A few more extensions of Putinar's Positivstellensatz to non-compact sets

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Abstract

We extend previous results about Putinar's Positivstellensatz for cylinders of type $S \times \mathbb{R}$ to sets of type $S \times \mathbb{R}^r$ in some special cases taking into account r and the degree of the polynomial with respect to the variables moving in $\mathbb{R}^r$ (this is to say, in the non-bounded directions). These special cases are in correspondence with the ones where the equality between the cone of non-negative polynomials and the cone of sums of squares holds. Degree bounds are provided.

1 Introduction

One of the most important results in the theory of sums of squares and certificates of non-negativity is Putinar's Positivstellensatz ([14]). This theorem states that given $g_1, \dots, g_s \in \mathbb{R}[\bar{X}] = \mathbb{R}[X_1, \dots, X_n]$ such that the quadratic module generated by $g_1, \dots, g_s$,

$$M(g_1, \dots, g_s) = \left\{ \sigma_0 + \sigma_1 g_1 + \dots + \sigma_s g_s \mid \sigma_0, \sigma_1, \dots, \sigma_s \in \sum \mathbb{R}[\bar{X}]^2 \right\} \subset \mathbb{R}[\bar{X}]$$

is archimedean, every $f \in \mathbb{R}[\bar{X}]$ positive on

$$S = \{\bar{x} \in \mathbb{R}^n \,|\, g_1(\bar{x}) \ge 0, \dots, g_s(\bar{x}) \ge 0\}$$

belongs to $M(g_1, \dots, g_s)$. An explicit expression of f as a member of $M(g_1, \dots, g_s)$ is a certificate of the non-negativity of f on S. Note that the condition of archimedeanity on $M(g_1, \dots, g_s)$ implies that S is compact (see for instance [9, Chapter 5] for the definition and equivalences of archimedeanity).

It is of interest to look for a bound for the degrees of all the different terms in the representation of f as an element of $M(g_1, \dots, g_s)$ which existence is ensured by Putinar's Positivstellensatz. An answer to this question was given by Nie and Schweighofer ([11, Theorem 6]). In the particular case where S is the hypercube $[0, 1]^n$, improved bounds were given in [3] and [8].

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Consider the norm $\|\cdot\|$ in $\mathbb{R}[\bar{X}]$ defined as follows.

$$\text{For } f = \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|\le d}} \binom{|\alpha|}{\alpha} a_\alpha \bar{X}^\alpha, \qquad \|f\| = \max\{|a_\alpha| \mid \alpha \in \mathbb{N}_0^n, |\alpha| \le d\}$$

where for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha_1! \dots \alpha_n!} \in \mathbb{N}$. Then [11, Theorem 6] is the following result.

Theorem 1 (Putinar's Positivstellensatz with degree bound) *Let $g_1, \dots, g_s \in \mathbb{R}[\bar{X}]$ such that*

$$\emptyset \neq S = \{\bar{x} \in \mathbb{R}^n \mid g_1(\bar{x}) \ge 0, \dots, g_s(\bar{x}) \ge 0\} \subset (-1, 1)^n,$$

and the quadratic module $M(g_1, \dots, g_s)$ is archimedean. There exists a positive constant c such that for every $f \in \mathbb{R}[\bar{X}]$ positive on S, if $\deg f = d$ and $\min\{f(\bar{x}) \mid \bar{x} \in S\} = f^* > 0$, then f can be written as

$$f = \sigma_0 + \sigma_1 g_1 + \cdots + \sigma_s g_s \in M(g_1, \dots, g_s)$$

with $\sigma_0, \sigma_1, \dots, \sigma_s \in \sum \mathbb{R}[\bar{X}]^2$ and

$$\deg(\sigma_0), \deg(\sigma_1 g_1), \dots, \deg(\sigma_s g_s) \le c\, \mathrm{e}^{\left(\frac{\|f\| d^2 n^d}{f^*}\right)^c}.$$

A natural question is if it is possible to relax the archimedeanity hyphothesis and to extend Putinar's Positivstellensatz to cases where S is non-compact. Even though it is well-known that this is not possible in full generality, results in this direction were given in [6], [7], [10] and [5]. We introduce the notation and definitions needed to state our main result from [5].

We note

$$C = \{(y, z) \in \mathbb{R}^2 \mid y^2 + z^2 = 1\}.$$

For

$$f = \sum_{0\le i\le m} f_i(\bar{X}) Y^i \in \mathbb{R}[\bar{X}, Y]$$

with $\deg_Y f = m$, we note

$$\bar{f} = \sum_{0\le i\le m} f_i(\bar{X}) Y^i Z^{m-i} \in \mathbb{R}[\bar{X}, Y, Z]$$

its homogenization only with respect to the variable Y. Such an f is said to be *fully m-ic* on S if for every $\bar{x} \in S$, $f_m(\bar{x}) \neq 0$. We define the norm $\|\cdot\|_\bullet$ on $\mathbb{R}[\bar{X}, Y]$ as follows.

$$\text{For } f = \sum_{0\le i\le m} \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|\le d}} \binom{|\alpha|}{\alpha} a_{\alpha,i} \bar{X}^\alpha Y^i, \qquad \|f\|_\bullet = \max\{|a_{\alpha,i}| \mid 0 \le i \le m, \alpha \in \mathbb{N}_0^n, |\alpha| \le d\}.$$

Note that in this norm, the variable Y is not considered in the same way than the variables $\bar{X}$. Finally, for $g_1, \dots, g_s \in \mathbb{R}[\bar{X}]$, we note

$$M_{\mathbb{R}[\bar{X},Y]}(g_1, \dots, g_s) = \left\{\sigma_0 + \sigma_1 g_1 + \cdots + \sigma_s g_s \mid \sigma_0, \sigma_1, \dots, \sigma_s \in \sum \mathbb{R}[\bar{X}, Y]^2\right\} \subset \mathbb{R}[\bar{X}, Y]$$

the quadratic module generated by $g_1, \dots, g_s$ in $\mathbb{R}[\bar{X}, Y]$, while the notation $M(g_1, \dots, g_s)$ is kept for the quadratic module generated by $g_1, \dots, g_s$ in $\mathbb{R}[\bar{X}]$.

In [5, Theorem 7], we prove an extension of Putinar's Positivstellensatz to cylinders of type $S \times \mathbb{R}$ which is the following result.

