Banach spaces-based mixed finite element method for nonlinear problems in fluid mechanic

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Stokes problem

Stokes problem in $\mathbb{R}^d$, $d \in \{2,3\}$

- **Equations**:

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \operatorname{div}\mathbf{u} &= \mathbf{0} \quad \text{in} \quad \Omega, \\ \mathbf{u} &= \mathbf{u}_D \quad \text{on} \quad \Gamma, \\ \int_\Omega p &= 0. \end{aligned}$$

- **Unknowns**: velocity $\mathbf{u}$ and pressure p.
- **Parameters and Data**: $\nu > 0$, viscosity, $\mathbf{f}$ external force on Ω and $\mathbf{u}_D$ velocity on Γ.
- **Compatibility condition on $\mathbf{u}_D$**:

$$\int_\Gamma \mathbf{u}_D \cdot \mathbf{n} = 0.$$

- **Geometry**: $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, a bounded domain with Lipschitz boundary Γ and $\mathbf{n}$ the outward unit normal vector.

Classical pseudostress-based formulation

Introducing

$$\boldsymbol{\sigma} := \nu \nabla \mathbf{u} - p\,\mathbb{I} \quad \text{in} \quad \Omega,$$

the Stokes equations can be rewritten as follows

$$\boldsymbol{\sigma}^{\mathrm{d}} = \nu \nabla \mathbf{u} \quad \text{in} \quad \Omega, \qquad -\mathbf{div}\,\boldsymbol{\sigma} = \mathbf{f} \quad \text{in} \quad \Omega, \qquad \mathbf{u} = \mathbf{u}_D \quad \text{on}\ \Gamma, \qquad \int_\Omega \mathrm{tr}\,(\boldsymbol{\sigma}) = 0.$$

Using the above equations, we derive the mixed variational formulation: Find $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div}\,;\Omega) := \{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}\,;\Omega) : \int_\Omega \mathrm{tr}\,(\boldsymbol{\sigma}) = 0\}$ and $\mathbf{u} \in \mathbf{L}^2(\Omega) = [\mathrm{L}^2(\Omega)]^d$ such that

$$\begin{aligned} \frac{1}{\nu}(\boldsymbol{\sigma}^{\mathrm{d}}, \boldsymbol{\tau}^{\mathrm{d}})_\Omega + (\mathbf{u}, \mathbf{div}\,\boldsymbol{\tau})_\Omega \quad &= \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D \rangle_\Gamma\,, \quad \forall \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}\,;\Omega), \\ (\mathbf{v}, \mathbf{div}\,\boldsymbol{\sigma})_\Omega \quad &= -(\mathbf{f}, \mathbf{v})_\Omega, \qquad \forall \mathbf{v} \in \mathbf{L}^2(\Omega). \end{aligned}$$

$$\boldsymbol{\sigma} \in RT_k \quad \text{and} \quad \mathbf{u} \in P_k^{disc} : \begin{cases} \Rightarrow & \text{momentum conservation} \\ \not\Rightarrow & \text{mass conservation} \end{cases}$$

$$+\ \nu^{-1}\text{–dependent error estimates.}$$

Z. Cai, C. Tong, P.S. Vassilevski, C. Wang, *Mixed finite element methods for incompressible flow: stationary Stokes equations*, Numer. Methods Partial Differential Equations 26 (2010), no. 4, 957-978.

G.N. Gatica , A. Márquez and M.A. Sánchez,*Pseudostress-based mixed finite element methods for the Stokes problem in R^n with Dirichlet boundary conditions. I: A priori error analysis*, Commun. Comput. Phys. 12 (2012), no. 1, 109134.

Helmholtz decomposition for $\mathbf{u}$

$$\mathbf{L}^2(\Omega) = \mathbf{H}(\mathrm{div}^0;\Omega) \oplus \nabla \mathrm{H}^1_0(\Omega)$$
$$\Downarrow$$
$$\mathbf{u} = \mathbf{w} + \nabla\varphi, \text{ with } \mathbf{w} \in \mathbf{H}(\mathrm{div}^0;\Omega) \text{ and } \varphi \in \mathrm{H}^1_0(\Omega)$$

