Some new results on circle graphs

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Abstract: The intersection graph of a set of chords on a circle is called a circle graph. This class of graphs admits some interesting subclasses, such as: Helly circle graphs, clique-Helly circle graphs, unit circle graphs and proper circular-arc graphs. In this paper, we prove some inclusion relations among these subclasses. A necessary condition for a graph being a Helly circle graph is shown and we conjecture that this condition is sufficient too. All possible intersections among these subclasses are analized. The number of regions generated is 10. We show a minimal example belonging to each one of them. Finally, some properties about minimal forbidden subgraphs for circle graphs are proved.

Keywords: circle graphs, clique-Helly circle graphs, Helly circle graphs, proper circular-arc graphs, unit circle graphs.

1- Introduction

Consider a finite family of non-empty sets. The *intersection graph* of this family is obtained by representing each set by a vertex, two vertices being connected by an edge if and only if the corresponding sets intersect. Intersection graphs have received much attention in the study of algorithmic graph theory and their applications [7]. Well known special classes of intersection graphs include interval graphs, chordal graphs, permutation graphs, circle graphs, circular-arc graphs, and so on.

A graph G is called a *circle graph* (CG) if there exists a set of chords L (a model) on a circle and a one-to-one correspondence between vertices of G and chords of L such that two distinct vertices are adjacent if and only if their corresponding chords intersect in the interior of the circle. That is, a circle graph is the intersection graph of a set of chords on a circle. Figure 1 shows a circle graph G and a model L for it.

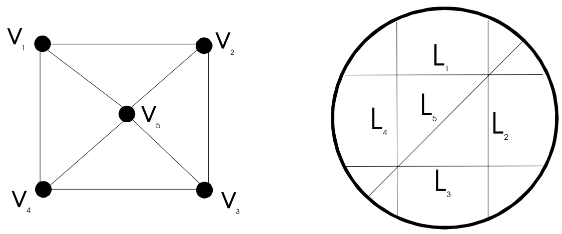

Figure 1

Circle graphs were introduced in [5], where an application to solve a problem about queues and stacks proposed by Knuth [10] is shown. Polynomial time algorithms for recognizing graphs in this class appear in [1,6,11,12].

A family of subsets S satisfies the *Helly property* when every subfamily of it consisting of pairwise intersecting subsets has a common element. A graph is *clique-Helly* when its cliques (maximal complete subgraphs) satisfy the Helly property. A *circular-arc graph* is the

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intersection graph of a family of arcs on a circle (we assume that all the arcs are open). A graph is *diamond-free* if it does not contain a diamond (Figure 2) as an induced subgraph

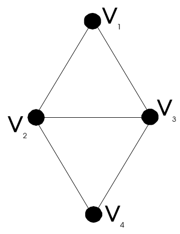

Figure 2

A *local complementation* of a graph G at vertex v consists in replacing $G_{N[v]}$ by its complement graph (where $G_{N[v]}$ is the induced subgraph by the neighbours of v). The graphs G_1 and G_2 are *locally equivalent* if there is a finite sequence of local complementations transforming G_1 into G_2.

Let $\overline{G}$ be the complement of a graph G and G^*, a graph G adding it an isolated vertex. Let C_j be an induced cycle with j vertices.

The proposal of this paper is to analyse different subclasses of circle graphs and to find some properties about minimal forbidden subgraphs for this class.

In Section 2, some inclusions among the subclasses of circle graphs that we define are proved. A necessary condition for a graph being a Helly circle graph is also shown and we conjecture that this condition is sufficient too.

In Section 3, all possible intersections among these subclasses of circle graphs are analysed, as it was done in [4] for circular-arc graphs and their subclasses: clique-Helly circular-arc graphs, Helly circular-arc graphs, proper circular-arc graphs and unit circular-arc graphs. The number of regions generated is 10. We show a minimal example belonging to each one of them. The minimality of the examples implies that any proper induced subgraph of them belongs to some other region.

