\mathcal{R}(P \leqslant 0, B)$, $\mathcal{R}(P \geqslant 0, B)$ and $\mathcal{R}(P = 0, B)$:

$$\cdots \to \mathbf{H}_i(\mathcal{R}(P = 0, B)) \to \mathbf{H}_i(\mathcal{R}(P \leqslant 0, B)) \oplus \mathbf{H}_i(\mathcal{R}(P \geqslant 0, B)) \to \mathbf{H}_i(B) \to \cdots$$

Note that each arrow in the above sequence represents an homomorphism of $\mathfrak{S}_n$-modules. Thus, by Schur's lemma they restrict to give an exact sequence between the $\mathbb{S}^\lambda$-isotypic components. Note that since B is assumed to be convex

$$\begin{aligned} \mathbf{H}_i(B) &\cong_{\mathfrak{S}_n} \mathbb{S}^{(n)}, \text{ for } i = 0, \\ &\cong_{\mathfrak{S}_n} 0 \text{ for } i > 0. \end{aligned}$$

Since $\lambda \neq (n)$, we obtain the following inequality for each $i \geqslant 0$:

$$m_{i,\lambda}(\mathcal{R}(P \geqslant 0, B)) + m_{i,\lambda}(\mathcal{R}(P \leqslant 0, B)) \leqslant m_{i,\lambda}(\mathcal{R}(P = 0, B)). \tag{10}$$

This together with (5) implies that for $0 \leqslant i \leqslant 2p-1$,

$$m_{i,\lambda}(\mathcal{R}(P \leqslant 0, B)) = 0. \tag{11}$$

The claim follows from (10) and (11). $\square$

Claim 3.4. *For each $\omega \in \Omega$, and $0 \leqslant i \leqslant 2p-1$,*

$$m_{i,\lambda}(\pi_1^{-1}(\omega)) = 0.$$

Proof of Claim 3.4. Follows immediately from Claim 3.3 noting that $\pi_1^{-1}(\omega)$ is equivariantly homeomorphic to $\mathcal{R}(\omega\underline{P} \leqslant 0, B)$, and

$$\omega\underline{P} \in \operatorname{span}(P_1, \ldots, P_q) \subset L.$$

$\square$

Claim 3.5. *For $0 \leqslant i \leqslant 2p-1$,*

$$m_{i,\lambda}(S(\underline{P}, B)) = 0.$$

Proof of Claim 3.5. Let $S = S(\underline{P}, B)$. There exists a first-quadrant spectral sequence, $E_r^{s,t}$ (the Leray spectral sequence of the map π_1), converging to $\mathbf{H}^{s+t}(S)$, whose E_2-term is given by

$$E_2^{s,t} = \mathbf{H}^s(\Omega, R^t\pi_{1*}(\mathbb{Q}_S)),$$

where $\mathbb{Q}_S$ denotes the constant $\mathbb{Q}$-sheaf on S. The sheaf $R^t\pi_{1*}(\mathbb{Q}_S)$ is the sheaf associated to the presheaf which associates to every open subset $U \subset \Omega$, the $\mathbb{Q}$-vector space,

$$\mathbf{H}^t(\pi_1^{-1}(U))$$

(see [18, Chapter II, Proposition 5.11]). The set $\pi_1^{-1}(U)$ is stable under the action of $\mathfrak{S}_n$, and so there exists an isotypic decomposition

$$\mathbf{H}^t(\pi_1^{-1}(U)) \cong_{\mathfrak{S}_n} \bigoplus_{\mu \vdash n} \left(\mathbf{H}^t(\pi_1^{-1}(U))\right)_\mu$$

(cf. Notation 2.4). Moreover, since the restriction homomorphisms of this presheaf are all $\mathfrak{S}_n$-equivariant, it follows from Schur's Lemma and the definition of the sheafification functor (see for instance [18, page 85]) that there is a direct sum decomposition of the sheaf $R^t\pi_{1*}(\mathbb{Q}_S))$ into its isotypic components $R^t\pi_{1*}(\mathbb{Q}_S))_\mu, \mu \vdash n$.