$$\begin{aligned}
\frac{1}{\nu}(\boldsymbol{\sigma}^{\mathrm{d}},\boldsymbol{\tau}^{\mathrm{d}})_\Omega + (\mathbf{w}+\nabla\varphi, \mathbf{div}\,\boldsymbol{\tau})_\Omega &= \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D\rangle_\Gamma\,, && \forall \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}\,;\Omega),\\
(\mathbf{v}+\nabla\varphi, \mathbf{div}\,\boldsymbol{\sigma})_\Omega &= -(\mathbf{f}, \mathbf{v}+\nabla\psi)_\Omega, && \forall (\mathbf{v},\psi) \in \mathbf{H}(\mathrm{div}^0;\Omega)\times \mathrm{H}^1_0(\Omega).
\end{aligned}$$

$$\text{Taking } \boldsymbol{\tau} = \Big(\psi - \tfrac{1}{|\Omega|}\int_\Omega \psi\Big)\mathbb{I}, \text{ with } \psi \in \mathrm{H}^1_0(\Omega)$$
$$\Downarrow$$
$$\frac{1}{\nu}(\boldsymbol{\sigma}^{\mathrm{d}},\boldsymbol{\tau}^{\mathrm{d}})_\Omega + (\mathbf{w}+\nabla\varphi, \mathbf{div}\,\boldsymbol{\tau})_\Omega = \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D\rangle_\Gamma \Leftrightarrow (\nabla\varphi,\nabla\psi)_\Omega = 0 \quad \forall \psi \in \mathrm{H}^1_0(\Omega)$$
$$\Downarrow$$
$$\varphi = 0 \text{ in } \Omega \text{ and } \mathbf{u} = \mathbf{w} \in \mathbf{H}(\mathrm{div}^0;\Omega).$$

New mass-conservative pseudostress-based formulation: first attempt

Find $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div}\,;\Omega)$, $\mathbf{u} \in \mathbf{H}(\mathrm{div}^0;\Omega)$ and $\varphi \in \mathrm{H}_0^1(\Omega)$, such that :

$$\begin{aligned}\frac{1}{\nu}(\boldsymbol{\sigma}^{\mathrm{d}},\boldsymbol{\tau}^{\mathrm{d}})_\Omega + \frac{1}{\nu}(\mathbf{div}\,\boldsymbol{\sigma},\mathbf{div}\,\boldsymbol{\tau})_\Omega + (\mathbf{u}+\nabla\varphi,\mathbf{div}\,\boldsymbol{\tau})_\Omega &= \quad \langle \boldsymbol{\tau}\mathbf{n},\mathbf{u}_D\rangle_\Gamma - \frac{1}{\nu}(\mathbf{f},\mathbf{div}\,\boldsymbol{\tau})_\Omega,\\ (\mathbf{v}+\nabla\psi,\mathbf{div}\,\boldsymbol{\sigma})_\Omega &= \quad -(\mathbf{f},\mathbf{v}+\nabla\psi)_\Omega,\end{aligned}$$

for all $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}\,;\Omega)$ and $(\mathbf{v},\psi) \in \mathbf{H}(\mathrm{div}^0;\Omega) \times \mathrm{H}_0^1(\Omega)$, where

$$\mathbf{H}(\mathrm{div}^0;\Omega) := \{\mathbf{v} \in \mathbf{H}(\mathrm{div};\Omega) : \mathrm{div}\,\mathbf{v} = 0 \text{ in } \Omega\}.$$

Given $\psi \in \mathrm{H}_0^1(\Omega)$:

$$\boldsymbol{\tau} = (\psi - \frac{1}{|\Omega|}\int_\Omega \psi)\mathbb{I} \Rightarrow (\mathbf{u}+\nabla\varphi,\mathbf{div}\,\boldsymbol{\tau})_\Omega = (\nabla\varphi,\nabla\psi) = 0 \Rightarrow \varphi = 0 \quad \text{in} \quad \Omega.$$

$$\Rightarrow \mathbf{L}^2(\Omega) \ni \mathbf{u} = \mathbf{u} \in \mathbf{H}(\mathrm{div}^0;\Omega) + \nabla 0 \in \mathrm{H}_0^1(\Omega) \quad (\mathbf{L}^2(\Omega) = \mathbf{H}(\mathrm{div}^0;\Omega) \oplus \mathrm{H}_0^1(\Omega)).$$

$$\boldsymbol{\sigma} \in BDM_{k+1},\ \mathbf{u} \in RT_k(\mathrm{div}^0) \quad \text{and} \quad \varphi \in P_{k+1}^{cont} : \begin{cases} \nRightarrow & \text{momentum conservation} \\ \Rightarrow & \text{mass conservation}\end{cases}$$

$+\ \nu^{-1}$–dependent error estimates.