In Section 4, a characterization of proper circular-arc graphs by forbidden subgraphs due to Tucker [16] is used to show some properties about minimal forbidden subgraphs for circle graphs.

2- Subclasses of circle graphs

We may define some interesting subclasses of circle graphs:

(1) *Helly circle graphs*: a graph G is a Helly circle (HC) graph if there is a model L for G such that the set of chords of L satisfies the Helly property. It is not known a characterization of this subclass.

(2) *Clique-Helly circle graphs*: a graph G is a clique-Helly circle (CHC) graph if G is a circle graph and a clique-Helly graph. Szwarcfiter [14] described a characterization of clique-Helly graphs leading to a polynomial time algorithm for recognizing them. This algorithm together with an efficient method for circle graphs results in a polynomial time algorithm for recognizing CHC graphs.

(3) *Unit circle graphs*: a graph G is a unit circle (UC) graph if there is a model L for G such that all the chords are of the same length. We will see that this subclass is equivalent to unit circular-arc (UCA) graphs, circular-arc graphs which have a representation where all the arcs are of the same length. These graphs can be also recognized in polynomial time [13].

(4) *Proper circular-arc graphs*: a graph G is a proper circular-arc (PCA) graph if there is a circular-arc representation of G such that no arc is properly contained in any other. It can be easily proved that proper circular-arc graphs are a subclass of circle graphs. The representation in arcs can be trivially transformed in the model in chords, using that no pair of arcs together covers the entire circle (a property of proper circular-arc graphs shown in [7]). Tucker [15] described a characterization and an efficient algorithm for recognizing PCA graphs.

First, we prove a necessary condition so that a graph is a Helly circle graph.

Theorem 1: Let G be a Helly circle graph. Then, G is a diamond free circle graph.

Proof: Let G be a Helly circle graph and suppose that G contains a diamond D (Figure 2) as an induced subgraph. Given V(D) = $\{v_1,v_2,v_3,v_4\}$ and let L_1, L_2, L_3 and L_4 be the chords corresponding to the vertices v_1, v_2, v_3 and v_4, respectively. As G is a circle graph and is Helly, L_1, L_2 and L_3 must have a common point P because v_1, v_2 and v_3 define a clique of D. Analogously, as G is a circle graph and is Helly, L_2, L_3 and L_4 must have a common point Q because v_2, v_3 and v_4 define a clique of D. If $P \neq Q$, then $L_2 = L_3$, a contradiction. If P = Q, then D is the clique with 4 vertices, a contradiction.□

We conjecture that the diamond is the only obstruction to a circle graph be a Helly circle graph.

Note: If this conjecture were true, we would have a polynomial recognition of the HC subclass, because we can check in polynomial time whether a graph has a diamond as an induced subgraph.

Now, we prove that HC $\subseteq$ CHC.

Theorem 2: Let G be a Helly circle graph. Then, G is a clique-Helly circle graph.

Proof: It was proved in [3] that diamond-free graphs are a subclass of clique-Helly graphs. And by Theorem 1, Helly circle graphs are included in diamond-free circle graphs. So, this theorem holds.□

It is interesting to remark that the analogous result for circular-arc graphs is not true. Helly circular-arc graphs are not necessarily clique-Helly [4].

Finally, it can be easily proved that unit circle graphs are equivalent to unit circular-arc graphs.

Theorem 3: A graph G is a unit circle graph if and only if G is a unit circular-arc graph.

Proof: Let us see the equivalence. The model in chords of an UC graph G can be transformed into a circular-arc representation of G in arcs of equal length using the same endpoints and joining them such that the arc has length at most π (assuming the radius r = 1). The converse is also true. Let G be an UCA graph. It is easy to see that we have a representation in unit arcs of G without common endpoints. We may suppose, without loss of generality, that each arc has length at most π, assuming again the radius r = 1 (otherwise, the graph is complete and, in consequence, it is a unit circle graph). Then, if we join the endpoints of each arc we have a circle model of G in chords of the same length.□

A trivial consequence of Theorem 3 is the following inclusion.

