Support bounds for the eliminant polynomial of a polynomial dynamical system

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EXACTAS .UBA

Motivation

We study polynomial dynamical systems with *differential parameters* $\mathbf{u} = (u_1, \ldots, u_m)$, namely systems of the form

$$\Sigma := \begin{cases} \mathbf{x}' = \mathbf{f}(\mathbf{x}, \mathbf{u}) \\ y = g(\mathbf{x}, \mathbf{u}), \end{cases}$$

where $\mathbf{f}(\mathbf{x}, \mathbf{u}) = (f_1, \ldots, f_n) \in K[\mathbf{x}, \mathbf{u}]$, $\mathbf{x} = (x_1, \ldots, x_n)$ are the state variables and y is an observable output.

The goal is to eliminate the state variables $\mathbf{x}$ and obtain a differential input–output equation involving only $\mathbf{u}, \mathbf{u}', \ldots, y, y', \ldots$

The outcome of this elimination process is the *eliminant polynomial*

$$\mathrm{Elim}(\Sigma, \mathbf{x}) \in K[\mathbf{u}, \mathbf{u}', \ldots, y, y', \ldots],$$

which encodes structural information about the system.

This problem is fundamental in:

- systems biology,
- chemical reaction networks,
- control theory,
- parameter identifiability.

The study of support bounds for eliminant polynomials has recently received renewed attention in the non-parametric and constant-parameter settings (see [1]).

Our objective is to obtain explicit and effective bounds for the support of this eliminant polynomial.

Our Approach

We transform our differential elimination problem into an algebraic elimination one. We first notice that:

$$\mathrm{Elim}(\Sigma, \mathbf{x}) \in K[\mathbf{u}^{(\leq \nu)}, y^{(\leq \nu)}] \text{ with } \mathrm{ord}(\mathrm{Elim}(\Sigma, \mathbf{x})) := \nu \leq n.$$

Then, by differentiating ν times the output equation $y = g(\mathbf{x}, \mathbf{u})$ and recursively substituting the equations of Σ, we obtain polynomials

$$g_0, \ldots, g_\nu \in K[\mathbf{x}, \mathbf{u}^{(\leq \nu)}].$$

We then show the image of the map $\varphi = (g_0, \ldots, g_\nu)$ is a hypersurface defined by $\mathrm{Elim}(\Sigma, \mathbf{x})$

Our approach is based on an adaptation of a Perron-type theorem due to D'Andrea et al. (see [2]), which incorporates geometric degree and mixed height estimates. This result provides support bounds for hypersurfaces arising as closures of images of polynomial maps.
Then, by estimating the degrees of the polynomials g_i, we obtain bounds for

$$\mathrm{Supp}(\mathrm{Elim}(\Sigma, \mathbf{x})).$$

i.e. the set of exponent vectors of the monomials appearing in $\mathrm{Elim}(\Sigma, \mathbf{x})$.

An example

For

$$\Sigma : \begin{cases} x_1' = 1 + 3x_1 - 5x_2 + u, \\ x_2' = -4 + 2x_1 + 7x_2 - 2u, \\ y = 5 + 11x_1 + 3x_2 + 3u + 4x_1u + 6x_2u + 11x_1^2 + 5x_2^2 + 27u^2, \end{cases}$$

The eliminant has 675 monomials in its support, while our bounds predict 697, yielding 96% accuracy. (see the table on the right for further examples).

Main Theorem

Let Σ be defined as before. For $\alpha \in \{\mathbf{x}, \mathbf{u}, \{\mathbf{x}, \mathbf{u}\}\}$, set

$$D_\alpha := \max_{1 \leq i \leq n} \deg_\alpha(f_i), \quad d_\alpha := \deg_\alpha(g), \text{ where } d_{\mathbf{x}}, D_{\mathbf{x}} > 0.$$

Then, every monomial $\left(\prod_{k=0}^{\nu} \prod_{j=1}^{m} (u_j^{(k)})^{\ell_{j,k}}\right) \prod_{i=0}^{\nu} (y^{(i)})^{e_i}$ appearing in $\mathrm{Elim}(\Sigma, \mathbf{x})$ satisfies:

1. $\displaystyle\sum_{k=0}^{\nu} \sum_{j=1}^{m} \ell_{j,k} + \sum_{i=0}^{\nu} \big(d_{\mathbf{x},\mathbf{u}} + i(D_{\mathbf{x},\mathbf{u}} - 1)\big) e_i \leq \prod_{i=0}^{\nu} \big(d_{\mathbf{x},\mathbf{u}} + i(D_{\mathbf{x},\mathbf{u}} - 1)\big),$
2. $\displaystyle\sum_{i=0}^{\nu} \big(d_{\mathbf{x}} + i(D_{\mathbf{x}} - 1)\big) e_i \leq \prod_{i=0}^{\nu} \big(d_{\mathbf{x}} + i(D_{\mathbf{x}} - 1)\big),$
3. $\displaystyle\sum_{k=0}^{\nu} \sum_{j=1}^{m} \ell_{j,k} + \sum_{i=0}^{\nu} (d_{\mathbf{u}} + iD_{\mathbf{u}}) e_i \leq \left(\prod_{i=0}^{\nu} \big(d_{\mathbf{x}} + i(D_{\mathbf{x}} - 1)\big)\right) \sum_{j=0}^{\nu} \frac{d_{\mathbf{u}} + jD_{\mathbf{u}}}{d_{\mathbf{x}} + j(D_{\mathbf{x}} - 1)}$
4. For $1 \leq r \leq \nu$:

$$\sum_{j=1}^{m} \ell_{j,r} + \sum_{i=1}^{\nu} \left\lfloor \frac{i}{r} \right\rfloor e_i \leq \left(\prod_{i=0}^{\nu} \big(d_{\mathbf{x}} + i(D_{\mathbf{x}} - 1)\big)\right) \sum_{j=r}^{\nu} \frac{\lfloor j/r \rfloor}{d_{\mathbf{x}} + j(D_{\mathbf{x}} - 1)}.$$

Advantages and New Features

- **Natural inclusion of differential parameters**

  Our framework allows the parameters $\mathbf{u}, \mathbf{u}', \mathbf{u}'', \ldots$ to behave as differential variables, while previous support bounds for eliminant polynomials mainly treated systems with $\mathbf{u}' = 0$ (constant parameters) or non-parametric systems. This extends the known support bounds to the differential-parameter setting (see inequalities 1-2-3) and produces entirely new inequalities controlling the contribution of derivatives of fixed order $\mathbf{u}^{(r)}$ (see inequalities of type 4).

- **Restriction to the constant-parameter setting**

  When restricted to constant parameters, our main theorem recovers all of the known bounds from the literature.

  Moreover, the resulting inequalities are slightly sharper than those in [1] and provide a natural candidate for the Newton polytope of the eliminant polynomial for a sufficiently generic system in the constant parameters setting.

- **Advantages of the geometric approach**

  After reducing the differential elimination problem to an algebraic elimination problem, the Perron-type framework provides an effective way to estimate the support of the eliminant polynomial. This geometric approach makes the extension to differential parameters comparatively direct and, as discussed before, naturally leads to new support bounds.

  Moreover, the method is flexible enough to incorporate additional structural information — for instance special forms of the output — while preserving an effective and relatively simple computational framework.

Computational Experiments

$[D_{\mathbf{x}}, D_{\mathbf{u}}, D_{\mathbf{x},\mathbf{u}}]$	$[d_{\mathbf{x}}, d_{\mathbf{u}}, d_{\mathbf{x},\mathbf{u}}]$	Main Theo	Monomials in $\mathrm{Elim}(\Sigma, \mathbf{x})$	%
$[1, 1, 1]$	$[1, 1, 1]$	7	7	100
$[1, 1, 1]$	$[2, 1, 2]$	697	655	94
$[1, 1, 1]$	$[2, 2, 2]$	697	675	98
$[2, 2, 2]$	$[1, 1, 1]$	292	115	39
$[3, 3, 3]$	$[1, 1, 1]$	5722	1144	20

KEY REFERENCES

[1] Y. Mukhina and G. Pogudin, *Support Bound for Differential Elimination in Polynomial Dynamical Systems*, in *Computer Algebra in Scientific Computing*, Springer Nature Switzerland, 2026, pp. 265–285.
[2] C. D'Andrea, T. Krick and M. Sombra, *Heights of varieties in multiprojective spaces and arithmetic Nullstellensätze*, Ann. Sci. ENS, Vol. 46, No. 4, 2013, pp. 549–627.

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