Thus, we have

$$R^t\pi_{1*}(\mathbb{Q}_S) \quad \cong \quad \bigoplus_{\mu \vdash n} (R^t\pi_{1*}(\mathbb{Q}_S))_\mu.$$

Since, $\pi_1 : S \to \Omega$ is a proper map, using the proper base change theorem (see for example [18, §3, Theorem 6.2]) we obtain that for $\omega \in \Omega$,

$$R^t\pi_{1*}(\mathbb{Q}_S)_\omega \cong \mathbf{H}^t(\pi_1^{-1}(\omega)),$$

and for $\mu \vdash n$,

$$(R^t\pi_{1*}(\mathbb{Q}_S))_\mu)_\omega \cong \mathbf{H}^t(\pi_1^{-1}(\omega))_\mu. \tag{12}$$

Using Claim 3.4 we have that for each $\omega \in \Omega$, and $0 \leqslant i \leqslant 2p-1$,

$$m_{i,\lambda}(\pi_1^{-1}(\omega)) = 0.$$

Taking $\mu = \lambda$ in Eqn. 12, we have $0 \leqslant t \leqslant 2p-1$,

$$(R^t\pi_{1*}(\mathbb{Q}_S))_\lambda)_\omega \cong \mathbf{H}^t(\pi_1^{-1}(\omega))_\lambda = 0,$$

which in turn implies that

$$R^t\pi_{1*}(\mathbb{Q}_S))_\lambda = 0. \tag{13}$$

Now,

$$\begin{aligned} E_2^{s,t} &\cong_{\mathfrak{S}_n} \mathbf{H}^s(\Omega, R^t\pi_{1*}(\mathbb{Q}_S)) \\ &\cong_{\mathfrak{S}_n} \mathbf{H}^s(\Omega, \bigoplus_{\mu \vdash n} (R^t\pi_{1*}(\mathbb{Q}_S))_\mu) \\ &\cong_{\mathfrak{S}_n} \bigoplus_{\mu \vdash n} \mathbf{H}^s(\Omega, (R^t\pi_{1*}(\mathbb{Q}_S))_\mu) \\ &= \bigoplus_{\mu \vdash n} (E_2^{s,t})_\mu, \end{aligned}$$

where

$$(E_2^{s,t})_\mu = \mathbf{H}^s(\Omega, (R^t\pi_{1*}(\mathbb{Q}_S))_\mu).$$

The differentials $d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$ in the spectral sequence $E_r^{s,t}$ are $\mathfrak{S}_n$-equivariant, and for each $\mu \vdash n$ using Schur's lemma yet again, we have for $r \geqslant 2$, $(E_r^{s,t})_\mu$ is a subquotient of $(E_2^{s,t})_\mu$.

It follows from the above and Eqn. (13) that for $0 \leqslant t \leqslant 2p-1$, and all $s \geqslant 0$ and $r \geqslant 2$,

$$(E_r^{s,t})_\lambda = 0.$$

This implies that for all $i, 0 \leqslant i \leqslant 2p-1$,

$$(\mathbf{H}^i(S))_\lambda = \bigoplus_{s+t=i} (E_\infty^{s,t})_\lambda = 0,$$

or equivalently,

$$m_{i,\lambda}(S) \quad = \quad 0 \text{ for } 0 \leqslant i \leqslant 2p-1.$$

□

Observe that Claims 3.2 and 3.5 together imply that for any $\underline{P} = (P_1, \ldots, P_q) \in L^q, q \geqslant 1$, and $0 \leqslant i \leqslant 2p-1$

$$m_{i,\lambda}(\pi_2(S(\underline{P}, R))) = 0.$$

Rewriting the above equation using Claim 3.1 we obtain that for $0 \leqslant i \leqslant 2p-1$

$$m_{i,\lambda}(\mathcal{R}(\bigvee_{j=1}^{q}(P_j \leqslant 0, B))) = 0. \tag{14}$$

We are now in a position to finish the proof of Proposition 3.1.

We now fix $\underline{P} = (P_1, \ldots, P_q) \in L^q$, and assume that $1 \leqslant q \leqslant p$. Observe that

$$\mathcal{R}(\bigwedge_{j=1}^{q} P_j = 0, B) = \mathcal{R}(\bigwedge_{j=1}^{q}((P_j \leqslant 0) \wedge (-P_j \leqslant 0)), B).$$

Let

$$\begin{aligned} V_j &= \mathcal{R}(P_j \leqslant 0, B), \text{ for } j = 1, \ldots, q, \\ V_j &= \mathcal{R}(-P_{j-q} \leqslant 0, B), \text{ for } j = q+1, \ldots, 2q. \end{aligned}$$