Theorem 2 *Let $g_1, \dots, g_s \in \mathbb{R}[\bar{X}]$ such that*

$$\emptyset \neq S = \{\bar{x} \in \mathbb{R}^n \mid g_1(\bar{x}) \geq 0, \dots, g_s(\bar{x}) \geq 0\} \subset (-1, 1)^n$$

and the quadratic module $M(g_1, \dots, g_s) \subset \mathbb{R}[\bar{X}]$ is archimedean. There exists a positive constant c such that for every $f \in \mathbb{R}[\bar{X}, Y]$ positive on $S \times \mathbb{R}$, if $\deg_{\bar{X}} f = d$, $\deg_Y f = m$ with f fully m-ic on S and

$$\min\{\bar{f}(\bar{x}, y, z) \mid \bar{x} \in S, (y, z) \in C\} = f^{\bullet} > 0,$$

then f can be written as

$$f = \sigma_0 + \sigma_1 g_1 + \dots + \sigma_s g_s \in M_{\mathbb{R}[\bar{X}, Y]}(g_1, \dots, g_s)$$

with $\sigma_0, \sigma_1, \dots, \sigma_s \in \sum \mathbb{R}[\bar{X}, Y]^2$ and

$$\deg(\sigma_0), \deg(\sigma_1 g_1), \dots, \deg(\sigma_s g_s) \leq c(m+1)2^{\frac{m}{2}} \mathrm{e}^{\left(\frac{\|f\|_{\bullet}(m+1)d^2(3n)^d}{f^{\bullet}}\right)^c}.$$

The condition of f being fully m-ic was defined originally in [12]. Under this assumption, in [12, Theorem 3] Powers obtains an extension of Schmüdgen's Positivstellensatz ([15]) to cylinders of type $S \times F$ with $S \subset \mathbb{R}^n$ a compact semialgebraic set and $F \subset \mathbb{R}$ an unbounded closed semialgebraic set. Indeed, the general idea to prove [5, Theorem 7] is the same as in [12, Theorem 3], which is to consider the variable Y as a parameter and to obtain for each specialization of $Y = y \in F$ a certificate of the non-negativity of $f(\bar{X}, y)$ on S, in a uniform way such that all these certificates can be glued together to obtain the desired representation for $f(\bar{X}, Y)$.

Both in [12, Theorem 3] (in the case $F = \mathbb{R}$) and in [5, Theorem 7], in the gluing process, it is used that every univariate polynomial non-negative on $\mathbb{R}$ is a sum of squares in $\mathbb{R}[Y]$. This suggests that the same approach can also be used in every other possible situation where the equality between the cone of non-negative polynomials and the cone of sums of squares holds, which are known to be the case of bivariate polynomials of degree 4 and the case of multivariate polynomials of degree 2 (see for instance [1, Chapter 6]). Indeed, if separate sets of variables are considered, then the equality also holds in the particular case of polynomials in two sets of variables, the first set consisting in a single variable, and the degree of the polynomial with respect to the second set of variables equal to 2 (see for instance [4] or [2, Section 7]).

We extend the notation introduced before. For $r \in \mathbb{N}$, we note $\bar{Y} = (Y_1, \dots, Y_r)$ and

$$C^r = \{(\bar{y}, z) \in \mathbb{R}^{r+1} \mid y_1^2 + \dots + y_r^2 + z^2 = 1\}.$$

Also, for even $m \in \mathbb{N}$ and

$$f = \sum_{\substack{\beta \in \mathbb{N}_0^r \\ |\beta| \leq m}} f_\beta(\bar{X}) \bar{Y}^\beta \in \mathbb{R}[\bar{X}, \bar{Y}]$$

with $\deg_{\bar{Y}} f = m$, we note

$$\bar{f} = \sum_{\substack{\beta \in \mathbb{N}_0^r \\ |\beta| \le m}} f_\beta(\bar{X}) \bar{Y}^\beta Z^{m-|\beta|} \in \mathbb{R}[\bar{X}, \bar{Y}, Z]$$

its homogenization only with respect to the variables $\bar{Y}$. For such an f, we say that f satisfies the condition (†) on S if for every $\bar{x} \in S$,

$$\sum_{\substack{\beta \in \mathbb{N}_0^r \\ |\beta| = m}} f_\beta(\bar{x}) \bar{Y}^\beta \text{ is a positive definite } m\text{-form in } \mathbb{R}^r.$$

Note that if f is positive in $S \times \mathbb{R}^r$ and satisfies condition (†) on S, then $\bar{f}$ is positive on $S \times C^r$.

We extend the definition of the norm $\|\cdot\|_\bullet$ to $\mathbb{R}[\bar{X}, \bar{Y}]$ as follows.

$$\text{For } f = \sum_{\substack{\beta \in \mathbb{N}_0^r \\ |\beta| \le m}} \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha| \le d}} \binom{|\alpha|}{\alpha} a_{\alpha,\beta} \bar{X}^\alpha \bar{Y}^\beta, \qquad \|f\|_\bullet = \max\{|a_{\alpha,\beta}| \mid \beta \in \mathbb{N}_0^r, |\beta| \le m, \alpha \in \mathbb{N}_0^n, |\alpha| \le d\}.$$

Finally, for $g_1, \dots, g_s \in \mathbb{R}[\bar{X}]$, we note

$$M_{\mathbb{R}[\bar{X},\bar{Y}]}(g_1, \dots, g_s) = \left\{ \sigma_0 + \sigma_1 g_1 + \cdots + \sigma_s g_s \mid \sigma_0, \sigma_1, \dots, \sigma_s \in \sum \mathbb{R}[\bar{X}, \bar{Y}]^2 \right\} \subset \mathbb{R}[\bar{X}, \bar{Y}]$$

the quadratic module generated by $g_1, \dots, g_s$ in $\mathbb{R}[\bar{X}, \bar{Y}]$.

Both the norm $\|\cdot\|_\bullet$ and the quadratic module $M_{\mathbb{R}[\bar{X},\bar{Y}]}$ will be used later on in this paper with many different vectors of variables $\bar{Y}$, but always distinguishing the same vector of variables $\bar{X}$.

In this setting, Theorem 2 ([5, Theorem 7]) is the extension of Putinar's Positivstellensatz corresponding to the case $r = 1$. We present below the two new extensions corresponding to the cases $r = 2, m = 4$ and $m = 2$.