Corollary 1: Let G be a unit circle graph. Then, G is a proper circular-arc graph.

3- Minimal examples

Figure 5 shows examples of minimal graphs belonging to the possible intersections defined by the inclusions among these classes. The examples can be checked with no difficulty by the reader. We present here only one proof of a minimal member belonging to the respective region.

First, we need some characterization theorems.

In order to characterize the PCA \ UCA region (which is equivalent to the PCA \ UC region), we need a definition due to Tucker [16]. Let j and k be two positive integers with j > k and let CI(j,k) be a circular-arc graph whose representation in circular arcs is built in the following way: let ε be a small positive number, $\varepsilon < k\pi/j$, and r=1 the radius of the circle. Draw j arcs $(A_0,A_1,...,A_{j-1})$ of length $l_1=2\pi k/j + \varepsilon$ such that each arc A_i begins in $2\pi i/j$ and finishes in $2\pi(i+k)/j + \varepsilon$ $(A_i = (2\pi i/j, 2\pi(i+k)/j + \varepsilon))$. Then, draw j new arcs $(B_0,B_1,...,B_{j-1})$ of length $l_2=2\pi k/j - \varepsilon$, such that each arc B_i begins in $(2\pi i + \pi k)/j$ and finishes in $(2\pi(i+k) + \pi k)/j - \varepsilon$ $(B_i = ((2\pi i + \pi k)/j, (2\pi(i+k) + \pi k)/j - \varepsilon))$. For example, the representation of Figure 3 generates CI(4,1) (Figure 4).

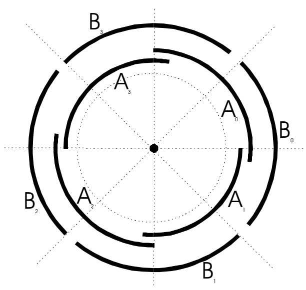

Figure 3

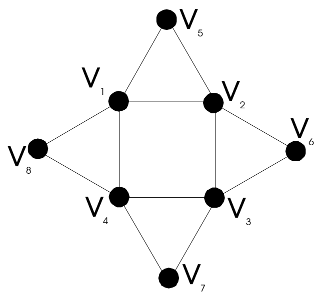

Figure 4

By construction, these graphs are proper circular-arc graphs [16].

Theorem 4 [16]: Let G be a proper circular-arc graph. Then G is a unit circular-arc graph if and only if G contains no CI(j,k) as an induced subgraph, where j and k are relatively prime and j > 2k.

This characterization of UCA graphs leads us to a polynomial time algorithm for its recognition [13].

A characterization of clique-Helly graphs is presented in [14]. We must define the concept of an extended triangle. Let G be a graph and T a triangle of G. The extended triangle of G, relative to T, is the subgraph of G induced by all the vertices which form a triangle with at least one edge of T. A vertex v is universal in a subgraph of G if v is adjacent to every other vertex of the subgraph.

Theorem 5 [14]: A graph G is a clique-Helly graph if and only if every of its extended triangles contains a universal vertex.

