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New Proofs of Euclid's and Euler's Theorems

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In this note we give a new proof of the existence of infinitely many prime numbers. There are several different proofs with many variants, and some of them can be found in [**1, 3, 4, 5, 6**]. This proof is based on a simple counting argument using the inclusion-exclusion principle combined with an explicit formula. A different proof based on counting arguments is due to Thue (1897) and can be found in [**6**] together with several generalizations, and a remarkable variant of it was given by Chaitin [**2**] using algorithmic information theory. Moreover, we prove that the series of reciprocals of the primes diverges. Our proofs arise from a connection between the inclusion-exclusion principle and the infinite product of Euler.

Let $\{p_i\}_i$ be the sequence of prime numbers, and let us define the following recurrence:

$$a_0 = 0, \qquad a_{k+1} = a_k + \frac{1 - a_k}{p_{k+1}}.$$

Let us note that the Nth term a_N generated by this recurrence coincides with

$$a_N = \sum_i \frac{1}{p_i} - \sum_{i<j} \frac{1}{p_i p_j} + \sum_{i<j<k} \frac{1}{p_i p_j p_k} - \cdots + (-1)^{N+1} \frac{1}{p_1 \ldots p_N},$$

and can be given in a closed form as

$$a_N = 1 - \prod_{i=1}^{N} \left(1 - \frac{1}{p_i}\right),$$

which implies that $0 < a_N < 1$, since each factor is strictly positive and less than one.

Now, we are ready to prove the classical Euclid's theorem:

Theorem 1. *There are infinitely many prime numbers.*

Proof. Let us suppose that $p_1 < p_2 < \cdots < p_N$ are all the primes. For any $x \geq 1$, and for $i = 1, \ldots, N$, let A_i be the set of integers in $[1, x]$ that are divisible by p_i. Then, the number of positive integers in $[1, x]$ is obtained by applying the inclusion-exclusion formula to find the cardinality of $\cup_{i=1}^{N} A_i$:

$$[x] = 1 + \sum_i \left[\frac{x}{p_i}\right] - \sum_{i<j} \left[\frac{x}{p_i p_j}\right] + \sum_{i<j<k} \left[\frac{x}{p_i p_j p_k}\right] - \cdots + (-1)^{N+1} \left[\frac{x}{p_1 \ldots p_N}\right],$$

where $[s]$ denotes the integral part of s as usual. Since

$$\lim_{x\to\infty} x^{-1}\left[\frac{x}{t}\right] = \frac{1}{t},$$

we reach a contradiction,

$$1 > a_N = \sum_i \frac{1}{p_i} - \sum_{i<j} \frac{1}{p_i p_j} + \sum_{i<j<k} \frac{1}{p_i p_j p_k} - \cdots + (-1)^{N+1} \frac{1}{p_1 \ldots p_N} = 1,$$

and the proof is finished. ■

Let us observe from the previous proof that the asymptotic density $D(p_1, \ldots, p_N)$ of the set of integers divisible by none of $p_1, \ldots, p_N$ is exactly

$$D(p_1, \ldots, p_N) = 1 - \sum_i \frac{1}{p_i} + \sum_{i<j} \frac{1}{p_i p_j} - \sum_{i<j<k} \frac{1}{p_i p_j p_k} + \cdots (-1)^N \frac{1}{p_1 \ldots p_N},$$

that is,

$$1 - a_N = D(p_1, \ldots, p_N) = \prod_{j=1}^{N} \left(1 - \frac{1}{p_j}\right),$$

and let us define $D = \lim_{N\to\infty} D(p_1, \ldots, p_N)$. Then, by taking logarithms, we obtain that

$$\sum_p \ln\left(1 - \frac{1}{p}\right)$$

converges if $D > 0$ and diverges if $D = 0$. Since $\sum_p \frac{1}{p}$ converges if and only if $\sum_p \ln\left(1 - \frac{1}{p}\right)$ does, it is enough to show that $D = 0$ to obtain:

Theorem 2. *The series* $\sum_p \frac{1}{p}$ *diverges.*

Proof. Let us show that $D > 0$ and the convergence of $\sum_p \frac{1}{p}$ cannot hold simultaneously. To this end, let us take $0 < \varepsilon < D$, and choose N big enough so that

$$\varepsilon < D(p_1, \ldots, p_N) \quad \text{and} \quad \sum_{p>p_N} \frac{1}{p} < \varepsilon.$$

Now the asymptotic density of the integers which are not divisible by any of the primes $p_1, \ldots, p_N$ is bounded below by ε. However, those integers must be divisible by some prime $p > p_N$, so their density is bounded above by

$$\sum_{p>p_N} \frac{1}{p} < \varepsilon,$$

a contradiction. Hence $D = 0$ and the series diverges. ■

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A Note on Covering a Square of Side Length $2 + \epsilon$ with Unit Squares

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In [**1**] Soifer posed the following problem: "Find the smallest number $\Pi(n)$ of unit squares that can cover a square of side length $n + \epsilon$ [for some $\epsilon > 0$]." For small values of n the estimates presented in [**1**] are: $5 \leq \Pi(2) \leq 7$ and $10 \leq \Pi(3) \leq 14$. The aim of this paper is to improve the lower bounds. We show that $\Pi(2) \geq 6$ and $\Pi(3) \geq 11$, i.e., we show that it is impossible to cover a square of side length greater than 2 with five unit squares and it is impossible to cover a square of side length greater than 3 with ten unit squares.

Lemma. *Let S be a square of side length 1, let $\delta > 0$, and let l_1 and l_2 be straight lines parallel to each other with distance $1 + \delta$. Moreover, let both l_1 and l_2 have a nonempty intersection with S. Denote by s_i the length of $l_i \cap S$ for $i \in \{1, 2\}$. Then $s_1 + s_2 < 1$.*

Proof. S has a nonempty intersection with l_1 and l_2. Therefore no side of S is parallel to l_1. We can assume that l_1 is on the left side of l_2 as in Figure 1. Denote by a the vertex of S that lies on the right side of l_2 (obviously, there is only one such vertex). Let p_1 denote the distance between l_2 and a and let w_1 denote the length of the longest segment which is contained in S and is parallel to l_2 and whose distance to a equals $1 + \delta$. It is easy to see that $w_1 = s_1 + s_2$ (see Figure 1). Let w denote the length of the longest segment which is contained in S and is parallel to l_2 and whose distance from a equals 1. From $\delta > 0$ we deduce that $w_1 < w$. To find w observe that $s_3 = \tan\frac{\alpha}{2}$ and $1 - s_3 = w\cos\alpha$ in Figure 1. Consequently,

$$w = \left(1 - \tan\frac{\alpha}{2}\right)\frac{1}{\cos\alpha} = \left(1 - \tan\frac{\alpha}{2}\right)\frac{1 + \tan^2\frac{\alpha}{2}}{1 - \tan^2\frac{\alpha}{2}} = \frac{1 + \tan^2\frac{\alpha}{2}}{1 + \tan\frac{\alpha}{2}}.$$

This value is smaller than 1, because $\tan\frac{\alpha}{2} < 1$ for $0^\circ < \alpha < 90^\circ$. Thus $s_1 + s_2 = w_1 < w < 1$. ■