Theorem 3 *Let $g_1, \dots, g_s \in \mathbb{R}[\bar{X}]$ such that*

$$\emptyset \ne S = \{\bar{x} \in \mathbb{R}^n \mid g_1(\bar{x}) \ge 0, \dots, g_s(\bar{x}) \ge 0\} \subset (-1, 1)^n$$

and the quadratic module $M(g_1, \dots, g_s) \subset \mathbb{R}[\bar{X}]$ is archimedean. There exists a positive constant c such that for every $f \in \mathbb{R}[\bar{X}, Y_1, Y_2]$ positive on $S \times \mathbb{R}^2$, if $\deg_{\bar{X}} f = d$, $\deg_{(Y_1,Y_2)} f = 4$, f satisfies condition (†) on S and

$$\min\left\{ \bar{f}(\bar{x}, y_1, y_2, z) \mid \bar{x} \in S, (y_1, y_2, z) \in C^2 \right\} = f^\bullet > 0,$$

then f can be written as

$$f = \sigma_0 + \sigma_1 g_1 + \cdots + \sigma_s g_s \in M_{\mathbb{R}[\bar{X},Y_1,Y_2]}(g_1, \dots, g_s)$$

with $\sigma_0, \sigma_1, \dots, \sigma_s \in \sum \mathbb{R}[\bar{X}, Y_1, Y_2]^2$ and

$$\deg(\sigma_0), \deg(\sigma_1 g_1), \dots, \deg(\sigma_s g_s) \le c e^{\left(\frac{\|f\|_\bullet d^2 (3n)^d}{f^\bullet} \right)^c}.$$

Theorem 4 *Let $g_1, \dots, g_s \in \mathbb{R}[\bar{X}]$ such that*

$$\emptyset \neq S = \{\bar{x} \in \mathbb{R}^n \mid g_1(\bar{x}) \geq 0, \dots, g_s(\bar{x}) \geq 0\} \subset (-1,1)^n$$

and the quadratic module $M(g_1, \dots, g_s) \subset \mathbb{R}[\bar{X}]$ is archimedean. There exists a positive constant c such that for every $f \in \mathbb{R}[\bar{X}, \bar{Y}]$ positive on $S \times \mathbb{R}^r$, if $\deg_{\bar{X}} f = d$, $\deg_{\bar{Y}} f = 2$, f satisfies condition $(\dagger)$ on S and

$$\min\left\{\bar{f}(\bar{x}, \bar{y}, z) \,|\, \bar{x} \in S, (\bar{y}, z) \in C^r\right\} = f^\bullet > 0,$$

then f can be written as

$$f = \sigma_0 + \sigma_1 g_1 + \dots + \sigma_s g_s \in M_{\mathbb{R}[\bar{X}, \bar{Y}]}(g_1, \dots, g_s)$$

with $\sigma_0, \sigma_1, \dots, \sigma_s \in \sum \mathbb{R}[\bar{X}, \bar{Y}]^2$ and

$$\deg(\sigma_0), \deg(\sigma_1 g_1), \dots, \deg(\sigma_s g_s) \leq c r^2 \mathrm{e}^{\left(\frac{\|f\|_\bullet r^2 d^2 (3n)^d}{f^\bullet}\right)^c}.$$

Finally, we introduce the notation we need to state the extension corresponding to the case of two separate sets of variables. For $r \in \mathbb{N}$, we note $\bar{Y}_2 = (Y_{21}, \dots, Y_{2r})$. Also, for even $m \in \mathbb{N}_0$ and

$$f = \sum_{0 \leq i \leq m} \sum_{\substack{\beta \in \mathbb{N}_0^r \\ |\beta| \leq 2}} f_{i,\beta}(\bar{X}) Y_1^i \bar{Y}_2^\beta \in \mathbb{R}[\bar{X}, Y_1, \bar{Y}_2]$$

with $\deg_{Y_1} f = m$ and $\deg_{\bar{Y}_2} f = 2$, we note

$$\bar{\bar{f}} = \sum_{0 \leq i \leq m} \sum_{\substack{\beta \in \mathbb{N}_0^r \\ |\beta| \leq 2}} f_{i,\beta}(\bar{X}) Y_1^i Z_1^{m-i} \bar{Y}_2^\beta Z_2^{2-|\beta|} \in \mathbb{R}[\bar{X}, Y_1, Z_1, \bar{Y}_2, Z_2]$$

its bihomogenization only with respect to the variables Y_1 and $\bar{Y}_2$, separately. The additional assumption playing the role of condition $(\dagger)$ in this case is the following one. We say that f satisfies the condition $(\ddagger)$ on S if for every $\bar{x} \in S$,

i) $\displaystyle\sum_{\substack{\beta \in \mathbb{N}_0^r \\ |\beta| \leq 2}} f_{m,\beta}(\bar{x}) \bar{Y}_2^\beta$ is positive in $\mathbb{R}^r$,

ii) $\displaystyle\sum_{\substack{\beta \in \mathbb{N}_0^r \\ |\beta| = 2}} f_{m,\beta}(\bar{x}) \bar{Y}_2^\beta$ is a positive definite quadratic form in $\mathbb{R}^r$,

iii) for every $y_1 \in \mathbb{R}$, $\displaystyle\sum_{0 \leq i \leq m} \sum_{\substack{\beta \in \mathbb{N}_0^r \\ |\beta| = 2}} f_{i,\beta}(\bar{x}) y_1^i \bar{Y}_2^\beta$ is a positive definite quadratic form in $\mathbb{R}^r$.

The definition of $(\ddagger)$ is made in such a way that if f is positive in $S \times \mathbb{R}^{r+1}$ and satisfies condition $(\ddagger)$ on S, then $\bar{\bar{f}}$ is positive on $S \times C \times C^r$.

The extension of Putinar's Positivstellensatz in this final case is the following result.

