Chapter 1
Introduction: A Metamathematical View of Differential Geometry

"Geometry is the ruler of all mental investigation".
M.V. Lomonosov

§ 1. Algebra and Geometry – the Duality of the Intellect

It is known from physiology that in the process of thinking the hemispheres of the human brain fulfil different functions. The left one is the site of the "rational" mind. In other words, this part of the brain carries out formal deductions, reasons logically, and so on. On the other hand, imagination, intuition, emotions and other components of the "irrational" mind are the product of the right hemisphere. This division of labour can have the following explanation.

The process of solving some problem or other by a human being or an artificial mechanism involves the need to draw correct conclusions from correct premises. The logical computations that carry out these functions in various specific circumstances can easily be formulated algorithmically and thereby carried out on modern computers. There are good reasons for supposing that the human brain acts in a similar way and the left hemisphere is its "logical block". However, the ability to argue logically is only half the problem, and apparently the simpler half. In fact, to solve any complicated problem it is necessary to construct a rather long chain, consisting of logically correct elements of the type "premise-conclusion". However, from given premises it is possible to draw very many correct conclusions. Therefore it is practically impossible to find the solution of a complicated problem by randomly building up logically correct chains of the form mentioned, in view of the large number of variants that arise. Thus the problem we are posing is: in which direction should we reason? We can solve it only if there are various mechanisms of selection and motivation, that is, mechanisms that induce the thinking apparatus to consider only expedient versions. Man solves this problem by using intuition and imagination. Thus, we can think that the process of evolution of nature has led to the two most important aspects of any thought process – the formally logical and the motivational – being provided by the two functional blocks of the brain – its left and right hemispheres, respectively.

Mathematics is the science that deals with pure thought. Therefore the two main aspects of mental activity mentioned above must be revealed in its structure. In fact, this is familiar to everyone from the school division of mathematics into algebra and geometry. Apparently we can regard it as established experimentally that an algebraist or analyst is a mathematician with a pronounced dominant left hemisphere, while for a geometer it is the right hemisphere. Thus, if we consider the body of mathematicians linked by various lines of communication, having a common memory, in which there are mechanisms of stimulation and repression, etc., as a thinking system, then geometry is the product of its right hemisphere.

It is clear a priori that successful functioning of the intellect (natural or artificial) can be ensured by balanced interaction of its right and left hemispheres, acting on different levels. Hypertrophy of the function of the left hemisphere leads to a phenomenon that can appropriately be called thought bureaucracy (formalism, scholasticism, and so on). On the other hand, hypertrophy of the right hemisphere leads to unsound fantasies and wandering in the clouds. For example, F. Klein (Klein [1926]) writes: "We state here as a principle that we will always combine the analytical and geometrical treatment of problems, and we shall not, like many mathematicians, take a one-sided point of view. An analytical treatment does not give a visual idea of the results obtained, while a geometrical examination can only give an approximate basis for proof . . . ".

We liken the development of thought in the solution of a problem to the process of propagation of an electromagnetic wave, as a consequence of the inductive connection of an electric field (E) and a magnetic field (H) (Fig. 1). We venture to compare the left hemisphere (algebra) with E, and the right hemisphere (geometry) with H, since a magnetic field does not have sources. Short-wave oscillations of algebra-geometry type inevitably arise when one or several investigators are working on a specific problem. Long waves of this kind, which often interfere with one another in an odd way, are the waves of mathematical history.

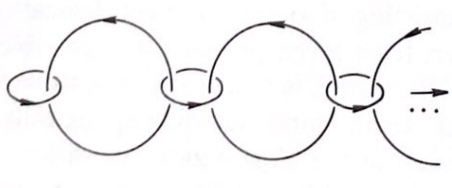

Fig. 1

If 20 years ago the formal algebraic spirit, personified by Nicolas Bourbaki, dominated, nowadays the geometrical spirit is on the crest of the wave, with the typical growth of interest shown by mathematicians in problems of physics and nature in general. The chain

$\boxed{\text{Analysis}}$ $\rightarrow$	$\boxed{\text{Differential topology}}$ $\rightarrow$	$\boxed{\text{Homological algebra}}$
$\downarrow$ initial "algebra"	"geometry" directing it	secondary "algebra" regulating this "geometry"

is an example of a large-scale wave in modern mathematics.

§2. Two Examples: Algebraic Geometry, Propositional Logic and Set Theory

2.1. In ancient manuscripts of geometrical content, instead of a proof of a theorem there is often displayed a convenient diagram together with the instruction "look". Thereby it is implicitly assumed that the contemplation of the diagram is capable of arranging the thoughts of the observer into a conclusive chain of deductions. Modern algebraic geometry is an approach to the solution of problems of commutative algebra by way of an intelligent and systematic construction of the necessary visual images. The source of this approach is an idea that goes back to Descartes: it is possible to obtain a visual representation of the set of solutions of a system of polynomial equations, interpreting it by introducing coordinates as an "algebraic subvariety" of affine space. In modern algebraic geometry the main way of visualizing is to interpret an arbitrary commutative algebra as the algebra of functions on some set. Its description in general outline reduces to the following (the definitions of the simplest concepts of commuatative algebra used below can be found, for example, in Volume 11 of the present series).

Let A be a commuatative algebra with unity over a field k. Consider the set Spec A of all prime ideals of this algebra. An element $a \in A$ can be thought of as a "function" on this set, whose value at a point $\mathfrak{p} \in \operatorname{Spec} A$ is equal to the image of a under the natural homomorphism $A \to A/\mathfrak{p}$. As a result we obtain the following picture: to each point $\mathfrak{p}$ of Spec A we "attach" an algebra $A/\mathfrak{p}$ without divisors of zero (since $\mathfrak{p}$ is a prime ideal) and the "function" mentioned above, associated with the element $a \in A$, is the map $\mathfrak{p} \mapsto [a] \in A/\mathfrak{p}$. The algebra $A/\mathfrak{p}$, generally speaking, depends on $\mathfrak{p}$. If this dependence did not exist, that is, all the algebras $A/\mathfrak{p}$ were the same, then the construction we have described would lead us to the usual $A/\mathfrak{p}$-valued functions on Spec A.

Example. Let $k = \mathbb{C}$ (the field of complex numbers) and let $A = \mathbb{C}[x]$ be the algebra of polynomials with complex coefficients in the variable x. Any non-zero prime ideal $\mathfrak{p} \subset A$ consists of polynomials divisible by $x - c$ (the number $c \in \mathbb{C}$ is fixed). Dividing a given polynomial $p(x) \in A$ by $x - c$ and taking the remainder, we obtain a unique representation in the form $p(x) = p_0 + (x - c)h(x)$, where $p_0 \in \mathbb{C}$. Then $A/\mathfrak{p} = \mathbb{C}$ if $\mathfrak{p} \neq \{0\}$, and we can represent the operation of factoriza-