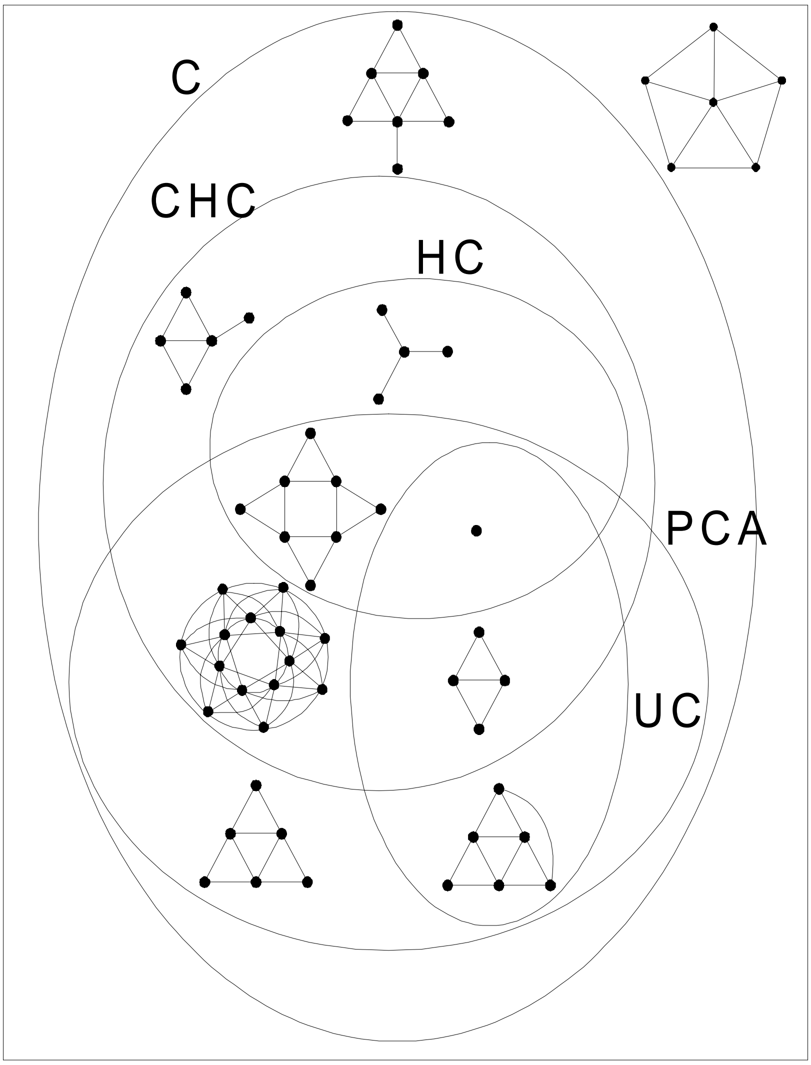

Figure 5

Now, we can prove the following result.

Proposition 1: Graph H (Figure 6) is a proper circular-arc graph and a clique-Helly circle graph but it is neither a unit circle graph nor a Helly circle graph.

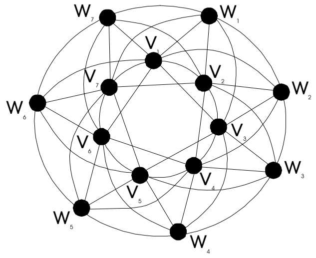

Figure 6

Proof:

(1) H is a proper circular arc graph but it is not a unit circle graph: by the definition of the graphs CI(j,k), H is isomorphic to CI(7,2) and this graph belongs to the PCA \ UCA region.
(2) H is a clique-Helly circle graph: by inspection, we can verify that every extended triangle of H contains a universal vertex. So, by Theorem 5, H is clique-Helly.
(3) H is not a Helly circle graph: the subgraph induced by v_1, v_2, v_3 and v_4 is a diamond.

In order to verify the minimality we prove that any proper induced subgraph H' of H is UCA. With this purpose, we show that H' does not contain CI(j,k) as an induced subgraph, where j and k are relatively prime and $j > 2k$. It is enough to prove this fact for CI(3,1), CI(4,1), CI(5,1), CI(6,1) and CI(5,2) because they are the graphs of this family with at most thirteen vertices.