Now Eqn. (14) applied to the various sub-tuples of the tuple

$$(P_1, \ldots, P_q, -P_1, \ldots, -P_q) \in L^{2q},$$

implies taking $i = 0$ that for all $J \subset [1, 2q]$, $m_{j-1,\lambda}(V^J) = 0$, where $j = \operatorname{card}(J)$ (noticing that $j - 1 \leqslant 2p - 1$, since $j = \operatorname{card}(J) \leqslant 2q \leqslant 2p$). Inequality (6) in Lemma 3.1 now implies that

$$\begin{aligned} m_{0,\lambda}(V_{[1,2q]}) &= m_{0,\lambda}(\mathrm{Z}(\{P_1, \ldots, P_q\}, B)) \\ &= 0. \end{aligned}$$

This finishes the proof of Proposition 3.1. □

Proposition 3.2. *Let B be a symmetric, convex, closed and bounded semi-algebraic set containing $\mathcal{B}_n$. For $k > 0$, and $n \geqslant 2k$, and each $\lambda = (n-j, j), 0 \leqslant j \leqslant k$, there exists $P \in \Sigma_{4,n}$, such that there exists $i, 0 \leqslant i \leqslant 5$,*

$$m_{i,\lambda}(\mathrm{Z}(P, B)) > 0.$$

Proof. Following Example 2.2 we let

$$\begin{aligned} P_1(X) &= \sigma_{1,n}(X) - k, \\ P_2(X) &= \sigma_{2,n}(X) - \tfrac{1}{2}k(k-1), \\ P_3(X) &= (4k-6)\sigma_{3,n}(X) - 4\sigma_4(X) - \tfrac{1}{2}k(k-1)^2(k-2). \end{aligned}$$

Then, $\mathrm{Z}(\{P_1, P_2, P_3\}, \mathrm{R}^n)$ is equal to the subset of $\mathcal{B}_n = \{0,1\}^n \subset B$ of cardinality $\binom{n}{k}$ consisting of points with exactly k 1's and $n-k$ 0's amongst its coordinates.

The $\mathfrak{S}_n$-module structure of $\mathbf{H}^0(\mathrm{Z}(\{P_1, P_2, P_3\}, B))$ is well-studied. It is isomorphic to the *Young module* $M^{(n-k,k)}$ [14, page 139] (see also [7, Example 1.19]). [5] The isotypic decomposition of the Young module $M^{(n-k,k)}$ is given by

$$M^{(n-k,k)} \cong_{\mathfrak{S}_n} \bigoplus_{j=0}^{k} \mathbb{S}^{n-j,j}$$

(see [14, page 141, Eqn. (3.72)]). Thus,

$$m_{0,\lambda}(\mathrm{Z}(\{P_1, P_2, P_3\}, B)) = 1 > 0, \tag{15}$$

for $\lambda = (n-j, j), 0 \leqslant j \leqslant k$.

Now suppose for the sake of contradiction that for all $P \in \Sigma_{4,n}$, and $\lambda = (n-j, j)$

$$m_{i,\lambda}(\mathrm{Z}(P, B)) = 0, \tag{16}$$

for $0 \leqslant i \leqslant 5$.

But Eqns. (16) and (15) together contradict Proposition 3.1 with $L = \Sigma_{4,n}$, and $p = q = 3$. ∎

In the proof of Theorem 3 we will also need the following lemma which is a straight-forward consequence of the hook formula.

Lemma 3.2. *For all $n \in \mathbb{N}$, and $\lambda = (n - \lfloor n/2 \rfloor, \lfloor n/2 \rfloor) \vdash n$*

$$\begin{aligned} \dim \mathbb{S}^\lambda &= \frac{1}{\lfloor n/2 \rfloor + 1}\binom{n}{\lfloor n/2 \rfloor} \text{ if } n \text{ is even}, \\ &= \frac{1}{2(\lfloor n/2 \rfloor + 2)}\binom{n}{\lfloor n/2 \rfloor} \text{ if } n \text{ is odd.} \end{aligned} \tag{17}$$

In particular, there exists $c > 1$ such that for all $n > 1$,

$$\dim \mathbb{S}^\lambda > c^n.$$

Proof. Eqn. (17) follows immediately from Eqn. 3 (hook length formula). The last statement is a consequence of the inequality

$$\frac{4^m}{2m+1} \leqslant \binom{2m}{m},$$

which is valid for all $m > 0$. ∎

We are finally in a position to prove Theorem 3.

