Classification of Leavitt path algebras: bivariant K-theory techniques

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IMaS UBA-CONICET

Frontiers in Leavitt Path Algebras and Related Topics

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Set-up & plan

- ℓ commutative ring;
- graph $r, s\colon E^1 \to E^0$ will be assumed to be finite;
- $L(E) = L_\ell(E)$ the LPA of E.

Plan:

1. (graded) K-theory;
2. (graded) kk-theory;
3. LPAs as objects in kk;
4. homotopy classification results.

K-theory

For a ring R,

$$\mathsf{proj}_R \rightsquigarrow K_0(R), K_1(R), K_2(R), \ldots$$

We will focus mainly on

$$K_0(R) = \frac{\mathbb{Z}[\,[P] : P \text{ proj. f.g. } R\text{-module}\,]}{\langle [P \oplus Q] = [P] + [Q] \rangle}.$$

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Here $BF(E) = coker(I - A_E^t)$ is the *Bowen-Franks* group of E.

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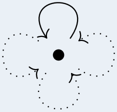
Here $BF^{gr}(E) = \mathrm{coker}(I - \sigma A_E^t)$ is the *Krieger dimension group* of E.

(Graded) K-theory (cont.)

Example

If $\mathcal{R}_n =$  , $L(\mathcal{R}_n) = L_n$, $A_{\mathcal{R}_n} = (n)$,

$$K_0(L_n) = \operatorname{coker}(1 - n \colon \mathbb{Z} \to \mathbb{Z}) = \mathbb{Z}/(n-1)\mathbb{Z}.$$

$$\begin{aligned} K_0^{gr}(L_n) &= \operatorname{coker}(1 - \sigma n \colon \mathbb{Z}[\sigma, \sigma^{-1}] \to \mathbb{Z}[\sigma, \sigma^{-1}]) \\ &= \mathbb{Z}[\sigma, \sigma^{-1}]/\langle \sigma n - 1 \rangle \cong \mathbb{Z}[1/n]. \end{aligned}$$

Classification conjectures

Classification question for SPI LPAs (Abrams, Ánh, Louly, Pardo)

Suppose E and F are SPI graphs, i.e. graphs for which $L(E)$ and $L(F)$ are simple purely infinite.

If there exists an isomorphism $K_0(L(E)) \xrightarrow{\sim} K_0(L(F))$ mapping $[L(E)]$ to $[L(F)]$, must $L(E)$ and $L(F)$ be isomorphic?

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Hazrat's conjecture

If E and F are two graphs, then $L(E) \cong_{gr} L(F)$ if and only if there is an ordered module isomorphism $K_0^{gr}(L(E)) \to K_0^{gr}(L(F))$ mapping $[L(E)] \mapsto [L(F)]$.

Towards *kk*-theory

$$\mathsf{Alg}_\ell \xrightarrow[j]{} \mathsf{kk} \longrightarrow \mathsf{Ab}, \qquad K_0(-) : \mathsf{Alg}_\ell \to \mathsf{Ab}$$

Plan:

- construct an "intermediate category" kk;
- classify LPAs as objects in kk;
- recover classification results in Alg_ℓ.

kk-theory, a first encounter

Algebraic bivariant *K*-theory was introduced by Cortiñas and Thom in analogy with Kasparov's *KK*-theory.

Some properties of kk:

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Some properties of kk:

- has the *same objects* as Alg_ℓ;
- is *additive*; hence $\mathrm{kk}(A, B) \in \mathrm{Ab}$;
- there is a *comparison functor*

  $$j\colon \mathrm{Alg}_\ell \to \mathrm{kk},$$

  which is the identity on objects.

Stability

In general the map

$$j\colon \hom_{\mathrm{Alg}_\ell}(A, B) \to \mathrm{kk}(A, B)$$

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$$f_0 \sim f_1 \iff \begin{array}{ccc} & & B[t] \\ & \overset{h}{\nearrow} & \Big\downarrow {\scriptstyle \mathrm{ev}_0,\mathrm{ev}_1} \\ A & \xrightarrow{f_0,f_1} & B \end{array}$$

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- matricial stability: $j(A \hookrightarrow M_\infty A)$ is an iso for all A.

$$A \sim_{Morita} B \qquad \rightsquigarrow \qquad j(A) \cong j(B).$$

The triangulated structure

- there is an *equivalence* $\Omega\colon \mathrm{kk} \to \mathrm{kk}$ given on objects by

$$\Omega A = t(t-1)A[t] = \ker(\mathrm{ev}_0) \cap \ker(\mathrm{ev}_1) \subset A[t].$$