Since H has a maximum independent set of size three, so H' cannot contain CI(4,1), CI(5,1) and CI(6,1), which have maximum independent set of size 4, 5 and 6, respectively. Now, suppose that H' contains CI(3,1) as an induced subgraph. Observe that there is no stable set of size three induced by the vertices v_i's (since the subgraph induced by all v_i's is the complement of C_7). So by symmetry, we may assume that every stable set contains w_3 and we have to analyse only two cases: the stable set of size three in CI(3,1) is formed either by $\{v_1,w_3,w_5\}$ or by $\{w_1,w_3,w_5\}$.

In the first case (Figure 7), the vertices which are adjacent to v_1 and w_5 simultaneously and non adjacent to w_3, are v_6, v_7 and w_6; adjacent to v_1 and w_3 and not to w_5, are v_2, v_3 and w_2; and adjacent to w_3 and w_5 and not to v_1, are v_4, v_5 and w_4. But we can not choose three of these vertices such that they induce a triangle. Then, H' cannot contain CI(3,1) as an induced subgraph.

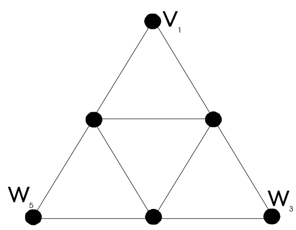

Figure 7

In the second case (Figure 8), the vertex adjacent to w_1 and w_5 and non adjacent to w_3 must be v_7; the vertex adjacent to w_1, w_3, v_7 and non adjacent to w_5 must be v_2. But now, we don't have a vertex adjacent to v_2, v_7, w_3 and w_5, and non adjacent to w_1. So, H' cannot contain CI(3,1) as an induced subgraph.

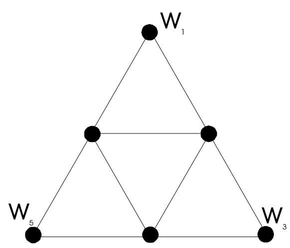

Figure 8

Finally, suppose that H' contains CI(5,2) as an induced subgraph. H' must contain the following structure included in CI(5,2) (for more detail about this graph, see [4]):

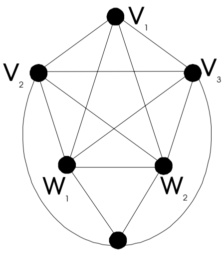

Figure 9

By the simmetry of CI(7,2), we may assume, without loss of generality, that one of the K_5 is induced by v_1,v_2,v_3,w_1 and w_2. But now, there is not any vertex in CI(7,2) adjacent to exactly four of these vertices. So, H' does not contain CI(5,2) as an induced subgraph.□

4- Minimal forbidden subgraphs

It is not known a characterization of circle graphs by forbidden subgraphs. An interesting approach in that sense is a characterization of this class proved by Bouchet [2].

Theorem 6 [2]: G is a circle graph if and only if no graph locally equivalent to G contains an induced subgraph isomorphic to W_5 or W_7 or BW_3 (Figure 10).

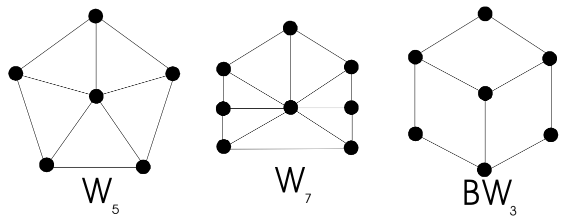

Figure 10

A trivial corollary of this theorem and the transitivity of the local equivalence is the following.

Corollary 2: Let H be a (not) circle graph. If a graph G is locally equivalent to H, then G is (not) a circle graph.

A characterization of proper circular-arc graphs by forbidden subgraphs due to Tucker [16] is used here to show some properties about minimal forbidden subgraphs for circle graphs. First, we need the definition of H_1, H_2, H_3, H_4 and H_5, the graphs of Figure 11.