[5] The Young module $M^{n-k,k}$ is isomorphic to the induced module $\mathrm{Ind}_{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}^{\mathfrak{S}_n} \mathbf{1}_{\mathfrak{S}_k} \boxtimes \mathbf{1}_{\mathfrak{S}_{n-k}}$.

Proof of Theorem 3. Since the set $\Sigma_{4,n}$ is invariant under simultaneous scaling of variables, and using the fact that B_n is assumed to be convex, symmetric, and contains the origin in its interior, we can assume without loss of generality that $B_n \supset \mathcal{B}_n$. It follows from Proposition 3.2 that for $k > 0$, $n \geqslant 2k$, and $\lambda = (n-k, k)$, there exists $i, 0 \leqslant i \leqslant 5$ and $P \in \Sigma_{4,n}$ such that

$$m_{i,\lambda}(\mathrm{Z}(P, B_n)) > 0.$$

Now choose $k = \lfloor n/2 \rfloor$ and use Lemma 3.2. □

3.3. **Proof of Theorem 5.** The proof is in two steps.

We first prove (Proposition 3.3) that since the dimensions of the cohomology modules $\mathbf{H}^0((\mathrm{Z}(\phi_n(f), \mathrm{R}^n))$ do not increase with n (using Theorem 2), for n large enough they cannot have Specht modules in their isotypic decomposition which correspond to partitions that are not equal to the trivial partition (n) or its transpose 1^n. We then use Theorem 4 to rule out the partition 1^n. This enables us to deduce that the $\mathbf{H}^0((\mathrm{Z}(\phi_n(f), \mathrm{R}^n))$ is a multiple of the trivial representation (i.e. $\mathbf{H}^0((\mathrm{Z}(\phi_n(f), \mathrm{R}^n)) = \mathbf{H}^0((\mathrm{Z}(\phi_n(f), \mathrm{R}^n))^{\mathfrak{S}_n})$ or equivalently that each semi-algebraically connected component of $\mathrm{Z}(\phi_n(f), \mathrm{R}^n)$ is stable under the action of $\mathfrak{S}_n$.

We next prove (Proposition 3.4 below) using Proposition 3.3 that the sequence of numbers $(b_0(\mathrm{Z}(\phi_n(f), \mathrm{R}^n))_{n>0}$ is non-increasing and so ultimately constant. Propositions 3.3 and Proposition 3.4 together suffices to prove Theorem 5.

Proposition 3.3. *Let $d, n \in \mathbb{N}$ with $d \geqslant 2, n > 2^{d-1}+1$ and let $P \in \mathrm{R}[X_1, \ldots, X_n]$ be a multi-affine symmetric polynomial with* $\deg P = d$. *Every semi-algebraic connected component of $\mathrm{Z}(P, \mathrm{R}^n)$ is stable under the action of $\mathfrak{S}_n$. This is to say, for every semi-algebraic connected component C of $Z(P, \mathrm{R}^n)$ and every $\alpha \in \mathfrak{S}_n$,*

$$C = \{(z_{\alpha(1)}, \ldots, z_{\alpha(n)}) \mid (z_1, \ldots, z_n) \in C\}.$$

Proof. Let $V = \mathrm{Z}(P, \mathrm{R}^n)$. First observe that $\mathbf{H}^0(V)^{\mathfrak{S}_n}$ is isomorphic (as a vector space) to the isotypic component of the trivial representation $\mathbb{S}^{(n)}$ in $\mathbf{H}^0(V)$. Second, each semi-algebraically connected component of V is stable under the action of $\mathfrak{S}_n$ if and only if

$$\mathbf{H}^0(V)^{\mathfrak{S}_n} = \mathbf{H}^0(V).$$

Thus, it suffices to prove that $\mathbf{H}^0(V)$ is isomorphic as an $\mathfrak{S}_n$-module to a multiple of trivial representation which is the same as proving that

$$m_{0,\lambda}(V) = 0,$$

for $\lambda \neq (n)$. Now it follows from Theorem 2 that

$$\begin{aligned} b_0(V) &= \dim \mathbf{H}^0(V) \\ &= \sum_{\lambda \vdash n} m_{0,\lambda}(V) \dim \mathbb{S}^\lambda \\ &\leqslant 2^{d-1}. \end{aligned} \tag{18}$$

It is an easy consequence of hook formula that

$$\dim S^\lambda = \begin{cases} 1 & \text{if } \lambda = (n),\ 1^n \\ \geqslant n-1 & \text{otherwise.} \end{cases} \tag{19}$$