(Remark: $RT_k(\mathrm{div}^0) + \nabla P_{k+1}^{cont} \subsetneq P_k^{disc} \nRightarrow$ momentum conservation)

J. C and R.Oyarzúa, *A conforming and mass conservative pseudostress-based mixed finite element method for Stokes*, Calcolo vol. 62, No. 31, (2025)

New mass and momentum conservative pseudostress-based formulation

Introducing

$$\boldsymbol{\sigma} := \nabla \mathbf{u} - \nu^{-1} p \, \mathbb{I} \quad \text{in} \quad \Omega,$$

the Stokes equations can be rewritten as follows

$$\boldsymbol{\sigma}^{\mathrm{d}} = \nabla \mathbf{u} \quad \text{in} \quad \Omega, \qquad -\,\mathbf{div}\,\boldsymbol{\sigma} = \nu^{-1}\mathbf{f} \quad \text{in} \quad \Omega, \qquad \mathbf{u} = \mathbf{u}_D \quad \text{on } \Gamma, \qquad \int_\Omega \mathrm{tr}\,(\boldsymbol{\sigma}) = 0.$$

Using the above equations, we derive the mixed variational formulation:
Find $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div}\,;\Omega)$, $\mathbf{u} \in \mathbf{H}(\mathrm{div}^0;\Omega)$ and $\varphi \in \mathrm{H}^1_0(\Omega)$, such that :

$$\begin{aligned}
(\boldsymbol{\sigma}^{\mathrm{d}}, \boldsymbol{\tau}^{\mathrm{d}})_\Omega + (\mathbf{u} + \nabla\varphi, \mathbf{div}\,\boldsymbol{\tau})_\Omega &= \quad \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D \rangle_\Gamma \,, \\
(\mathbf{v} + \nabla\psi, \mathbf{div}\,\boldsymbol{\sigma})_\Omega &= \quad -\nu^{-1}(\mathbf{f}, \mathbf{v} + \nabla\psi)_\Omega,
\end{aligned}$$

for all $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}\,;\Omega)$ and $(\mathbf{v}, \psi) \in \mathbf{H}(\mathrm{div}^0;\Omega) \times \mathrm{H}^1_0(\Omega)$.

$$\boldsymbol{\sigma} \in BDM_1, \;\; \mathbf{u} \in RT_0(\mathrm{div}^0) \quad \text{and} \quad \varphi \in CR \;: \begin{cases} \Rightarrow \quad \text{momentum conservation} \\ \\ \Rightarrow \quad \text{mass conservation} \end{cases}$$

$+\ \nu^{-1}$–independent error estimates.

(Remark: $RT_0(\mathrm{div}^0) \oplus \nabla_h CR = P_0 \;\Rightarrow$ momentum conservation)

Navier-Stokes problem in $\mathbb{R}^2$

Navier–Stokes problem in $\mathbb{R}^2$

- **Equations**:

$$\begin{aligned} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} && \text{in} \quad \Omega, \\ \operatorname{div}\mathbf{u} &= 0 && \text{in} \quad \Omega, \\ \mathbf{u} &= \mathbf{u}_D && \text{on} \quad \Gamma, \\ (p,1)_\Omega &= 0. \end{aligned}$$

- **Unknowns**: velocity $\mathbf{u}$ and pressure p.
- **Parameters and Data**: $\nu > 0$, viscosity, $\mathbf{f}$ external force on Ω and $\mathbf{u}_D$ velocity on Γ.
- **Compatibility condition on $\mathbf{u}_D$**:

$$\int_\Gamma \mathbf{u}_D \cdot \mathbf{n} = 0.$$

- **Geometry**: $\Omega \subset \mathbb{R}^2$, a bounded domain with Lipschitz boundary Γ and $\mathbf{n}$ the outward unit normal vector.

A reformulation of the pseudostress tensor

Introducing

$$\boldsymbol{\sigma} := \nabla \mathbf{u} - \nu^{-1}(\mathbf{u} \otimes \mathbf{u}) + \nu^{-1} c_{\mathbf{u}}\, \mathbb{I} - \nu^{-1} p\, \mathbb{I} \quad \text{in} \quad \Omega$$

with $c_{\mathbf{u}} := \dfrac{1}{2|\Omega|}(\mathrm{tr}\,(\mathbf{u} \otimes \mathbf{u}), 1)_\Omega \in \mathbb{R}$ we obtain

$$p = -\frac{\nu}{2}\mathrm{tr}\,(\boldsymbol{\sigma}) - \frac{1}{2}\mathrm{tr}\,(\mathbf{u} \otimes \mathbf{u}) + c_{\mathbf{u}} \quad \text{in} \quad \Omega,$$

