1 Newton's Method

In this section we present the classical Newton method for solving the nonlinear problem: Find $\phi \in \mathbb{X}$ such that

$$A(\phi, \psi) + C(\phi; \phi, \psi) = F(\psi),$$

for all $\psi \in \mathbb{X}$. Before describing the method for the particular nonlinear formulation considered in this work, we briefly recall Newton's method in an abstract setting and introduce the notion of the Gâteaux derivative, which plays a fundamental role in its construction.

1.1 Abstract Framework

Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces and let $x_0 \in \mathbb{X}$ be a given point. For positive constants $0 < r < R$, define the open and closed balls centered at x_0 by

$$B_R := \{x \in \mathbb{X} : \|x - x_0\|_{\mathbb{X}} < R\},$$

and

$$\overline{B}_r := \{x \in \mathbb{X} : \|x - x_0\|_{\mathbb{X}} \le r\}.$$

Assume that the nonlinear operator

$$P : B_R \subset \mathbb{X} \longrightarrow \mathbb{Y}$$

has a zero $x^* \in B_R$, that is,

$$P(x^*) = 0,$$

and that P has a continuous derivative in B_R. Then Newton's method is described as follows:

Given an initial approximation $x_0 \in B_R$ of x^*, compute

$$x_m = x_{m-1} - \big(P'(x_{m-1})\big)^{-1} P(x_{m-1}), \qquad m \ge 1,$$

provided that the inverse of $P'(x_{m-1})$ exists.

The convergence of this iteration is guaranteed under suitable assumptions by the classical Kantorovich Theorem.

1.2 The Gâteaux Derivative

Before deriving Newton's method for our nonlinear problem, we recall the definition of the Gâteaux derivative.

Definition 1 (Gâteaux Derivative)**.** *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces and let*

$$P : \mathbb{X} \to \mathbb{Y}.$$

The operator P is said to be Gâteaux differentiable *at $\phi \in \mathbb{X}$ if there exists a bounded linear operator*

$$P'(\phi) : \mathbb{X} \to \mathbb{Y}$$

such that, for every direction $\zeta \in \mathbb{X}$,

$$P'(\phi)(\zeta) = \lim_{t \to 0} \frac{P(\phi + t\zeta) - P(\phi)}{t},$$

whenever the above limit exists.

The operator $P'(\phi)$ is called the Gâteaux derivative *of P at ϕ.*

1.3 Application to the Nonlinear Problem

Let $\zeta \in \mathbb{X}$ be fixed. We introduce the linear operator

$$N(\zeta, \cdot) : \mathbb{X} \longrightarrow \mathbb{X}',$$

defined by

$$\langle N(\zeta, \varphi), \psi \rangle = A(\varphi, \psi) + C(\zeta; \varphi, \psi), \qquad \forall\, \varphi, \psi \in \mathbb{X}, \tag{1}$$

and let the functional $F \in \mathbb{X}'$ be defined by

$$\langle F, \psi \rangle = F(\psi), \qquad \forall\, \psi \in \mathbb{X}. \tag{2}$$

Then, we define the nonlinear operator

$$P : \mathbb{X} \longrightarrow \mathbb{X}', \qquad P(\phi) = N(\phi, \phi) - F. \tag{3}$$

Equivalently,

$$\langle P(\phi), \psi \rangle = A(\phi, \psi) + C(\phi; \phi, \psi) - F(\psi), \qquad \forall\, \psi \in \mathbb{X}.$$

Hence, $\phi \in \mathbb{X}$ is a solution of the nonlinear problem if and only if

$$P(\phi) = 0 \quad \text{in } \mathbb{X}'.$$

1.4 Computation of the Gâteaux Derivative

We now compute the Gâteaux derivative of the operator P.

Let $\phi, \zeta \in \mathbb{X}$. By definition,

$$P'(\phi)(\zeta) = \lim_{t \to 0} \frac{P(\phi + t\zeta) - P(\phi)}{t}.$$

For every $\psi \in \mathbb{X}$,

$$\begin{aligned}
&\left\langle \frac{P(\phi + t\zeta) - P(\phi)}{t}, \psi \right\rangle \\
&= \frac{1}{t}\Big(A(\phi + t\zeta, \psi) + C(\phi + t\zeta; \phi + t\zeta, \psi) - F(\psi) \\
&\qquad\qquad - A(\phi, \psi) - C(\phi; \phi, \psi) + F(\psi) \Big).
\end{aligned}$$

Using the linearity of A and the bilinearity of the trilinear form C with respect to its first two arguments, we obtain

$$\begin{aligned}
&\left\langle \frac{P(\phi + t\zeta) - P(\phi)}{t}, \psi \right\rangle \\
&= A(\zeta, \psi) + C(\zeta; \phi, \psi) + C(\phi; \zeta, \psi) + t\, C(\zeta; \zeta, \psi).
\end{aligned}$$

Passing to the limit as $t \to 0$, the last term vanishes, yielding

$$\boxed{\langle (P'(\phi))(\zeta), \psi\rangle = A(\zeta,\psi) + C(\zeta;\phi,\psi) + C(\phi;\zeta,\psi), \qquad \forall\, \phi,\zeta,\psi \in \mathbb{X}.}$$

Therefore,

$$\langle (P'(\phi))(\zeta), \psi\rangle = A(\zeta,\psi) + C(\zeta;\phi,\psi) + C(\phi;\zeta,\psi), \qquad \forall\, \phi,\zeta,\psi \in \mathbb{X}. \tag{4}$$

1.5 Newton's Method for the Nonlinear Problem

Applying Newton's method to the nonlinear operator P, we obtain the following iterative scheme.

Given an initial approximation $\phi_0 \in \mathbb{X}$, for $m \geq 1$, find $\phi_m \in \mathbb{X}$ such that

$$\big\langle (P'(\phi_{m-1}))(\phi_m - \phi_{m-1}), \psi \big\rangle = -\langle P(\phi_{m-1}), \psi\rangle, \qquad \forall\, \psi \in \mathbb{X}. \tag{5}$$

Substituting the expressions of P and its Gâteaux derivative into (5), after straightforward algebraic manipulations, the Newton iteration can be rewritten as follows.

Find $\phi_m \in \mathbb{X}$ satisfying

$$\boxed{A(\phi_m,\psi) + C(\phi_m;\phi_{m-1},\psi) + C(\phi_{m-1};\phi_m,\psi) = C(\phi_{m-1};\phi_{m-1},\psi) + F(\psi),}$$

for every test function $\psi \in \mathbb{X}$.

This formulation is particularly convenient for the implementation of Newton's method since, at each iteration, only a linear problem must be solved.