Since,

$$n > 2^{d-1} + 1,$$

we have that

$$n - 1 \geqslant 2^{d-1} + 1 > b_0(V).$$

It now follows from (18) and (19) that

$$m_{0,\lambda}(V) = 0, \text{if } \lambda \neq (n), 1^n. \tag{20}$$

However, since $d \geqslant 2$, and hence

$$\text{length}(1^n) = n > 2^{d-1} + 1 \geqslant 0 + 2d - 1,$$

it follows from Theorem 4 that

$$m_{0,\lambda}(V) = 0 \text{ if } \lambda = 1^n. \tag{21}$$

The proposition now follows from (20) and (21). □

Lemma 3.3. *Let $d, n \in \mathbb{N}$ with $n \geqslant 2^{d-1} + 1$ and let $P \in \mathrm{R}[X_1, \ldots, X_n]$ be a multi-affine symmetric polynomial with $\deg P = d$. Every semi-algebraic connected component of $\mathrm{Z}(P, \mathrm{R}^n)$ intersects the hyperplane $\mathrm{Z}(X_n, \mathrm{R}^n)$.*

Proof. Suppose

$$P(X_1, \ldots, X_n) = X_n Q(X_1, \ldots, X_{n-1}) + R(X_1, \ldots, X_{n-1}),$$

$$Q(X_1, \ldots, X_{n-1}) = X_{n-1} S(X_1, \ldots, X_{n-2}) + T(X_1, \ldots, X_{n-2}),$$

with Q, R, S, T multi-affine. Notice that Q and R are symmetric as elements in $\mathrm{R}[X_1, \ldots, X_{n-1}]$ and S and T are symmetric as elements in $\mathrm{R}[X_1, \ldots, X_{n-2}]$. Let C be a semi-algebraic connected component of $\mathrm{Z}(P, \mathrm{R}^n)$. We consider the following cases:

- There exists $z = (z_1, \ldots, z_n) \in C$ and $1 \leqslant i \leqslant n$ with $z_i = 0$:
  In this case, $(z_1, \ldots, z_{i-1}, z_n, z_{i+1}, \ldots, z_{n-1}, 0) \in C \cap \mathrm{Z}(X_n, \mathrm{R}^n)$ by Proposition 3.3.
- There exists $1 \leqslant i < j \leqslant n$ and $z = (z_1, \ldots, z_n) \in C$ with z_i and z_j of opposite non-zero sign:
  Without loss of generality suppose $z_i > 0$ and $z_j < 0$. By Proposition 3.3, if we consider z' which is obtained from z by swapping coordinates z_i and z_j, then z' also lies in C. Since C is semi-algebraically arc-connected, there exists $z'' = (z''_1, \ldots, z''_n)$ in C with $z''_i = 0$, and then we proceed as in the first case.
- There exists $z = (z_1, \ldots, z_n) \in C$ and $1 \leqslant i \leqslant n$ with

  $$Q(z_1, \ldots, \widehat{z_i}, \ldots, z_n) = 0 :$$

  [6] Since

  $$\begin{aligned} 0 &= P(z) \\ &= z_i Q(z_1, \ldots, \widehat{z_i}, \ldots, z_n) + R(z_1, \ldots, \widehat{z_i}, \ldots, z_n), \end{aligned}$$

  we have that $R(z_1, \ldots, \widehat{z_i}, \ldots, z_n) = 0$ and therefore the line $z + \langle e_i \rangle$ is included in C. In particular $(z_1, \ldots, z_{i-1}, 0, z_{i+1}, \ldots, z_n) \in C$ and we proceed as in the first case.

[6] Here and elsewhere $\widehat{\ }$ denotes omission.

- $C \subset (0,+\infty)^n$ and for every $z = (z_1,\ldots,z_n) \in C$ and $1 \leqslant i \leqslant n$, $Q(z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_n) \neq 0$:

  This assumption implies that for $1 \leqslant i \leqslant n$, the polynomial

  $$Q(X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_n)$$

  has a constant sign on C. Let us consider a fixed value of $1 \leqslant i \leqslant n$ and see that for every $1 \leqslant j \leqslant n$ with $j \neq i$, the polynomial

  $$S(X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_{j-1},X_{j+1},\ldots,X_n)$$

  never vanishes on C, and it has the same sign as $Q(X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_n)$. Notice that this implies that the sign of $Q(X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_n)$ on C is independent of i.

