Graded homotopy classification of Leavitt path algebras

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Leavitt path algebras

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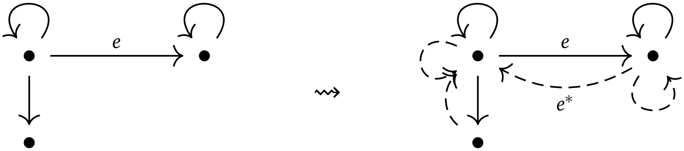
The ***Leavitt path ℓ-algebra*** $L(E)$ of E is a quotient of the path algebra of the ***double*** graph of E.



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Examples:

- $M_n(\ell)$
- $M_n(\ell[t, t^{-1}])$
- $\ell\{x, x^* : x^*x = 1\}$
- $L_2 = \frac{\ell\{x_1, x_2, x_1^*, x_2^*\}}{\langle x_i^* x_i - 1,\ x_1^* x_2,\ x_2^* x_1,\ x_1 x_1^* + x_2 x_2^* - 1\rangle}$.

The graded classification conjecture

The graded classification conjecture states that LPAs can be characterized by their ***graded Grothendieck group***:

$$K_0^{\mathrm{gr}}(L(E)) = \frac{\mathbb{Z}\{[P] : P \text{ graded projective } L(E)\text{-module}\}}{\langle [P \oplus Q] = [P] + [Q] \rangle}.$$

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Conjecture (Hazrat, '13)

If E and F are two finite graphs, then $L(E) \cong_{\mathrm{gr}} L(F)$ if and only if $K_0^{\mathrm{gr}}(L(E)) \cong K_0^{\mathrm{gr}}(L(F))$ as pointed preordered modules.

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One first considers ***elementary homotopies*** between graded ℓ-algebra homomorphisms $f_0, f_1 : A \to B$.

$$f_0 \sim f_1 \quad \Longleftrightarrow \quad \begin{tikzcd} & B[t] \arrow[d, "ev_i"] \\ A \arrow[ur, "\exists h"] \arrow[r, "f_i"'] & B \end{tikzcd} \quad i = 0, 1.$$

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Two maps are ***homotopic*** if there is a finite chain

$$f \approx g \iff f = f_0 \sim f_1 \sim \cdots \sim f_n = g.$$

The graded homotopy classification conjecture

Conjecture

Let E and F be two finite graphs. The following statements are equivalent:

(i) There is a pointed preordered module isomorphism $K_0^{\mathrm{gr}}(L(E)) \cong K_0^{\mathrm{gr}}(L(F))$.

(ii) There are unital, graded algebra homomorphisms $f : L(E) \longleftrightarrow L(F) : g$ such that $gf \approx 1_{L(E)}$ and $fg \approx 1_{L(F)}$.

Primitive graphs

The ***adjacency matrix*** $A_E \in \mathbb{N}_0^{E^0 \times E^0}$ of a graph E is defined as

$$(A_E)_{v,w} = \#\left\{ \bullet_v \xrightarrow{e} \bullet_w \right\}.$$

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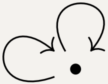
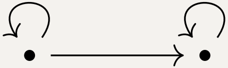
We will assume all graphs to be ***primitive***, meaning that there exists some $N > 1$ such that all entries of A_E^N are positive.

Example



$\begin{pmatrix} 2 \end{pmatrix}$

Non-example



$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Primitive graphs

The tools that we use come from the (ungraded) homotopy classification of purely infinite simple LPAs (Cortiñas, Montero, '20, Cortiñas '22).

We consider primitive graphs in order to be able to adapt some of these techniques.

Lemma

If E is a primitive graph and $e \in E^1$ then ee^* is a full idempotent of $L(E)_0$.

Goal: prove the graded homotopy classification conjecture for primitive graphs.

Graded bivariant K-theory

To understand the assingment

$$\hom_{gr-Alg}(L(E), L(F)) \longrightarrow \hom_{\mathbb{Z}[\sigma]}(K_0^{\mathrm{gr}}(L(E)), K_0^{\mathrm{gr}}(L(F)))$$

we will consider an intermediate category, graded algebraic bivariant K-theory (Ellis, '14).