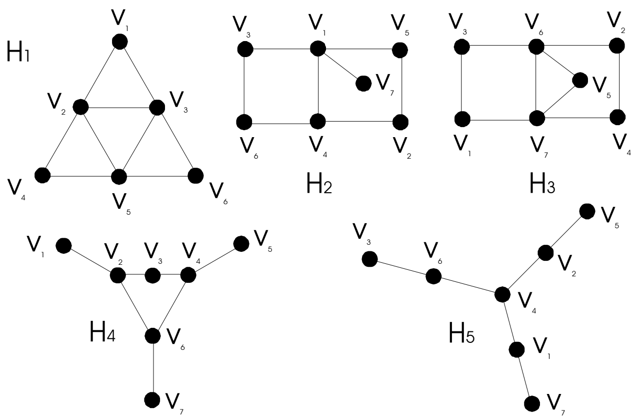

Figure 11

Theorem 7 [16]: A graph G is a proper circular-arc graph if and only if it contains as induced subgraphs none of these graphs: $\overline{H_1}$, $\overline{H_2}$, $\overline{H_3}$, $\overline{H_4}$, $\overline{H_5}$, H_1^*, $\{C_n^*\}_{n\geq4}$, $\{\overline{C_{2j+1}^*}\}_{j\geq1}$ and $\{\overline{C_{2j}}\}_{j\geq3}$.

We have to analyse which of the forbidden subgraphs of Theorem 7 are circle graphs and which are not.

Lemma 1: $\overline{H_1}$, $\overline{H_3}$, H_1^*, $\overline{C_3^*}$ and $\{C_n^*\}_{n\geq4}$ are circle graphs.

Proof: It is easy to find circle models for $\overline{H_1}$, H_1^*, $\overline{C_3^*}$ and $\{C_n^*\}_{n\geq4}$. Let us see the circle model for $\overline{H_3}$ (Figure 12)

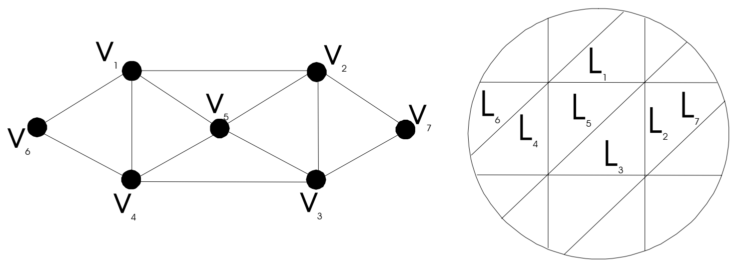

Figure 12

Lemma 2: $\overline{H_2}$, $\overline{H_4}$, $\overline{H_5}$, $\{\overline{C_{2j+1}^*}\}_{j\geq2}$ and $\{\overline{C_{2j}}\}_{j\geq3}$ are not circle graphs.

Proof:

- $\overline{H_2}$ (Figure 13) is not a circle graph. If we apply the operation of local complementation first at vertex v_4 and then at vertex v_1, we obtain a graph isomorphic to BW_3.

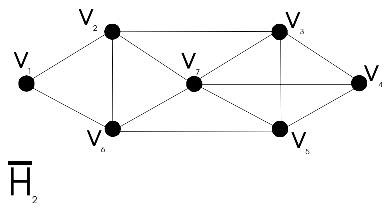

Figure 13

- $\overline{H_4}$ (Figure 14) is not a circle graph. If we apply the operation of local complementation first at vertex v_6 and then at vertex v_3, the subgraph induced by the vertices v_1, v_2, v_4, v_5, v_6 and v_7 is isomorphic to W_5.

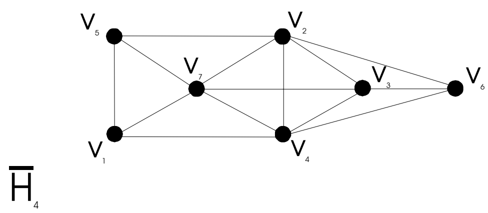

Figure 14

- $\overline{H_5}$ (Figure 15) is not a circle graph. If we apply the operation of local complementation successively at vertices v_5, v_3 and v_7, we obtain a graph isomorphic to BW_3 again.