and then, the Navier–Stokes equations can be rewritten as follows

$$\boldsymbol{\sigma}^{\mathrm{d}} = \nabla \mathbf{u} - \nu^{-1}(\mathbf{u} \otimes \mathbf{u})^{\mathrm{d}} \quad \text{in} \quad \Omega, \quad -\mathbf{div}\, \boldsymbol{\sigma} = \nu^{-1}\, \mathbf{f} \quad \text{in} \quad \Omega,$$
$$\mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma, \quad (\mathrm{tr}\,(\boldsymbol{\sigma}), 1)_\Omega = 0.$$

The mixed variational formulation

Find $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times \mathbf{L}^4(\Omega)$, such that

$$\begin{aligned} (\boldsymbol{\sigma}^{\mathrm{d}}, \boldsymbol{\tau}^{\mathrm{d}})_\Omega + (\mathbf{div}\,\boldsymbol{\tau}, \mathbf{u})_\Omega + \frac{1}{\nu}(\mathbf{u}\otimes\mathbf{u}, \boldsymbol{\tau}^{\mathrm{d}})_\Omega &= \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D\rangle_\Gamma, \quad \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \\ (\mathbf{div}\,\boldsymbol{\sigma}, \mathbf{v})_\Omega &= -\frac{1}{\nu}(\mathbf{f}, \mathbf{v})_\Omega \quad \forall \mathbf{v} \in \mathbf{L}^4(\Omega), \end{aligned} \tag{1}$$

where $\mathbb{H}_0(\mathbf{div}_{4/3};\Omega) := \{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) \,:\, \mathbf{div}\,\boldsymbol{\tau} \in \mathbf{L}^{4/3}(\Omega) \quad \text{and} \quad \int_\Omega \mathrm{tr}\,(\boldsymbol{\tau}) = 0\}$.

Theorem

Let $\mathbf{f} \in \mathbf{L}^{4/3}(\Omega)$ *and* $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$, *such that*

$$C\left(\nu^{-1}\|\mathbf{u}_D\|_{1/2,\Gamma} + \nu^{-2}\|\mathbf{f}\|_{0,4/3;\Omega}\right) < 1,$$

where $C > 0$ *only depends on* Ω. *Then, there exists a unique* $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times \mathbf{L}^4(\Omega)$ *solution to* (1). *In addition, the following estimate holds:*

$$\|\boldsymbol{\sigma}\|_{\mathbf{div}_{4/3},\Omega} + \|\mathbf{u}\|_{0,4;\Omega} \lesssim \|\mathbf{u}_D\|_{1/2,\Gamma} + \nu^{-1}\|\mathbf{f}\|_{0,4/3;\Omega}.$$

J. C, C. García and R.Oyarzúa. *Analysis of a momentum conservative mixed-FEM for the stationary Navier–Stokes problem.*

Numerical Methods for Partial Differential Equations, 2021, vol. 37, no 5, p. 2895-2923.

Since $\Omega \subset \mathbb{R}^2$ is simply connected, there exists $\epsilon > 0$, depending only on Γ, such that, for every $p \in (4/3 - \epsilon,\, 4 + \epsilon)$, the following Helmholtz–Weyl decomposition holds and is stable:

$$\mathbf{L}^p(\Omega) = \operatorname{curl} \widetilde{\mathrm{W}}^{1,p}(\Omega) \oplus \nabla \mathrm{W}_0^{1,p}(\Omega)^1,$$

where

$$\widetilde{\mathrm{W}}^{1,p}(\Omega) := \{\theta \in \mathrm{W}^{1,p}(\Omega) : (\theta, 1)_\Omega = 0\},$$

and for any scalar field θ, we define the operator curl by

$$\operatorname{curl} \theta := \left(\frac{\partial \theta}{\partial x_2}, -\frac{\partial \theta}{\partial x_1}\right)^t.$$

More precisely, for each $\mathbf{z} \in \mathbf{L}^p(\Omega)$, there exist unique $\xi \in \widetilde{\mathrm{W}}^{1,p}(\Omega)$ and $\psi \in \mathrm{W}_0^{1,p}(\Omega)$ such that

$$\mathbf{z} = \operatorname{curl} \xi + \nabla \psi \quad \text{and} \quad |\xi|_{1,p;\Omega} + |\psi|_{1,p;\Omega} \lesssim \|\mathbf{z}\|_{0,p;\Omega}.$$

[1] D. Mitrea, *Sharp L^p–Hodge decompositions for Lipschitz domains in $\mathbb{R}^2$*, Advances in Differential Equations, 2002, vol. 7, no 3, p. 343-364.