  - If there exists $z = (z_1,\ldots,z_n) \in C$ such that

    $$S(z_1,\ldots,\widehat{z_i},\ldots,\widehat{z_j},\ldots,z_n) = 0,$$

    since

    $$0 \neq Q(z_1,\ldots,\widehat{z_i},\ldots,z_n) =$$
    $$z_j S(z_1,\ldots,\widehat{z_i},\ldots,\widehat{z_j},\ldots,z_n) + T(z_1,\ldots,\widehat{z_i},\ldots\widehat{z_j},\ldots,z_n),$$

    we have that $T(z_1,\ldots,\widehat{z_i},\ldots,\widehat{z_j},\ldots,z_n) \neq 0$ and therefore for every $t \in \mathrm{R}$,

    $$Q(z_1,\ldots,\widehat{z_i},\ldots,z_{j-1},t,z_{j+1},\ldots z_n) \neq 0.$$

    This implies that for each $t \in \mathrm{R}$ the point

    $$(z_1,\ldots,z_{i-1},a_t,z_{i+1},\ldots,z_{j-1},t,z_{j+1},\ldots,z_n)$$

    where

    $$a_t = \frac{-R(z_1,\ldots,\widehat{z_i},\ldots,z_{j-1},t,z_{j+1},\ldots z_n)}{Q(z_1,\ldots,\widehat{z_i},\ldots,z_{j-1},t,z_{j+1},\ldots z_n)},$$

    belongs to C, which contradicts the fact that $C \subset (0,\infty)^n$.
  - If there exists $z = (z_1,\ldots,z_n) \in C$ such that $S(z_1,\ldots,\widehat{z_i},\ldots,\widehat{z_j},\ldots,z_n)$ has opposite sign to $Q(z_1,\ldots,\widehat{z_i},\ldots,z_n)$, then for $t \leqslant z_j$ we have that

    $$Q(z_1,\ldots,\widehat{z_i},\ldots,z_{j-1},t,z_{j+1},\ldots,z_n) =$$
    $$(t-z_j)S(z_1,\ldots,\widehat{z_i},\ldots,\widehat{z_j},\ldots,z_n) + Q(z_1,\ldots,\widehat{z_i},\ldots z_{j-1},z_j,z_{j+1},\ldots,z_n)$$

    is different from zero since it has the same sign as

    $$Q(z_1,\ldots,\widehat{z_i},\ldots z_{j-1},z_j,z_{j+1},\ldots,z_n).$$

    As before, this implies that that for each $t \in (-\infty,z_j]$, the point

    $$(z_1,\ldots,z_{i-1},a_t,z_{i+1},\ldots,z_{j-1},t,z_{j+1},\ldots,z_n)$$

    where

    $$a_t = \frac{-R(z_1,\ldots,\widehat{z_i},\ldots,z_{j-1},t,z_{j+1},\ldots z_n)}{Q(z_1,\ldots,\widehat{z_i},\ldots,z_{j-1},t,z_{j+1},\ldots,z_n)}$$

    belongs to C, which contradicts the fact that $C \subset (0,\infty)^n$.

Now, let us prove that if $z = (z_1, \dots, z_n) \in C$ and we take $(z'_1, \dots, z'_{n-1}) \in \mathrm{R}^{n-1}$ with $z'_1 \geqslant z_1, \dots, z'_{n-1} \geqslant z_{n-1}$, then $Q(z'_1, \dots, z'_{n-1}) \neq 0$ and

$$\left(z'_1, \dots, z'_{n-1}, \frac{-R(z'_1, \dots, z'_{n-1})}{Q(z'_1, \dots, z'_{n-1})}\right) \in C.$$

We proceed by induction. Suppose that we know already that for some $1 \leqslant i \leqslant n$, $Q(z'_1, \dots, z'_{i-1}, z_i, \dots, z_{n-1}) \neq 0$ and

$$\left(z'_1, \dots, z'_{i-1}, z_i, \dots, z_{n-1}, \frac{-R(z'_1, \dots, z'_{i-1}, z_i, \dots, z_{n-1})}{Q(z'_1, \dots, z'_{i-1}, z_i, \dots, z_{n-1})}\right) \in C.$$

