Nonexistence of graded unital homomorphisms between Leavitt algebras and their Cuntz splices

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Leavitt path algebras and graded K-theory

Leavitt path algebras

Throughout the talk ℓ will be a commutative unital ring.

The Leavitt path algebra of a graph $s, r\colon E^1 \to E^0$ is the associative ℓ-algebra with generators $\{v, e, e^* : v \in E^0, e \in E^1\}$ subject to the Cuntz-Krieger relations:

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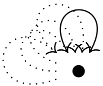
$$f^* \cdot g = \delta_{f,g} \cdot r(f), \tag{CK1}$$

$$v = \sum_{e \in s^{-1}(v)} e \cdot e^* \tag{CK2}$$

for each $f, g \in E^1$ and regular vertex v.

Leavitt algebras and their Cuntz splices

Given $n \geq 1$, the Leavitt algebra L_n is the Leavitt path algebra of the rose of n petals,

$\mathcal{R}_n =$  , $L_n = L(\mathcal{R}_n)$.

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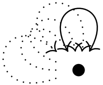
$\mathcal{R}_n =$  , $\qquad L_n = L(\mathcal{R}_n).$

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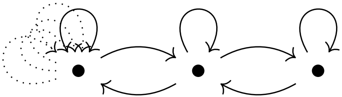
$\mathcal{R}_{n^-} =$  .

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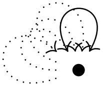
$\mathcal{R}_n =$  , $\qquad L_n = L(\mathcal{R}_n)$.

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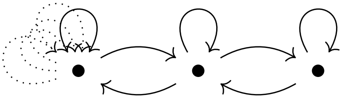
$\mathcal{R}_{n^-} =$  .

We will write L_{n^-} for its Leavitt path algebra. It is an open question to determine whether L_n and L_{n^-} are isomorphic.

Leavitt path algebras and graded K-theory

The algebra $L(E)$ is $\mathbb{Z}$-graded by setting $|v| = 0$, $|e| = 1$, $|e^*| = -1$ for each $v \in E^0$, $e \in E^1$.

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In particular, for each $m \geq 2$ this induces a grading over $C_m \simeq \mathbb{Z}/m\mathbb{Z}$,

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Given $m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$, the C_m-graded K-theory $K_0^{C_m-\mathrm{gr}}(L(E))$ of $L(E)$ is the group completion of the monoid of isomorphism classes of projective f.g. C_m-graded modules.
The grading shift of modules induces a C_m-module structure on $K_0^{C_m-\mathrm{gr}}(L(E))$.

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Objective

We shall see that there are no unital maps $L_n \leftrightarrow L_{n^-}$ that preserve the C_m-grading.

Leavitt path algebras and (graded) K-theory

The (graded) K-theory of $L(E)$ can be computed in terms of the adjacency matrix of E,

$$A_E \in \mathbb{N}_0^{\mathrm{reg}(E)\times E^0}, (A_E)_{v,w} = \#\{e \in E^1 : s(e) = v, r(e) = w\}.$$

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Theorem

If ℓ is regular supercoherent and the map $\mathbb{Z} \to K_0(\ell)$ is an isomorphism, then for any row-finite graph E we have

$$K_0(L(E)) = \operatorname{coker}(I - A_E^t). \qquad (I_{v,w} = \delta_{v,w}).$$

From now on we will assume that ℓ satisfies the hypotheses of the previous theorem (e.g. ℓ can be a PID).

Leavitt path algebras and (graded) K-theory

Theorem

For any finite regular graph E and $m \geq 2$ there is an isomorphism

$$K_0^{C_m-\mathrm{gr}}(L(E)) \simeq \mathrm{coker}(I - (A_E^m)^t), \qquad [L(E)] \mapsto 1_E := \sum_{v \in E^0} [v].$$

The C_m-module structure is induced by multiplication by A_E^{m-1}.

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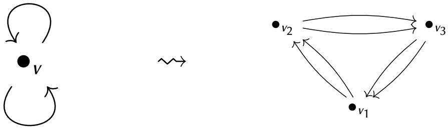
By a result of Ara, Hazrat, Li and Sims, this amounts to computing the K_0 of the "m-sheeted covering of E". For example, when $m = 3$ and $E = \mathcal{R}_2$ we have the following picture:



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Write τ for the generator of C_m. We define the Bowen-Franks C_m-module of a finite graph E as $\mathfrak{BF}_m(E) = \mathrm{coker}(I - \tau \cdot A_E^t)$.

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Theorem

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Strategy

Show there are no C_m-module maps $\mathfrak{BF}_m(\mathcal{R}_n) \to \mathfrak{BF}_m(\mathcal{R}_{n^-})$ sending $1_{\mathcal{R}_n} \mapsto 1_{\mathcal{R}_{n^-}}$ and likewise in the opposite direction.

Bowen-Franks C_m-modules

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Example

Since the adjacency matrix of $\mathcal{R}_n$ is $(n) \in M_1(\mathbb{Z})$, we have

$$\mathfrak{BF}_m(\mathcal{R}_n) = \operatorname{coker}(\mathbb{Z} \xrightarrow{1-n^m} \mathbb{Z}) \simeq \mathbb{Z}/(n^m - 1)\mathbb{Z}$$

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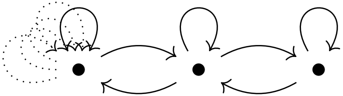
We will now compute $\mathfrak{BF}_m(\mathcal{R}_{n^-})$. Since $\mathcal{R}_{n^-} =$  , its adjacency matrix is

$$A_{\mathcal{R}_{n^-}} = \begin{pmatrix} n & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Bowen-Franks C_m-modules

By definition

$$\mathfrak{BF}_m(\mathcal{R}_{n^-}) = \operatorname{coker}(I - \tau A^t_{\mathcal{R}_{n^-}}) = \operatorname{coker} \begin{pmatrix} 1 - n \cdot \tau & -\tau & 0 \\ -\tau & 1 - \tau & -\tau \\ 0 & -\tau & 1 - \tau \end{pmatrix}.$$

This matrix can be thought of as the projection of the matrix $I - X \cdot A^t_{\mathcal{R}_{n^-}} \in \mathbb{Z}[X]$.