Theorem 5 *Let $g_1, \ldots, g_s \in \mathbb{R}[\bar{X}]$ such that*

$$\emptyset \neq S = \{\bar{x} \in \mathbb{R}^n \mid g_1(\bar{x}) \geq 0, \ldots, g_s(\bar{x}) \geq 0\} \subset (-1,1)^n$$

and the quadratic module $M(g_1, \ldots, g_s) \subset \mathbb{R}[\bar{X}]$ is archimedean. There exists a positive constant c such that for every $f \in \mathbb{R}[\bar{X}, Y_1, \bar{Y}_2]$ positive on $S \times \mathbb{R}^{r+1}$, if $\deg_{\bar{X}} f = d$, $\deg_{Y_1} f = m$, $\deg_{\bar{Y}} f = 2$, f satisfies condition (‡) on S and

$$\min\left\{\bar{\bar{f}}(\bar{x}, y_1, z_1, \bar{y}_2, z_2) \,|\, \bar{x} \in S, (y_1, z_1) \in C, (\bar{y}_2, z_2) \in C^r\right\} = f^\bullet > 0,$$

then f can be written as

$$f = \sigma_0 + \sigma_1 g_1 + \cdots + \sigma_s g_s \in M_{\mathbb{R}[\bar{X}, Y_1, \bar{Y}_2]}(g_1, \ldots, g_s)$$

with $\sigma_0, \sigma_1, \ldots, \sigma_s \in \sum \mathbb{R}[\bar{X}, Y_1, \bar{Y}_2]^2$ and

$$\deg(\sigma_0), \deg(\sigma_1 g_1), \ldots, \deg(\sigma_s g_s) \leq c(m+1)2^{\frac{m}{2}} r^2 \mathrm{e}^{\left(\frac{\|f\|_\bullet (m+1) r^2 d^2 (3n)^d}{f^\bullet}\right)^c}.$$

2 Proof of the main results

The proofs of Theorems 3, 4 and 5 follow the same path than the proof of Theorem 2 ([5, Theorem 7]), which is itself mainly a combination and reorganization of techniques from [11], [12] and [16].
For $n \in \mathbb{N}$, we denote by $\widetilde{\Delta}_n$ the simplex

$$\widetilde{\Delta}_n = \left\{\bar{x} \in \mathbb{R}^n \mid \sum_{1 \leq i \leq n} x_i \leq 1 \text{ and } x_i \geq 0 \text{ for } 1 \leq i \leq n\right\}$$

and by Δ_n the standard simplex

$$\Delta_n = \left\{(x_0, \bar{x}) \in \mathbb{R}^{n+1} \mid \sum_{0 \leq i \leq n} x_i = 1 \text{ and } x_i \geq 0 \text{ for } 0 \leq i \leq n\right\}.$$

The following two lemmas are slight variations of [5, Lemma 16] and [11, Lemma 11] (see also [5, Lemma 17]).

Lemma 6 *Let $f \in \mathbb{R}[\bar{X}, \bar{Y}]$ such that $\deg_{\bar{X}} f = d$ and $\deg_{\bar{Y}} f = m$. For every $\bar{x} \in \widetilde{\Delta}_n$ and $(\bar{y}, z) \in C^r$,*

$$|\bar{f}(\bar{x}, \bar{y}, z)| \leq \|f\|_\bullet \binom{m+r}{r} (d+1).$$

Lemma 7 *Let $f \in \mathbb{R}[\bar{X}, \bar{Y}]$ such that $\deg_{\bar{X}} f = d$ and $\deg_{\bar{Y}} f = m$. For every $\bar{x}_1, \bar{x}_2 \in \widetilde{\Delta}_n$ and $(\bar{y}, z) \in C^r$,*

$$|\bar{f}(\bar{x}_1, \bar{y}, z) - \bar{f}(\bar{x}_2, \bar{y}, z)| \leq \frac{1}{2}\sqrt{n} \|f\|_\bullet \binom{m+r}{r} d(d+1) \|\bar{x}_1 - \bar{x}_2\|.$$

We focus on the proof of Theorem 3. Under the stronger assumption of $S \subset \widetilde{\Delta}_n^\circ$, Proposition 8 below is just a slight variant of Theorem 3 with a substantially better degree bound (which unfortunately does not have good rescaling properties). Once Proposition 8 is proved, Theorem 3 simply follows by composing with a linear change of variables.

Proposition 8 *Let $g_1, \dots, g_s \in \mathbb{R}[\bar{X}]$ such that*

$$\emptyset \neq S = \{\bar{x} \in \mathbb{R}^n \mid g_1(\bar{x}) \geq 0, \dots, g_s(\bar{x}) \geq 0\} \subset \widetilde{\Delta}_n^\circ$$

and the quadratic module $M(g_1, \dots, g_s)$ is archimedean. There exists a positive constant c such that for every $f \in \mathbb{R}[\bar{X}, Y_1, Y_2]$ positive on $S \times \mathbb{R}^2$, if $\deg_{\bar{X}} f = d$, $\deg_{(Y_1,Y_2)} f = 4$, f satisfies condition (†) on S and

$$\min\{\bar{f}(\bar{x}, y_1, y_2, z) \,|\, \bar{x} \in S, (y_1, y_2, z) \in C^2\} = f^\bullet > 0,$$

then f can be written as

$$f = \sigma_0 + \sigma_1 g_1 + \dots + \sigma_s g_s \in M_{\mathbb{R}[\bar{X},Y_1,Y_2]}(g_1, \dots, g_s)$$

with $\sigma_0, \sigma_1, \dots, \sigma_s \in \sum \mathbb{R}[\bar{X}, Y_1, Y_2]^2$ and

$$\deg(\sigma_0), \deg(\sigma_1 g_1), \dots, \deg(\sigma_s g_s) \leq c e^{\left(\frac{\|f\|_\bullet d^2}{f^\bullet}\right)^c}.$$

Proof: Without loss of generality we suppose $\deg g_i \geq 1$ and $|g_i| \leq 1$ in $\widetilde{\Delta}_n$ for $1 \leq i \leq s$.
If $d = 0$ then $f \in \mathbb{R}[Y_1, Y_2]$ is positive on $\mathbb{R}^2$, and since $\deg_{(Y_1,Y_2)} f = 4$, $f \in \sum \mathbb{R}[Y_1, Y_2]^2$ and any constant $c \geq 4$ works. So from now on, we suppose $d \geq 1$ and in the case that the final constant c we find turns out to be less than 4, we just replace it by the result of applying [5, Lemma 18] to the 6-uple $(4, 0, c, 0, 1, c)$.
We prove first that there exist $\lambda \in \mathbb{R}_{>0}$ and $k \in \mathbb{N}_0$ such that

$$h = \bar{f} - \lambda \big(Y_1^2 + Y_2^2 + Z^2\big)^2 \sum_{1 \leq i \leq s} g_i \cdot (g_i - 1)^{2k} \in \mathbb{R}[\bar{X}, Y_1, Y_2, Z]$$

satisfies $h \geq \frac{1}{2} f^\bullet$ in $\widetilde{\Delta}_n \times C^2$.
For each $(y_1, y_2, z) \in C^2$ we consider

$$A_{y_1,y_2,z} = \left\{ \bar{x} \in \widetilde{\Delta}_n \mid \bar{f}(\bar{x}, y_1, y_2, z) \leq \frac{3}{4} f^\bullet \right\}.$$