Then, for $t \geqslant z_i$,

$$Q(z'_1, \dots, z'_{i-1}, t, z_{i+1}, \dots, z_{n-1}) =$$
$$= (t - z_i) S(z'_1, \dots, z'_{i-1}, z_{i+1}, \dots, z_{n-1}) + Q(z'_1, \dots, z'_{i-1}, z_i, z_{i+1}, \dots, z_{n-1})$$

is different from zero since it has the same sign as $Q(z'_1, \dots, z'_{i-1}, z_i, z_{i+1}, \dots, x_{n-1})$. This implies that for each $t \in [z_i, z'_i]$, the point

$$(z'_1, \dots, z'_{i-1}, t, z_{i+1}, \dots, z_{n-1}, a_t)$$

where

$$a_t = \frac{-R(z'_1, \dots, z'_{i-1}, t, z_{i+1}, \dots, z_{n-1})}{Q(z'_1, \dots, z'_{i-1}, t, z_{i+1}, \dots, z_{n-1})}$$

belongs to C.

Finally, take any $z = (z_1, \dots, z_n) \in C$. For every $t \geqslant 0$, $Q(z_1 + t, \dots, z_{n-1} + t) \neq 0$ and

$$\left(z_1 + t, \dots, z_{n-1} + t, \frac{-R(z_1 + t, \dots, z_{n-1} + t)}{Q(z_1 + t, \dots, z_{n-1} + t)}\right) \in C.$$

This is impossible because since P is symmetric and $n \geqslant 2^{d-1} + 1 \geqslant d + 1$, it can be easily seen that $\deg R = d$, $\deg Q = d - 1$ and

$$\lim_{t \to +\infty} \frac{-R(z_1 + t, \dots, z_{n-1} + t)}{Q(z_1 + t, \dots, z_{n-1} + t)} = -\infty,$$

which contradicts the assumption that $C \subset (0, +\infty)^n$.

- $C \subset (-\infty, 0)^n$ and for every $z = (z_1, \dots, z_n) \in C$ and $1 \leqslant i \leqslant n$, $Q(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n) \neq 0$:
  In this case we proceed as in the previous one.

□

From now on we consider fixed $d \in \mathbb{N}$ and $a_0, \dots, a_d \in \mathrm{R}$ with $a_d \neq 0$. For $n \geqslant d$, let

$$P_n = \sum_{0 \leqslant \ell \leqslant d} a_\ell \sigma_{\ell,n} \in \mathrm{R}[X_1, \dots, X_n].$$

Proposition 3.4. *The sequence $(b_0(\mathrm{Z}(P_n, \mathrm{R}^n))_{n \geqslant d}$ is eventually decreasing, and therefore eventually constant.*

Proof. By Lemma 3.3, if $n \geqslant 2^{d-1} + 1$, every semi-algebraic connected component of $Z(P, \mathrm{R}^n)$ intersects the hyperplane $Z(X_n, \mathrm{R}^n)$. Since $P_n(X_1, \dots, X_{n-1}, 0) = P_{n-1}(X_1, \dots, X_{n-1})$, we have that

$$b_0(\mathrm{Z}(P_{n-1}, \mathrm{R}^{n-1})) \geqslant b_0(\mathrm{Z}(P_n, \mathrm{R}^n)).$$

□

Proof of Theorem 5. Theorem 5 follows from Propositions 3.3 and 3.4. □

We finish this section by showing two examples of ideals $I \subset \Lambda$ such that

$$\lim_{n\to\infty} m_{0,\{\lambda\}_n}(V_n(I)) > 1,$$

for $\lambda = ()$. First, we include an auxiliary lemma.

Lemma 3.4. *Let $n \geqslant 3$. For $x \in \mathrm{R}^n$ with $N_1(x) = 0$,*

$$N_3(x)^2 \leqslant \frac{(n-2)^2}{n(n-1)} N_2(x)^3.$$

Proof. The inequality holds if $x = 0$. If $x \neq 0$, we take $R^2 = N_2(x)$ and then the inequality can be checked using Lagrange Multipliers to find the extreme values of $N_3(x)$ subject to the restrictions $N_1(x) = 0$, $N_2(x) = R^2$. □

Now let $f = \sigma_2 - 1, g = \sigma_3 - \sigma_1, I = (f), J = (g)$ and $\lambda = ()$. We will show that

$$\begin{aligned} \lim_{n\to\infty} m_{0,\{\lambda\}_n}(V_n(I)) &= 2, \\ \lim_{n\to\infty} m_{0,\{\lambda\}_n}(V_n(J)) &= 3. \end{aligned}$$

Indeed, using Theorem 5, it is enough to show that for $n \geqslant 3$, $b_0(\phi_n(f), \mathrm{R}^n) = 2$ and $b_0(\phi_n(g), \mathrm{R}^n) = 3$.