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- it is *triangulated* and j is *excisive*; every linearly split extension

$$K \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} Q \qquad (\mathcal{E})$$

gets mapped to a *triangle*

$$\Omega Q \xrightarrow{\partial_{\mathcal{E}}} K \xrightarrow{j(i)} E \xrightarrow{j(p)} Q.$$

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Fact

Writing $\mathrm{kk}_n(D,C) = \mathrm{kk}(D,\Omega^n C)$ *and* $\mathrm{kk}_{-n}(D,C) = \mathrm{kk}(\Omega^n D, C)$*, for all algebras D there exists a long exact sequence*

$$\cdots \to \mathrm{kk}_1(D,E) \to \mathrm{kk}_1(D,Q) \to \mathrm{kk}_0(D,K) \to \mathrm{kk}_0(D,E) \to \mathrm{kk}_0(D,Q) \to \cdots$$

The universal property

The functor $j\colon \mathrm{Alg}_\ell \to \mathrm{kk}$ is the intial homotopy invariant, matricially stable, excisive functor with values in a triangulated category:

$$\begin{array}{ccc} \mathrm{Alg}_\ell & \xrightarrow{\ j\ } & \mathrm{kk} \\ & {\scriptstyle F}\searrow & \downarrow{\scriptstyle \exists !} \\ & & T \end{array}$$

kk vs. K

Algebraic kk-theory recovers C. Weibel's homotopy K-theory.

For LPAs over a field, it coincides with K-theory.

Theorem (Cortiñas-Thom, '07)

$$\mathrm{kk}_n(\ell, A) \cong KH_n(A).$$

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Corollary

$$\mathrm{kk}_n(\ell, L(E)) \cong K_n(L(E)).$$

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For simplicity, for the rest of the talk we assume ℓ to be a field.

Classification of LPAs in kk

Theorem (Cortiñas-Montero '18)

TFAE:

i) $K_0(L(E)) \cong K_0(L(F))$ *and* $\# \operatorname{sing}(E) = \# \operatorname{sing}(F)$

ii) $j(L(E)) \cong j(L(F))$.

$\square$

A glimpse at the proof strategy

There is a linearly split extension

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They also show that under the identification

$$\mathrm{kk}(\ell^{\operatorname{reg}(E)}, \ell^{E^0}) \cong K_0(\ell)^{\operatorname{reg}(E)\times E^0} = \mathbb{Z}^{\operatorname{reg}(E)\times E^0}$$

the map $K(E) \to C(E)$ corresponds to $I - A_E^t$.

A glimpse at the proof strategy (cont.)

We thus have a distinguished triangle in kk of the form

$$\Omega L(E) \to \ell^{\mathrm{reg}(E)} \xrightarrow{I - A_E^t} \ell^{E^0} \to L(E).$$

From here on, the proof is completed using abstract nonsense of triangulated categories.

Homotopy classification

Theorem (Cortiñas-Montero '18)

Let E and F be purely infinte simple graphs. The following statements are equivalent:

i) $K_0(L(E)) \cong K_0(L(F))$;

ii) *there exists algebra homomorphisms* $f\colon L(E) \longleftrightarrow L(F)\colon g$ *such that* $f \circ g \approx \mathrm{id}_{L(F)}$, $g \circ f \approx \mathrm{id}_{L(E)}$.

□

A glimpse at the proof strategy, II

The proof involves studying the map

$$j\colon \hom_{\mathsf{Alg}_\ell}(L(E), L(F)) \to \mathsf{kk}(L(E), L(F))$$

and proving that it is:

- surjective;
- injective up to the notion of homotopy defined above.

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Writing $BF^\vee(E) = \operatorname{coker}(I - A_E)$, there is a SES

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Surjectivity relies upon the fact that a map $K_0(L(E)) \to K_0(L(F))$ lifts to an algebra map.

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- and can lift maps $K_0^{gr}(L(E)) \to K_0^{gr}(L(F))$ to graded algebra maps (A. '23, Vaš '23).

Theorem (A. '23)

Given E and F two finite, primitive graphs, the following are equivalent:

(i) $K_0^{gr}(L(E)) \cong K_0^{gr}(L(F))$ *as pointed ordered modules;*

(ii) $L(E)$ *and* $L(F)$ *are* *graded* *homotopy equivalent.*

Thank you!