Note that $A_{y_1,y_2,z} \cap S = \emptyset$. To exhibit sufficient conditions for λ and k, we consider separately the cases $\bar{x} \in \widetilde{\Delta}_n \setminus A_{y_1,y_2,z}$ and $\bar{x} \in A_{y_1,y_2,z}$.
If $\bar{x} \in \widetilde{\Delta}_n \setminus A_{y_1,y_2,z}$

$$\begin{aligned} h(\bar{x}, y_1, y_2, z) &= \bar{f}(\bar{x}, y_1, y_2, z) - \lambda \big(y_1^2 + y_2^2 + z^2\big)^2 \sum_{1 \leq i \leq s} g_i(\bar{x}) \cdot (g_i(\bar{x}) - 1)^{2k} \\ &\geq \bar{f}(\bar{x}, y_1, y_2, z) - \lambda \sum_{1 \leq i \leq s} |g_i(\bar{x})| \cdot (|g_i(\bar{x})| - 1)^{2k} \\ &> \frac{3}{4} f^\bullet - \frac{\lambda s}{2k+1} \end{aligned}$$

using [11, Remark 12]. Therefore the condition $h(\bar{x}, y_1, y_2, z) \geq \frac{1}{2} f^\bullet$ is ensured if

$$2k + 1 \geq \frac{4\lambda s}{f^\bullet}. \tag{1}$$

If $\bar{x} \in A_{y_1,y_2,z}$, for any $\bar{x}_0 \in S$, by Lemma 7 with $r = 2$ and $m = 4$, we have

$$\frac{f^\bullet}{4} \leq \bar{f}(\bar{x}_0, y_1, y_2, z) - \bar{f}(\bar{x}, y_1, y_2, z) \leq \frac{15}{2}\sqrt{n}\|f\|_\bullet d(d+1)\|\bar{x}_0 - \bar{x}\|,$$

then

$$\frac{f^\bullet}{30\sqrt{n}\|f\|_\bullet d(d+1)} \leq \|\bar{x}_0 - \bar{x}\|$$

and therefore

$$\frac{f^\bullet}{30\sqrt{n}\|f\|_\bullet d(d+1)} \leq \operatorname{dist}(\bar{x}, S). \tag{2}$$

Using [5, Remark 13], there exist $c_1, c_2 > 0$ and $1 \leq i_0 \leq s$ such that $g_{i_0}(\bar{x}) < 0$ and

$$\operatorname{dist}(\bar{x}, S)^{c_1} \leq -c_2 g_{i_0}(\bar{x}). \tag{3}$$

By (2) and (3) we have

$$g_{i_0}(\bar{x}) \leq -\delta. \tag{4}$$

with

$$\delta = \frac{1}{c_2}\left(\frac{f^\bullet}{30\sqrt{n}\|f\|_\bullet d(d+1)}\right)^{c_1} > 0.$$

On the other hand, defining $f^\bullet_{y_1,y_2,z} = \min\{\bar{f}(\bar{x}, y_1, y_2, z) \,|\, \bar{x} \in S\}$, again by Lemma 7 with $r = 2$ and $m = 4$, we have that

$$|\bar{f}(\bar{x}, y_1, y_2, z) - f^\bullet_{y_1,y_2,z}| \leq \frac{15}{2}\sqrt{n}\|f\|_\bullet d(d+1)\operatorname{diam}(\widetilde{\Delta}_n) = \frac{15}{\sqrt{2}}\sqrt{n}\|f\|_\bullet d(d+1). \tag{5}$$

Then, using again [11, Remark 12], (4) and (5) we have

$$\begin{aligned}
h(\bar{x}, y_1, y_2, z) &\geq \bar{f}(\bar{x}, y_1, y_2, z) - \lambda g_{i_0}(\bar{x})(g_{i_0}(\bar{x}) - 1)^{2k} - \frac{\lambda(s-1)}{2k+1} \\
&\geq \bar{f}(\bar{x}, y_1, y_2, z) - f^\bullet_{y_1,y_2,z} + f^\bullet_{y_1,y_2,z} + \lambda\delta - \frac{\lambda(s-1)}{2k+1} \\
&\geq -\frac{15}{\sqrt{2}}\sqrt{n}\|f\|_\bullet d(d+1) + f^\bullet + \lambda\delta - \frac{\lambda(s-1)}{2k+1}.
\end{aligned}$$

Finally, the condition $h(\bar{x}, y_1, y_2, z) \geq \frac{1}{2} f^\bullet$ is ensured if

$$\lambda \geq \frac{15\sqrt{n}\|f\|_\bullet d(d+1)}{\sqrt{2}\delta} = \frac{c_2 2^{c_1}(15\sqrt{n}\|f\|_\bullet d(d+1))^{c_1+1}}{\sqrt{2} f^{\bullet c_1}} \tag{6}$$

and

$$2k + 1 \geq \frac{2\lambda(s-1)}{f^\bullet}. \tag{7}$$

Since (1) implies (7), it is enough for λ and k to satisfy (1) and (6). So for the rest of the proof we take

$$\lambda = \frac{c_2 2^{c_1}(15\sqrt{n}\|f\|_\bullet d(d+1))^{c_1+1}}{\sqrt{2} f^{\bullet c_1}} = c_3 \frac{(15\|f\|_\bullet d(d+1))^{c_1+1}}{f^{\bullet c_1}} > 0$$

with $c_3 = \frac{c_2 2^{c_1} \sqrt{n}^{c_1+1}}{\sqrt{2}}$ and

$$k = \left\lceil \frac{1}{2}\left(\frac{4\lambda s}{f^\bullet} - 1\right)\right\rceil \in \mathbb{N}_0.$$

In this way,

$$\begin{aligned} k &\le \frac{1}{2}\left(\frac{4\lambda s}{f^\bullet} - 1\right) + 1 \\ &= 2c_3 s \left(\frac{15\|f\|_\bullet d(d+1)}{f^\bullet}\right)^{c_1+1} + \frac{1}{2} \\ &\le c_4 \left(\frac{15\|f\|_\bullet d(d+1)}{f^\bullet}\right)^{c_1+1} \end{aligned} \tag{8}$$

with $c_4 = 2c_3 s + 1$. Here (and also several times after here) we use Lemma 6 with $r = 2$ and $m = 4$ to ensure

$$\frac{15\|f\|_\bullet (d+1)}{f^\bullet} \ge 1.$$

Also, if we define $\ell = \deg_{\bar{X}} h$, we have

$$\begin{aligned} \ell &\le \max\{d, (2k+1) \max_{1\le i\le s} \deg g_i\} \\ &\le \max\left\{d, \left(2c_4\left(\frac{15\|f\|_\bullet d(d+1)}{f^\bullet}\right)^{c_1+1} + 1\right) \max_{1\le i\le s} \deg g_i\right\} \\ &\le c_5 \left(\frac{15\|f\|_\bullet d(d+1)}{f^\bullet}\right)^{c_1+1} \end{aligned} \tag{9}$$

with $c_5 = (2c_4 + 1) \max\limits_{1\le i\le s} \deg g_i$.