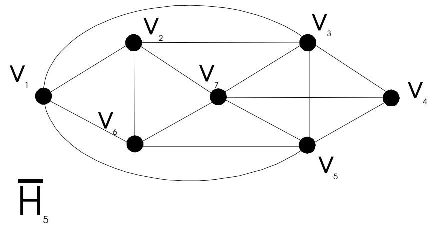

Figure 15

- Graphs $\{\overline{C_{2j+1}^{*}}\}_{j\geq 2}$ are not circle graphs. The local complementation at the universal vertex lead us to the wheel W_{2j+1}. It is easy to see that W_{2j+1} is not a circle graph, for $j \geq 2$. Then, this part of the lemma holds by Corollary 2.

- Graphs $\{\overline{C_{2j}}\}_{j\geq 3}$ are not circle graphs. Let j be an integer number, $j \geq 3$. Suppose that $\overline{C_{2j}}$ is a circle graph and let $v_1,v_2,...,v_{2j}$ be the vertices of the induced cycle C_{2j}. First, we have to prove that the extreme points a_i, b_i of the odd chords L_i (chords corresponding to vertices with odd index) are in the following clockwise (or counterclockwise) order: $a_1,a_3,...,a_{2j-1},b_1,b_3,...,b_{2j-1}$ (Figure 16).

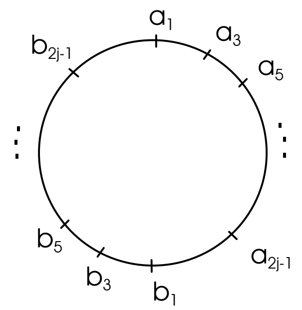

Figure 16

Vertices v_1, v_3 and v_5 form a complete subgraph, so without loss of generality its corresponding extreme points are drawn as in Figure 16. Suppose that the extreme points of the odd chords $L_1,L_3,L_5,...,L_{2k-1}$ ($k \geq 3$) are in the clockwise order of Figure 17 and we have to draw the chord L_{2k+1}, corresponding to vertex v_{2k+1}.

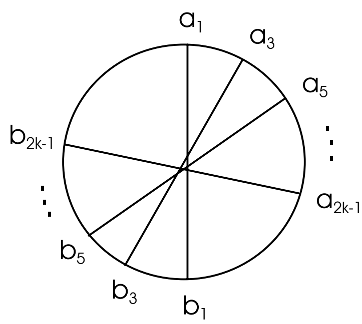

Figure 17

The following facts assert that one of the extreme points of L_{2k+1} is between a_{2k-1} and b_1, and the other one is between b_{2k-1} and a_1:

a) v_{2k+1} forms a complete subgraph with $v_1, v_3, \ldots, v_{2k-1}$.

b) v_{2k} is adjacent to any odd vertex, except v_{2k-1} and v_{2k+1}.

c) v_{2k-2} is adjacent to any odd vertex, except v_{2k-3} and v_{2k-1}.

We have to draw now even chords. Analogously, they verify the same property of odd chords. First, draw the chord L_2 corresponding to vertex v_2. As v_2 is adjacent to any odd vertex except v_1 and v_3, we may assume without loss of generality that its corresponding chord has one of the extremes between a_3 and a_5, and the other one between a_{2j-1} and b_1. Repeat this analysis for $L_4, L_6, \ldots, L_{2j-2}$ and conclude that we cannot draw the chord L_{2j}.□

Now, the following result about minimal forbidden subgraphs for circle graphs can be proved.

Theorem 7: Let H be a minimal forbidden subgraph for circle graphs. Then either H contains properly some of the graphs of Lemma 1 or H is one of the graphs of Lemma 2.

Proof: It is a direct consequence of Theorem 7, Lemmas 1 and 2 and the fact of proper circular-arc graphs are a subclass of circle graphs.□

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