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$$\begin{tikzcd}
\hom_{gr-Alg}(L(E), L(F)) \arrow[r, "j"'] \arrow[rr, bend left, "K_0^{\mathrm{gr}}"] & kk^{\mathrm{gr}}(j(L(E)), j(L(F))) \arrow[r, dashed, "ev"'] & \hom_{\mathbb{Z}[\sigma]}(K_0^{\mathrm{gr}}(E), K_0^{\mathrm{gr}}(F))
\end{tikzcd}$$

Graded bivariant K-theory

The objects of kk^{gr} are graded algebras; the description of morphisms is more complicated.

We can characterize this category as the “smallest” triangulated category recieving a comparison functor $j\colon gr-Alg \to kk^{\mathrm{gr}}$ satisfying the following properties:

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Theorem (A., Cortiñas, '22)

We have:

(i) $kk^{\mathrm{gr}}(\ell, L(E)) \cong K_0^{\mathrm{gr}}(L(E))$;

(ii) $K_0^{\mathrm{gr}}(E) \cong K_0^{\mathrm{gr}}(F)$ *as modules if and only if* $j(L(E)) \cong j(L(F))$.

Grdaded homotopy classification

The starting point for graded homotopy classification is the following “universal coefficient theorem”:

Theorem (A., Cortiñas, ’22)

There is a short exact sequence:

$$0 \longrightarrow K_0^{\mathrm{gr}}(E_t) \otimes_{\mathbb{Z}[\sigma]} K_1^{\mathrm{gr}}(L(F)) \xrightarrow{\partial} kk^{\mathrm{gr}}(L(E), L(F)) \longrightarrow \hom_{\mathbb{Z}[\sigma]}(K_0^{\mathrm{gr}}(E), K_0^{\mathrm{gr}}(L(F))) \longrightarrow 0$$

Here E_t is the ***dual*** *graph of E.*

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(1) every arrow in $kk^{\mathrm{gr}}(L(E), L(F))$ that induces a preordered module map is of the form $j(f)$ for some unital graded algebra homomorphism $f\colon L(E) \to L(F)$. (***"surjectivity"***)

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Let us first see why (1) and (2) together imply that K_0^{gr} classifies LPAs up to graded homotopy.

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- by (1), there are graded algebra homomorphisms $f : L(E) \longleftrightarrow L(F) : g$ such that $j(f) = \xi$ and $j(g) = \xi^{-1}$.
- since $j(fg) = j(f)j(g) = 1_{L(F)}$ and likewise $j(gf) = 1_{L(E)}$, by (2) we have that both fg and gf are homotopic to conjugation by some degree zero unit.

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- this implies that f is a graded homotopy equivalence.

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We now enumerate the main tools used in the proof.

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Theorem (A. '22, Vaš '22)

Preordered module maps $K_0^{\mathrm{gr}}(L(E)) \to K_0^{\mathrm{gr}}(L(F))$ can be lifted to graded unital maps $L(E) \to L(F)$.

We also need a further understanding of the map ∂ and of $K_1^{\mathrm{gr}}(L(F))$.

Poincaré duality

To understand ∂ we used a graded analogoue of the algebraic version of Poincaré duality for LPAs (Cortiñas '22).

Theorem (A. '23)

Given R and S two graded algebras we have a natural isomorphism

$$kk^{\mathrm{gr}}(R \otimes_\ell L(E), S) \cong kk^{\mathrm{gr}}(R, S \otimes_\ell L(E_t)[+1]).$$

Corollary

We have an isomorphism $kk^{\mathrm{gr}}(L(E), L(F)) \cong KH_1^{\mathrm{gr}}(L(F) \otimes_\ell L(E))$.

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Lemma

If R is a strongly graded ring such that R_0 is ultramatricial, then $K_1^{\mathrm{gr}}(R) \cong (R_0)^{\times}_{\mathsf{ab}}$. In particular $K_1^{\mathrm{gr}}(L(E)) \cong (L(E)_0)^{\times}_{\mathsf{ab}}$.