Lemma

Let $\epsilon > 0$ be the constant given in [D. Mitrea, Proposition 2.2], let $p \in (4/3-\epsilon, 4+\epsilon)$ be such that the L^p-decomposition holds, and let $q \in \mathbb{R}$ be the conjugate exponent of p, that is, $1/p + 1/q = 1$. Then there exists a constant $\beta_0 > 0$ such that

$$\sup_{0 \neq \psi \in \mathrm{W}_0^{1,p}(\Omega)} \frac{(\nabla\psi, \nabla\varphi)_\Omega}{|\psi|_{1,p;\Omega}} \geq \beta_0 \, |\varphi|_{1,q;\Omega} \quad \forall \varphi \in \mathrm{W}_0^{1,q}(\Omega). \tag{2}$$

Proof:

- Given $\varphi \in \mathrm{W}_0^{1,q}(\Omega)$, we first observe that

  $$(\left||\nabla\varphi|^{q-2}\nabla\varphi\right|^p, 1)_\Omega = (|\nabla\varphi|^q, 1)_\Omega < +\infty,$$

  which implies that $|\nabla\varphi|^{q-2}\nabla\varphi \in \mathbf{L}^p(\Omega)$.
- Then, from the L^p-decomposition, we deduce that there exist unique $\xi \in \widetilde{\mathrm{W}}^{1,p}(\Omega)$ and $\chi \in \mathrm{W}_0^{1,p}(\Omega)$, such that

  $$|\nabla\varphi|^{q-2}\nabla\varphi = \operatorname{curl}\xi + \nabla\chi,$$

  and also, since the sum is direct and topological, the following estimate holds

  $$|\xi|_{1,p;\Omega} + |\chi|_{1,p;\Omega} \lesssim \||\nabla\varphi|^{q-2}\nabla\varphi\|_{0,p;\Omega} = |\varphi|_{1,q;\Omega}^{q-1}.$$

- Then, we obtain

$$\sup_{0\neq\psi\in \mathrm{W}_0^{1,p}(\Omega)} \frac{(\nabla\psi,\nabla\varphi)_\Omega}{|\psi|_{1,p;\Omega}} \geq \frac{(\nabla\chi,\nabla\varphi)_\Omega}{|\chi|_{1,p;\Omega}} = \frac{((|\nabla\varphi|^{q-2}\nabla\varphi - \mathrm{curl}\,\xi),\nabla\varphi)_\Omega}{|\chi|_{1,p;\Omega}}$$

$$\gtrsim \frac{(|\nabla\varphi|^{q-2}\nabla\varphi,\nabla\varphi)_\Omega - (\mathrm{curl}\,\xi,\nabla\varphi)_\Omega}{|\varphi|_{1,q;\Omega}^{q-1}} = \frac{|\varphi|_{1,q;\Omega}^{q}}{|\varphi|_{1,q;\Omega}^{q-1}},$$

which completes the proof.

Equivalent formulation

Find $(\boldsymbol{\sigma}, (\omega, \varphi)) \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times (\widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}_0^{1,4}(\Omega))$, such that:

$$\begin{aligned}
(\boldsymbol{\sigma}^{\mathrm{d}}, \boldsymbol{\tau}^{\mathrm{d}})_\Omega + (\mathbf{div}\ \boldsymbol{\tau}, \operatorname{curl}\omega + \nabla\varphi)_\Omega + \frac{1}{\nu}(\operatorname{curl}\omega \otimes \operatorname{curl}\omega, \boldsymbol{\tau}^{\mathrm{d}})_\Omega &= \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D \rangle_\Gamma, \\
(\mathbf{div}\ \boldsymbol{\sigma}, \operatorname{curl}\theta + \nabla\psi)_\Omega &= -\frac{1}{\nu}(\mathbf{f}, \operatorname{curl}\theta + \nabla\psi)_\Omega,
\end{aligned}$$

for all $(\boldsymbol{\tau}, (\theta, \psi)) \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times (\widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}_0^{1,4}(\Omega))$.