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From this we are able to obtain that

$$\mathfrak{B}\mathfrak{F}_m(\mathcal{R}_{n^-}) \simeq \frac{\mathbb{Z}[C_m]}{\langle \xi_n(\tau)\rangle}, \qquad \xi_n(X) = X^3 + (2n-1)X^2 - (n+2)X + 1$$

and $1_{\mathcal{R}_{n^-}} \mapsto 1 - n\cdot\tau$.

There are no graded unital maps $L_{n^-} \to L_n$

By what we have seen, to see that there are no unital C_m-graded maps $L_n \leftrightarrow L_{n^-}$ it suffices to prove that there are no pointed C_m-module maps between

$$(\mathbb{Z}/(n^m - 1)\mathbb{Z}, 1) \qquad \text{and} \qquad \left(\frac{\mathbb{Z}[C_m]}{\langle \xi_n(\tau) \rangle}, 1 - n \cdot \tau\right).$$

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Proposition

Let $m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ and $n \geq 2$. There are no C_m-graded unital maps $L_{n^-} \to L_n$.

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Proof.

We may assume $m < \infty$. A pointed C_m-module map $\phi\colon \mathfrak{B}\mathfrak{F}_m(\mathcal{R}_{n^-}) \to \mathfrak{B}\mathfrak{F}_m(\mathcal{R}_n)$ should satisfy $1 = \phi([1 - n \cdot \tau]) = (1 - n \cdot \tau)\phi([1]) = (1 - n^m)\phi([1]) = 0$, a contradiction. □

There are no graded unital maps $L_n \to L_{n^-}$

A non-triviality criterion

For the nonexistence of maps $L_n \to L_{n^-}$ we shall need a non-triviality criterion for Bowen-Franks modules.

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Lemma

Let E be a finite regular graph. Assume that all complex roots of $\chi_{A_E}(X) \in \mathbb{Z}[X]$ are real. If $\mathfrak{BF}_2(E)$ is finite and nontrivial, then $\infty > |\mathfrak{BF}_m(E)| > |\mathfrak{BF}_2(E)| > 1$ for all $m > 2$.

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Sketch of proof.

By the Smith normal form, we know that $|\chi_{A_E^m}(1)| = |\det(I - (A_E^m)^t)|$ is either zero, in which case $\mathfrak{B}\mathfrak{F}_m(E)$ is infinite, or it coincides with $|\mathfrak{B}\mathfrak{F}_m(E)|$.

Since the roots of $\chi_{A_E^m}$ are m-powers of the roots of χ_{A_E}, the result follows from the hypotheses on χ_{A_E} and $\mathfrak{B}\mathfrak{F}_2(E)$. □

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Corollary

For each $n, m \geq 2$ we have $\mathfrak{BF}_m(\mathcal{R}_{n^-}) \geq 3n^2 - 2n - 1$.

Some preliminary lemmas

Lemma

If there exists a C_m-graded unital map $\phi\colon L_n \to L_{n^-}$, then

$$(1 - nX)^2 \in \langle X^m - 1, \xi_n(X) \rangle$$

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Proof.

The existence of a C_m-module map $\phi\colon \mathbb{Z}/(n^m - 1)\mathbb{Z} \to \mathbb{Z}[X]/\langle X^m - 1, \xi_n(X)\rangle$ mapping $1 \mapsto [1 - n \cdot X]$ would imply that

$$[(1 - n \cdot X)^2] = (1 - n \cdot \tau)[(1 - n \cdot X)] = (1 - n \cdot \tau)\phi(1) = \phi(1 - n^m) = 0.$$

□

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For each $n, m \geq 2$, put $I_{n,m} := \langle X^m - 1, \xi_n(X) \rangle$. If $(1 - nX)^2 \in I_{n,m}$, we have

$$I_{n,m} = \langle (X-1)^2, m(X-1), (3n-1)(x-1) + n - 1 \rangle.$$

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Step 1: $(n-1)^2 \in I_{n,m}$. There are $p_n(X), q_n(X) \in \mathbb{Z}[X]$ such that

$$p_n(X) \cdot \xi_n(X) + q_n(X) \cdot (1 - nX)^2 = (n-1)^2,$$

and their coefficients depend polynomially on n. We may find them interpolating.

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Step 3: since $(X-1)^2 \in I_{n,m}$, we may add it as a generator and reduce $X^m - 1$ and $\xi_n(X)$ modulo $(X-1)^2$.

□

The main result

Theorem (A. - Cortiñas)

There are no C_m-graded unital maps $L_n \to L_{n^-}$ for any $n \geq 2$ and $m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$.

The main result

Theorem (A. - Cortiñas)

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Sketch of proof.

Assuming $(1 - nX)^2 \in \langle X^m - 1, \xi_n(X) \rangle$, we have

$$\mathfrak{B}\mathfrak{F}_m(\mathcal{R}_{n^-}) \simeq \mathbb{Z}[X]/\langle (X-1)^2, m(X-1), (3n-1)(X-1) + n - 1 \rangle.$$

If there is a C_m-graded unital map, it is also C_d-graded for each $d \mid m$, so we may assume that m is a prime.

A contradiction is then drawn by manipulating the ideal to contradict the lower bound on $|\mathfrak{B}\mathfrak{F}_m(\mathcal{R}_{n^-})|$. □