On the other hand, using conveniently [11, Proposition 14] and (8),

$$\begin{aligned} \|h\|_\bullet &\le \|f\|_\bullet + 2\lambda s \max_{1\le i\le s}\{(\deg g_i + 1)(\|g_i\| + 1)\}^{2k+1} \\ &= \|f\|_\bullet + 2c_3 s \frac{(15\|f\|_\bullet d(d+1))^{c_1+1}}{f^{\bullet c_1}} \max_{1\le i\le s}\{(\deg g_i + 1)(\|g_i\| + 1)\}^{2k+1} \\ &\le (2c_3 s + 1) \max_{1\le i\le s}\{(\deg g_i + 1)(\|g_i\| + 1)\} \cdot \\ &\quad \cdot \frac{(15\|f\|_\bullet d(d+1))^{c_1+1}}{f^{\bullet c_1}} \max_{1\le i\le s}\{(\deg g_i + 1)(\|g_i\| + 1)\}^{2c_4\left(\frac{15\|f\|_\bullet d(d+1)}{f^\bullet}\right)^{c_1+1}} \\ &= c_6 \frac{(15\|f\|_\bullet d(d+1))^{c_1+1}}{f^{\bullet c_1}} \mathrm{e}^{c_7\left(\frac{15\|f\|_\bullet d(d+1)}{f^\bullet}\right)^{c_1+1}} \end{aligned} \tag{10}$$

with $c_6 = (2c_3 s + 1)\max_{1\le i\le s}\{(\deg g_i + 1)(\|g_i\| + 1)\}$ and $c_7 = \log\left(\max_{1\le i\le s}\{(\deg g_i + 1)(\|g_i\| + 1)\}^{2c_4}\right)$.

So far we have found λ and k such that that $h \ge \frac{1}{2}f^\bullet$ in $\widetilde{\Delta}_n \times C^2$, together with bounds for $k, \ell = \deg_{\bar{X}} h$ and $\|h\|_\bullet$. Now, we introduce a new variable X_0 to homogenize with respect to the variables $\bar{X}$ and use Pólya's Theorem. Let

$$h = \sum_{\substack{\beta\in\mathbb{N}_0^2\\|\beta|\le 4}}\sum_{0\le j\le \ell} h_{j,\beta}(\bar{X})Y_1^{\beta_1}Y_2^{\beta_2}Z^{4-|\beta|}$$

with $h_{j,\beta} \in \mathbb{R}[\bar{X}]$ equal to zero or homogeneous of degree j for $\beta \in \mathbb{N}_0^2$, $0 \le |\beta| \le 4$ and $0 \le j \le \ell$. We define

$$H = \sum_{\substack{\beta\in\mathbb{N}_0^2\\|\beta|\le 4}}\sum_{0\le j\le \ell} h_{j,\beta}(\bar{X})(X_0 + X_1 \cdots + X_n)^{\ell-j}Y_1^{\beta_1}Y_2^{\beta_2}Z^{4-|\beta|} \in \mathbb{R}[X_0, \bar{X}, Y_1, Y_2, Z]$$

which is bihomogeneous in $(X_0, \bar{X})$ and (Y_1, Y_2, Z) of bidegree $(\ell, 4)$.

Since $H(x_0, \bar{x}, y_1, y_2, z) = h(\bar{x}, y_1, y_2, z)$ for every $(x_0, \bar{x}, y_1, y_2, z) \in \Delta_n \times C^2$, it is clear that $H \ge \frac{1}{2}f^\bullet$ in $\Delta_n \times C^2$.

On the other hand, for each $(y_1, y_2, z) \in C^2$, we consider $H(X_0, X, y_1, y_2, z) \in \mathbb{R}[X_0, \bar{X}]$. Using again [11, Proposition 14] we have

$$\begin{aligned}
\|H(X_0, X, y_1, y_2, z)\| &\le \sum_{\substack{\beta\in\mathbb{N}_0^2\\|\beta|\le 4}}\sum_{0\le j\le \ell} \|h_{j,\beta}(\bar{X})(X_0 + \cdots + X_n)^{\ell-j}y_1^{\beta_1}y_2^{\beta_2}z^{4-|\beta|}\| \\
&\le \sum_{\substack{\beta\in\mathbb{N}_0^2\\|\beta|\le 4}}\sum_{0\le j\le \ell} \|h_{j,\beta}(\bar{X})(X_0 + \cdots + X_n)^{\ell-j}\| \\
&\le \sum_{\substack{\beta\in\mathbb{N}_0^2\\|\beta|\le 4}}\sum_{0\le j\le \ell} \|h_{j,\beta}(\bar{X})\| \\
&\le 15(\ell + 1)\|h\|_\bullet.
\end{aligned}$$

We use now the bound for Pólya's Theorem from [13, Theorem 1]. Take $N \in \mathbb{N}$ given by

$$N = \left\lfloor \frac{15(\ell+1)\ell(\ell-1)\|h\|_\bullet}{f^\bullet} - \ell \right\rfloor + 1.$$

Then for each $(y_1, y_2, z) \in C^2$ we have that $H\big(X_0, \bar{X}, y_1, y_2, z\big)\,(X_0 + X_1 + \cdots + X_n)^N \in \mathbb{R}[X_0, \bar{X}]$ is a homogeneous polynomial such that all its coefficients are positive. More precisely, suppose we write

$$H\big(X_0, \bar{X}, Y, Z\big)\,(X_0 + X_1 + \cdots + X_n)^N = \sum_{\substack{\alpha=(\alpha_0,\bar{\alpha})\in\mathbb{N}_0^{n+1}\\|\alpha|=N+\ell}} b_\alpha(Y_1, Y_2, Z)X_0^{\alpha_0}\bar{X}^{\bar{\alpha}} \in \mathbb{R}[X_0, \bar{X}, Y, Z] \quad (11)$$

with $b_\alpha \in \mathbb{R}[Y_1, Y_2, Z]$ homogeneous of degree 4. The conclusion is that for every $\alpha \in \mathbb{N}_0^{n+1}$ with $|\alpha| = N + \ell$, the polynomial b_α is positive in C^2, and therefore, since it is a homogenous polynomial, b_α is non-negative in $\mathbb{R}^3$.