Lemma

If $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times \mathbf{L}^4(\Omega)$ *is a solution to the initial variational formulation, then* $\mathbf{u} = \operatorname{curl} \omega$ *for some* $\omega \in \widetilde{\mathrm{W}}^{1,4}(\Omega)$*, and* $(\boldsymbol{\sigma}, (\omega, 0))$ *is a solution to the equivalent formulation. Conversely, if* $(\boldsymbol{\sigma}, (\omega, \varphi)) \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times (\widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}_0^{1,4}(\Omega))$ *is a solution to the equivalent formulation, then* $\varphi = 0$ *in* Ω *and* $(\boldsymbol{\sigma}, \mathbf{u}) = (\boldsymbol{\sigma}, \operatorname{curl} \omega)$ *is a solution to the initial variational formulation.*

Proof:

- Let $(\boldsymbol{\sigma}, (\omega, \varphi)) \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times \big(\widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}_0^{1,4}(\Omega)\big)$ be a solution of the equivalent formulation, and let

$$\boldsymbol{\tau}_\phi := (\phi - |\Omega|^{-1} (\phi, 1)_\Omega)\, \mathbb{I} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega), \quad \text{with} \quad \phi \in \mathrm{W}_0^{1,4/3}(\Omega).$$

- Observe that the compatibility condition, together with the fact that $\phi|_\Gamma = 0$, implies

$$\langle \boldsymbol{\tau}_\phi \mathbf{n}, \mathbf{u}_D \rangle_\Gamma = 0.$$

- Therefore, taking $\boldsymbol{\tau} = \boldsymbol{\tau}_\phi$ in the first equation of the equivalent formulation, and using that $\boldsymbol{\tau}_\phi^{\mathrm{d}} = \mathbf{0}$ and $\mathbf{div}\,(\boldsymbol{\tau}_\phi) = \nabla \phi$, we obtain

$$(\nabla \phi, \nabla \varphi)_\Omega = 0.$$

- Since $\phi \in \mathrm{W}_0^{1,4/3}(\Omega)$ is arbitrary, it follows from the inf–sup condition, with $p = 4/3$ and $q = 4$, that $|\varphi|_{1,4;\Omega} = 0$. Consequently, owing to the Poincaré inequality, we deduce that $\varphi = 0$ in Ω.
- Setting $\mathbf{u} = \operatorname{curl} \omega \in \mathbf{L}^4(\Omega)$, we conclude that $(\boldsymbol{\sigma}, \mathbf{u})$ satisfies the first equation of the initial variational formulation.
- Moreover, by the L^p-decomposition, the second equation of the equivalent formulation immediately yields the second equation of the initial variational formulation. Therefore, $(\boldsymbol{\sigma}, \mathbf{u}) = (\boldsymbol{\sigma}, \operatorname{curl} \omega)$ is a solution of the initial variational formulation.

- Conversely, let $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times \mathbf{L}^4(\Omega)$ be the unique solution of the initial variational formulation, and let $\omega \in \widetilde{\mathrm{W}}^{1,4}(\Omega)$ and $\varphi \in \mathrm{W}_0^{1,4}(\Omega)$ be such that $\mathbf{u} = \mathrm{curl}\,\omega + \nabla\varphi$.
- Then, the first equation of the initial variational formulation can be equivalently rewritten as

$$(\boldsymbol{\sigma}^{\mathrm{d}}, \boldsymbol{\tau}^{\mathrm{d}})_\Omega + (\mathbf{div}\ \boldsymbol{\tau}, \mathrm{curl}\,\omega + \nabla\varphi)_\Omega + \frac{1}{\nu}\big((\mathrm{curl}\,\omega + \nabla\varphi) \otimes (\mathrm{curl}\,\omega + \nabla\varphi), \boldsymbol{\tau}^{\mathrm{d}}\big)_\Omega = \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D \rangle_\Gamma$$

  for all $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$. But, proceeding exactly as above, that is, by taking $\boldsymbol{\tau}_\phi := (\phi - |\Omega|^{-1}(\phi, 1)_\Omega)\,\mathbb{I} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$, with arbitrary $\phi \in \mathrm{W}_0^{1,4/3}(\Omega)$, we deduce that $\varphi = 0$ in Ω, and consequently, $(\boldsymbol{\sigma}, (\omega, \varphi))$ satisfies the first equation of the equivalent formulation.
- Moreover, by applying again the L^p-decomposition , we infer that the second equation of the initial variational form immediately yields the second equation of the equivalent formulation. Therefore, $(\boldsymbol{\sigma}, (\omega, \varphi))$ is a solution of the equivalent formulation.