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Sketch of proof.

By Dade's theorem $K_1^{\mathrm{gr}}(R) \cong K_1(R_0)$. Both K_1 and $(-)^{\times}_{\mathrm{ab}}$ commute with finite products and unions; hence, the result boils down to the fact that $K_1(M_n(\ell)) = M_n(\ell)^{\times}_{\mathrm{ab}} = \ell^{\times}$. □

The action on K_1 for corner skew Laurent polynomial rings

We wish to see how the isomorphism $K_1^{\mathrm{gr}}(R) \cong (R_0)^\times_{\mathsf{ab}}$ translates the shift action on $K_1^{\mathrm{gr}}(R)$ to an action on $(R_0)^\times_{\mathsf{ab}}$.

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A graded ring R is a ***corner skew Laurent polynomial ring*** if there exist $t_+ \in R_1$, $t_- \in R_{-1}$ such that $t_- t_+ = 1$. Writing $p = t_+ t_-$, we have an endomorphism

$$\alpha \colon R_0 \to R_0, \qquad x \mapsto t_+ x t_-$$

which corestricts to an isomorphism $R_0 \cong pR_0p$.

The action on K_1 for corner skew Laurent polynomial rings

We wish to see how the isomorphism $K_1^{\mathrm{gr}}(R) \cong (R_0)^{\times}_{\mathsf{ab}}$ translates the shift action on $K_1^{\mathrm{gr}}(R)$ to an action on $(R_0)^{\times}_{\mathsf{ab}}$.

A graded ring R is a ***corner skew Laurent polynomial ring*** if there exist $t_+ \in R_1$, $t_- \in R_{-1}$ such that $t_-t_+ = 1$. Writing $p = t_+t_-$, we have an endomorphism

$$\alpha \colon R_0 \to R_0, \qquad x \mapsto t_+xt_-$$

which corestricts to an isomorphism $R_0 \cong pR_0p$.

Example

Let E be an essential graph and consider $\{e_v : v \in E^0\} \subset E^1$ such that each edge e_v ends at v. The elements $t_+ = \sum_{v\in E^0} e_v$ and $t_- = t_+^*$ satisfy $t_-t_+ = 1$, hence $L(E)$ is a corner skew Laurent polynomial ring.

The action on K_1 for corner skew Laurent polynomial rings

Let E be an essential graph and $\alpha \colon L(E)_0 \to L(E)_0$ the endomorphism associated to its corner skew Laurent polynomial structure.

Theorem (Ara, Pardo, '14)

Under the isomorphism $K_0^{\mathrm{gr}}(L(E)) \cong K_0(L(E)_0)$ the shift automorphism on $K_0^{\mathrm{gr}}(L(E))$ corresponds to $K_0(\alpha) \colon K_0(L(E)_0) \to K_0(L(E)_0)$.

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Theorem (A. '23)

If E is an essential graph, then under the isomorphism $K_1^{\mathrm{gr}}(L(E)) \cong K_1(L(E)_0)$ the shift automorphism on $K_1^{\mathrm{gr}}(L(E))$ corresponds to $K_1(\alpha)\colon K_1(L(E)_0) \to K_1(L(E)_0)$.

Question

These theorems have been proved at the level of (graded) K-theory groups. Can we lift them to a statement at the level of categories of (graded) modules?

Graded homotopy classification

With all of this in place, we have the following classification result.

Theorem (A. '23)

Let ℓ be a field and let E and F be two finite, primitve graphs. The following statements are equivalent:

(i) *there is an isomorphism* $(K_0^{\mathrm{gr}}(L(E)), K_0^{\mathrm{gr}}(L(E))_+, [L(E)]) \cong (K_0^{\mathrm{gr}}(L(F)), K_0^{\mathrm{gr}}(L(F))_+, [L(F)])$;

(ii) *there is a unital graded homotopy equivalence* $f\colon L_\ell(E) \to L_\ell(F)$.