Before going on, we bound $N + \ell$ using (9) and (10) as follows.

$$
\begin{aligned}
N+\ell &\le \frac{15(\ell+1)\ell(\ell-1)\|h\|_\bullet}{f^\bullet}+1 \\
&\le \frac{15\ell^3\|h\|_\bullet}{f^\bullet}+1 \\
&\le 15c_5^3c_6\left(\frac{15\|f\|_\bullet d(d+1)}{f^\bullet}\right)^{4(c_1+1)} \mathrm{e}^{c_7\left(\frac{15\|f\|_\bullet d(d+1)}{f^\bullet}\right)^{c_1+1}}+1 \\
&\le c_8\left(\frac{15\|f\|_\bullet d(d+1)}{f^\bullet}\right)^{4(c_1+1)} \mathrm{e}^{c_7\left(\frac{15\|f\|_\bullet d(d+1)}{f^\bullet}\right)^{c_1+1}}
\end{aligned}
\tag{12}
$$

with $c_8 = 15c_5^3c_6 + 1$.

Now we substitute $X_0 = 1 - X_1 - \cdots - X_n$ and $Z = 1$ in (11) and we obtain

$$
f = \lambda\big(Y_1^2+Y_2^2+1\big)^2 \sum_{1\le i\le s} g_i(g_i-1)^{2k} + \sum_{\substack{\alpha=(\alpha_0,\bar\alpha)\in\mathbb{N}_0^{n+1}\\ |\alpha|=N+\ell}} b_\alpha(Y_1,Y_2,1)(1-X_1-\cdots-X_n)^{\alpha_0}\bar X^{\bar\alpha} \in \mathbb{R}[\bar X, Y_1, Y_2].
\tag{13}
$$

From (13) we want to conclude that $f \in M_{\mathbb{R}[\bar X,Y_1,Y_2]}(g_1\ldots,g_s)$ and to find the positive constant c such that the degree bound holds.

The first term on the right hand side of (13) clearly belongs to $M_{\mathbb{R}[\bar X,Y_1,Y_2]}(g_1\ldots,g_s)$. Moreover, for $1 \le i \le s$,

$$
\deg\big(Y_1^2+Y_2^2+1\big)^2 g_i(g_i-1)^{2k} = 4 + (2k+1)\deg g_i. \tag{14}
$$

Now we focus on the second term on the right hand side of (13), which is itself a sum. Take a fixed $\alpha \in \mathbb{N}_0^{n+1}$ with $|\alpha| = N + \ell$.

Since $b_\alpha(Y_1,Y_2,1)$ is non-negative in $\mathbb{R}^2$ and $\deg_{(Y_1,Y_2)} b_\alpha(Y_1,Y_2,1) \le 4$, $b_\alpha(Y_1,Y_2,1) \in \sum \mathbb{R}[Y_1,Y_2]^2$. Moreover, we can write $b_\alpha(Y_1,Y_2,1)$ as a sum of squares with the degree of each square bounded by 4.

Also, take $v(\alpha) = (v_0,\bar v) \in \{0,1\}^{n+1}$ such that $\alpha_i \equiv v_i \pmod 2$ for $0 \le i \le n$. Denoting $g_0 = 1 \in \mathbb{R}[\bar X]$, since $S \subset \widetilde\Delta_n^\circ$, by Putinar's classical Positivstellensatz we have representations

$$
(1-X_1-\cdots-X_n)^{v_0}\bar X^{\bar v} = \sum_{0\le i\le s} \sigma_{v(\alpha)i}\, g_i,
$$

with $\sigma_{v(\alpha)i} \in \sum \mathbb{R}[\bar X]^2$ for $0 \le i \le s$, and then

$$
(1-X_1-\cdots-X_n)^{\alpha_0}\bar X^{\bar\alpha} = (1-X_1-\cdots-X_n)^{\alpha_0-v_0}\bar X^{\bar\alpha-\bar v} \sum_{0\le i\le s} \sigma_{v(\alpha)i}\, g_i
$$

belongs to $M(g_1,\ldots,g_s)$ since $(1-X_1-\cdots-X_n)^{\alpha_0-v_0}\bar X^{\bar\alpha-\bar v} \in \mathbb{R}[\bar X]^2$.

We conclude that each term in the sum belongs to $M_{\mathbb{R}[\bar X,Y_1,Y_2]}(g_1,\ldots,g_s)$. In addition, for $0 \le i \le s$ we have

$$
\deg b_\alpha(Y_1,Y_2,1)(1-X_1-\cdots-X_n)^{\alpha_0-v_0}\bar X^{\bar\alpha-\bar v}\sigma_{v(\alpha)i}\, g_i \le 4 + N + \ell + c_9 \tag{15}
$$

with $c_9 = \max\{\deg \sigma_{vi} g_i \mid v \in \{0,1\}^{n+1}, 0 \le i \le s\}$.

To finish the proof, we only need to bound simultaneously the right hand side of (14) and (15).

On the one hand, using (8),

$$
\begin{aligned}
4+(2k+1)\max_{1\le i\le s}\deg g_i &\le 4+\left(2c_4\left(\frac{15\|f\|_\bullet d(d+1)}{f^\bullet}\right)^{c_1+1}+1\right)\max_{1\le i\le s}\deg g_i\\
&\le c_{10}\left(\frac{\|f\|_\bullet d^2}{f^\bullet}\right)^{c_1+1}
\end{aligned}
$$

with $c_{10}=(2c_4+5)30^{c_1+1}\max\limits_{1\le i\le s}\deg g_i$, since $d\ge 1$.

On the other hand, using (12),

$$
\begin{aligned}
4+N+\ell+c_9 &\le 4+c_8\left(\frac{15\|f\|_\bullet d(d+1)}{f^\bullet}\right)^{4(c_1+1)}\mathrm{e}^{c_7\left(\frac{15\|\bar f\|_\bullet d(d+1)}{f^\bullet}\right)^{c_1+1}}+c_9\\
&\le c_{11}\left(\frac{\|f\|_\bullet d^2}{f^\bullet}\right)^{4(c_1+1)}\mathrm{e}^{c_{12}\left(\frac{\|f\|_\bullet d^2}{f^\bullet}\right)^{c_1+1}}
\end{aligned}
$$

with $c_{11}=(4+c_8+c_9)30^{4(c_1+1)}$ and $c_{12}=c_7 30^{c_1+1}$, again since $d\ge 1$.