Variational problem

Find $(\boldsymbol{\sigma}, (\omega, \varphi)) \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times (\widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}_0^{1,4}(\Omega))$, such that:

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau}, (\omega, \varphi)) + \mathbf{c}(\omega; \omega, \boldsymbol{\tau}) &= F(\boldsymbol{\tau}) & \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega), \\ \mathbf{b}(\boldsymbol{\sigma}, (\theta, \psi)) &= G(\theta, \psi) & \forall\, (\theta, \psi) \in \widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}_0^{1,4}(\Omega). \end{aligned} \tag{3}$$

Bilinear forms

$$\mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) := (\boldsymbol{\sigma}^{\mathrm{d}}, \boldsymbol{\tau}^{\mathrm{d}})_\Omega \quad \text{and} \quad \mathbf{b}(\boldsymbol{\tau}, (\theta, \psi)) := (\mathbf{div}\ \boldsymbol{\tau}, \operatorname{curl} \theta + \nabla \psi)_\Omega .$$

Trilinear form

$$\mathbf{c}(\widehat{\omega}; \omega, \boldsymbol{\tau}) := \nu^{-1} (\operatorname{curl} \widehat{\omega} \otimes \operatorname{curl} \omega, \boldsymbol{\tau}^{\mathrm{d}})_\Omega .$$

Functionals

$$F(\boldsymbol{\tau}) := \langle \boldsymbol{\tau} \mathbf{n}, \mathbf{u}_D \rangle_\Gamma \quad \text{and} \quad G(\theta, \psi) := -\nu^{-1} (\mathbf{f}, \operatorname{curl} \theta + \nabla \psi)_\Omega .$$

The fixed-point operator

- Now, define the operator:

$$\mathcal{J} : \widetilde{\mathrm{W}}^{1,4}(\Omega) \to \widetilde{\mathrm{W}}^{1,4}(\Omega), \quad \widehat{\omega} \to \mathcal{J}(\widehat{\omega}) = \omega,$$

  where, given $\widehat{\omega} \in \widetilde{\mathrm{W}}^{1,4}(\Omega)$, ω is the second component of the solution to the linearized version of the variational problem:
  Find $(\boldsymbol{\sigma}, (\omega, \varphi)) \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times (\widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}_0^{1,4}(\Omega))$, such that

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau}, (\omega, \varphi)) + \mathbf{c}(\widehat{\omega}; \omega, \boldsymbol{\tau}) &= F(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega), \\ \mathbf{b}(\boldsymbol{\sigma}, (\theta, \psi)) &= G(\theta, \psi) \quad \forall\, (\theta, \psi) \in \widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}_0^{1,4}(\Omega). \end{aligned}$$

- $(\boldsymbol{\sigma}, (\omega, \varphi)) \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times (\widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}_0^{1,4}(\Omega))$ is a solution of the variational problem if and only if ω is a fixed point of $\mathcal{J}$, that is

$$\mathcal{J}(\omega) = \omega.$$

Well-definiteness of $\mathcal{J}$

- Kernel of $\mathbf{b}$

$$\mathbb{V} := \{\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \,:\, \mathbf{b}(\boldsymbol{\tau},(\omega,\psi)) = 0, \quad \forall\,(\omega,\psi) \in \widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}_0^{1,4}(\Omega)\}.$$

- $\mathbb{V}$ can be characterized as

$$\mathbb{V} = \{\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \,:\, \mathbf{div}\,\boldsymbol{\tau} = \mathbf{0} \quad \text{in} \quad \Omega\}.$$

- Ellipticity of $\mathbf{a}$ on $\mathbb{V}$

$$\mathbf{a}(\boldsymbol{\tau},\boldsymbol{\tau}) \gtrsim \|\boldsymbol{\tau}\|^2_{\mathbf{div}_{4/3};\Omega} \quad \forall\,\boldsymbol{\tau} \in \mathbb{V}.$$

- Inf–sup condition of $\mathbf{b}$

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)} \frac{\mathbf{b}(\boldsymbol{\tau},(\theta,\psi))}{\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega}} \gtrsim \|(\theta,\psi)\| \qquad \forall\,(\theta,\psi) \in \widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}_0^{1,4}(\Omega).$$

- Continuity of the functionals F and G

$$|F(\boldsymbol{\tau})| \le C_F \|\mathbf{u}_D\|_{1/2,\Gamma}\,\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega}, \qquad |G(\omega,\psi)| \le \frac{1}{\nu}\|\mathbf{f}\|_{0,4/3;\Omega}\,\|(\omega,\psi)\|.$$

Well-definiteness of $\mathcal{J}$

- Now we define the bilinear form $\mathbf{A} : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ given by

$$\mathbf{A}((\boldsymbol{\sigma}, \omega, \varphi), (\boldsymbol{\tau}, \theta, \psi)) := \mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau}, (\omega, \varphi)) + \mathbf{b}(\boldsymbol{\sigma}, (\theta, \psi)),$$

  with $\mathbb{X} := \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times \widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}_0^{1,4}(\Omega)$.