We take a fixed value of $x \in \mathrm{R}^n$ with $N_1(x) = \sigma_{1,n}(x) = 0$ and consider the polynomials

$$f_x(t) = \sigma_{2,n}(x_1 + t, \ldots, x_n + t) - 1 = \binom{n}{2} t^2 - \left(\frac{1}{2} N_2(x) + 1\right)$$

and

$$g_x(t) = \sigma_{3,n}(x_1 + t, \ldots, x_n + t) - \sigma_{1,n}(x_1 + t, \ldots, x_n + t) =$$

$$= \binom{n}{3} t^3 - \left(\frac{n-2}{2} N_2(x) + n\right) t + \frac{1}{3} N_3(x).$$

It is clear that f_x has a positive discriminant, and on the other hand, $\mathrm{Disc}(g_x)$ is also positive since it is a positive multiple of

$$4\left(\frac{n-2}{2} N_2(x) + n\right)^3 - 3\binom{n}{3} N_3(x)^2 \;>\; 4\left(\frac{n-2}{2} N_2(x)\right)^3 - 3\binom{n}{3} N_3(x)^2 \;\geqslant\; 0$$

using Lemma 3.4.

Finally, we split R^n as

$$\mathrm{R}^n = \bigcup_{x \in \mathrm{R}^n, \sigma_1(x) = 0} \{(x_1 + t, \ldots, x_n + t) \mid t \in \mathrm{R}\}$$

and then the claim follows using the continuity of roots with respect to the coefficients of a polynomial of fixed degree outside the region where the discriminant vanishes.

4. Conclusion and open problems

We have proved an upper bound of 2^{d-1} on the number of semi-algebraically connected components of a real hypersurface in R^n defined by a multi-affine polynomial of degree d. Moreover, we have shown that no bound which grows only polynomially with n exists for the higher Betti numbers of such hypersurfaces inside a closed ball.

Finally, we have proved a special case of a stability conjecture due to Basu and Riener on the cohomology modules of symmetric real algebraic sets.

There are several open questions that are suggested by our results.

1. Does the upper bound in Theorem 2 extend to the bounded case? More precisely, is there a bound on $\beta_{\mathbf{A}_d,\mathbf{B},0}(n)$ which is independent of n for some natural sequence $\mathbf{B}$, for example $\mathbf{B} = ([-1,1]^n)_{n>0}$? At the same time it would be interesting to extend Theorem 3 to the unbounded case. More precisely, does there exist $c > 1$, such that $\beta_{\mathbf{A}_d,p}(n) > c^n$ for some $d, p > 0$?
2. Can one prove a bound on the number of connected components of a real algebraic set in R^n defined by *two* multi-affine polynomials of degree at most d which is independent of n? We have shown that no such bound exists for real algebraic sets defined by three or more multi-affine polynomials. It would be satisfactory to be able fill this gap.
3. Multi-affine polynomials that arise in practice (such as the basis generating polynomial of a matroid) often have special properties such as real stability or being Lorentzian [4, 13]). It would be interesting to study the topology of real hypersurfaces defined by such polynomials from a quantitative point of view.
4. The algorithmic problem of computing the number of semi-algebraically connected components of a given real algebraic set in R^n has attracted wide attention. The main tool for solving this problem is via computation of a one-dimensional semi-algebraic subset (called a roadmap of V). While there has been a steady improvement in the complexity of algorithms for computing roadmaps of semi-algebraic sets [5, 9, 8], the complexities of all known algorithms are exponential in n. This is not unexpected as the number of semi-algebraically connected components of real algebraic sets in R^n defined by polynomials of degree at most d, grows exponentially in n in the worst case for $d > 2$. However, in this paper we have proved that the number of semi-algebraically connected components of hypersurfaces defined by multi-affine polynomial is small. This suggests the problem of finding a more efficient algorithm (say with polynomial complexity) for computing this number (maybe without resorting to a roadmap algorithm). In the *symmetric case* such an algorithm (with polynomial complexity with the degree being considered fixed) was shown to exist in [11].

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Department of Mathematics, Purdue University, West Lafayette, IN 47906, U.S.A.
Email address: `sbasu@math.purdue.edu`

Departamento de Matemática Facultad de Ciencias Exactas y Naturales Universidad de Buenos Aires, and IMAS UBA-CONICET, Buenos Aires, Argentina.
Email address: `perrucci@dm.uba.ar`