Finally, we define c as the positive constant obtained applying [5, Lemma 18] to the 6-uple $(c_{10}, c_1+1, c_{11}, 4(c_1+1), c_{12}, c_1+1)$. $\square$

We are ready now to prove Theorem 3. The proof consists basically in a linear change of variables and the application of Proposition 8, as in the proof of [5, Theorem 7].

Proof of Theorem 3: We consider the affine change of variables $\ell:\mathbb{R}^n\to\mathbb{R}^n$ given by

$$
\ell(X_1,\dots,X_n)=\left(\frac{X_1+1}{2n},\dots,\frac{X_n+1}{2n}\right).
$$

For $0\le i\le s$, we take $\tilde g_i(\bar X)=g_i(\ell^{-1}(\bar X))\in\mathbb{R}[\bar X]$ and we define

$$
\widetilde{S}=\{\bar x\in\mathbb{R}^n \mid \tilde g_1(\bar x)\ge 0,\dots,\tilde g_s(\bar x)\ge 0\}.
$$

It is easy to see that

$$
\emptyset\ne\widetilde{S}=\ell(S)\subseteq\widetilde{\Delta}_n^\circ.
$$

Moreover, since $M(g_1,\dots,g_n)$ is archimedean, $M(\tilde g_1,\dots,\tilde g_s)$ is also archimedean (see [5, Proof of Theorem 7]).

Let $f\in\mathbb{R}[\bar X,Y_1,Y_2]$ be as in the statement of Theorem 3 and let $\tilde f(\bar X,Y_1,Y_2)=f(\ell^{-1}(\bar X),Y_1,Y_2)\in\mathbb{R}[\bar X,Y_1,Y_2]$. It can be easily seen that $\tilde f$ is positive on $\tilde S\times\mathbb{R}^2$, $\deg_{\bar X}\tilde f=\deg_{\bar X} f=d$, $\deg_{(Y_1,Y_2)}\tilde f=\deg_{(Y_1,Y_2)} f=4$, $\tilde f$ satisfies condition (†) on $\widetilde{S}$ and

$$
\min\{\bar{\tilde f}(\bar x,y_1,y_2,z)\,|\,\bar x\in\widetilde{S},(y_1,y_2,z)\in C^2\}=\min\{\bar f(\bar x,y_1,y_2,z)\,|\,\bar x\in S,(y_1,y_2,z)\in C^2\}=f^\bullet>0.
$$

In addition, $\|\tilde f\|_\bullet\le\|f\|_\bullet(3n)^d$ (again, see [5, Proof of Theorem 7]).

Take c as the positive constant from Proposition 8 applied to $\tilde g_1,\dots,\tilde g_s$. Therefore, $\tilde f$ can be written as

$$
\tilde f=\tilde\sigma_0+\tilde\sigma_1\tilde g_1+\cdots+\tilde\sigma_s\tilde g_s\in M_{\mathbb{R}[\bar X,Y]}(\tilde g_1,\dots,\tilde g_s)
$$

with $\tilde{\sigma}_0, \tilde{\sigma}_1, \dots, \tilde{\sigma}_s \in \sum \mathbb{R}[\bar{X}, Y_1, Y_2]^2$ and

$$\deg(\tilde{\sigma}_0), \deg(\tilde{\sigma}_1 \tilde{g}_1), \dots, \deg(\tilde{\sigma}_s \tilde{g}_s) \leq c e^{\left(\frac{\|f\|_\bullet d^2 (3n)^d}{f^\bullet}\right)^c}$$

and the final representation for f is simply obtained by composing with ℓ. □

The proof of Theorem 4 is a straightforward adaptation of the proof of Theorem 3 (and Proposition 8), and we omit it.

To prove Theorem 5, we need the following two auxilary lemmas, which are again slight variations of [5, Lemma 16] and [11, Lemma 11].

Lemma 9 *Let $f \in \mathbb{R}[\bar{X}, Y_1, \bar{Y}_2]$ such that $\deg_{\bar{X}} f = d$ $\deg_{Y_1} f = m$, and $\deg_{\bar{Y}_2} f = 2$. For every $\bar{x} \in \widetilde{\Delta}_n$, $(y_1, z_1) \in C$ and $(\bar{y}_2, z_2) \in C^r$,*

$$|\bar{\bar{f}}(\bar{x}, y_1, z_1, \bar{y}_2, z_2)| \leq \frac{1}{2} \|f\|_\bullet (m+1)(r+1)(r+2)(d+1).$$

Lemma 10 *Let $f \in \mathbb{R}[\bar{X}, Y_1, \bar{Y}_2]$ such that $\deg_{\bar{X}} f = d$ $\deg_{Y_1} f = m$, and $\deg_{\bar{Y}_2} f = 2$. For every $\bar{x}_1, \bar{x}_2 \in \widetilde{\Delta}_n$ $(y_1, z_1) \in C$ and $(\bar{y}_2, z_2) \in C^r$,*

$$|\bar{\bar{f}}(\bar{x}_1, y_1, z_1, \bar{y}_2, z_2) - \bar{\bar{f}}(\bar{x}_2, y_1, z_1, \bar{y}_2, z_2)| \leq \frac{1}{4} \sqrt{n} \|f\|_\bullet (m+1)(r+1)(r+2) d(d+1) \|\bar{x}_1 - \bar{x}_2\|.$$

Then, the proof of Theorem 5 is also a straightforward adaptation of the proof of Theorem 3 (and Proposition 8), with the only caveat that the auxiliary polynomial h (at the beginning of the proof of Proposition 8) should be defined as

$$h = \bar{\bar{f}} - \lambda \big(Y_1^2 + Z_1^2\big)^{\frac{m}{2}} \big(Y_{21}^2 + \cdots + Y_{2r}^2 + Z_2^2\big) \sum_{1 \leq i \leq s} g_i \cdot (g_i - 1)^{2k} \in \mathbb{R}[\bar{X}, Y_1, Z_1, \bar{Y}_2, Z_2]$$

and then

$$\|h\|_\bullet \leq \|f\|_\bullet + \lambda s 2^{\frac{m}{2}} \max_{1 \leq i \leq s} \{(\deg g_i + 1)(\|g_i\| + 1)\}^{2k+1}.$$

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