- $\mathbf{A}$ satisfies the following inf-sup condition

$$\sup_{\mathbf{0} \neq (\boldsymbol{\tau}, (\theta, \psi)) \in \mathbb{X}} \frac{\mathbf{A}((\boldsymbol{\zeta}, \vartheta, \phi), (\boldsymbol{\tau}, \theta, \psi))}{\|(\boldsymbol{\tau}, \theta, \psi)\|} \geq \gamma \|(\boldsymbol{\zeta}, \vartheta, \phi)\| \quad \forall (\boldsymbol{\zeta}, \vartheta, \phi) \in \mathbb{X},$$

  where γ is a positive constant independent of ν.

Lemma

Let $\mathrm{K} \subseteq \widetilde{\mathrm{W}}^{1,4}(\Omega)$ *be the convex and bounded set given by*

$$\mathrm{K} := \left\{ \widehat{\omega} \in \widetilde{\mathrm{W}}^{1,4}(\Omega) \;:\quad \|\widehat{\omega}\|_{1,4;\Omega} \leq \frac{2}{\gamma} \left(C_F \|\mathbf{u}_D\|_{1/2,\Gamma} + \frac{1}{\nu} \|\mathbf{f}\|_{0,4/3;\Omega} \right) \right\}.$$

Assume that

$$\frac{4}{\nu\gamma^2} \left(C_F \|\mathbf{u}_D\|_{1/2,\Gamma} + \frac{1}{\nu} \|\mathbf{f}\|_{0,4/3;\Omega} \right) \leq 1.$$

Then, given $\widehat{\omega} \in \mathrm{K}$*, there exists a unique* $\omega \in \mathrm{K}$ *such that* $\mathcal{J}(\widehat{\omega}) = \omega$*.*

Proof: We apply the Banach–Nečas–Babuška theorem (Extension to Banach spaces of the Babuška–Brezzi Theorem) to operator $\mathbf{A}((\cdot,\cdot,\cdot),(\cdot,\cdot,\cdot)) + \mathbf{c}(\widehat{\omega};\cdot,\cdot)$

Theorem

Let $\mathbf{f} \in \mathbf{L}^{4/3}(\Omega)$ *and* $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$ *be such that*

$$\frac{4}{\nu\gamma^2}\left(C_F\|\mathbf{u}_D\|_{1/2,\Gamma} + \frac{1}{\nu}\|\mathbf{f}\|_{0,4/3;\Omega}\right) < 1.$$

Then, there exists a unique $(\boldsymbol{\sigma}, \omega, \varphi) \in \mathbb{X} = \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times \widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}_0^{1,4}(\Omega)$ *solution to* (3). *Moreover, the solution satisfies*

$$\|(\boldsymbol{\sigma}, \omega, \varphi)\| \leq \frac{2}{\gamma}\left(C_F\|\mathbf{u}_D\|_{1/2,\Gamma} + \frac{1}{\nu}\|\mathbf{f}\|_{0,4/3;\Omega}\right).$$

Proof: For $\widehat{\omega}_1, \widehat{\omega}_2, \omega_1, \omega_2 \in \mathrm{K}$, such that $\mathcal{J}(\widehat{\omega}_1) = \omega_1$ and $\mathcal{J}(\widehat{\omega}_2) = \omega_2$ we observe that

$$\|\omega_1 - \omega_2\|_{1,4;\Omega} \leq \frac{4}{\nu\gamma^2}\left(C_F\|\mathbf{u}_D\|_{1/2,\Gamma} + \frac{1}{\nu}\|\mathbf{f}\|_{0,4/3;\Omega}\right)\|\widehat{\omega}_1 - \widehat{\omega}_2\|_{1,4;\Omega}.$$

and use the hypotheses and the classical Banach's fixed point theorem.

Galerkin scheme

- Discrete spaces

$$\mathbb{H}_h^{\boldsymbol{\sigma}} := \quad \left\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div}_{4/3};\Omega) \,:\, \mathbf{c}^t \boldsymbol{\tau}_h|_T \in \mathbf{RT}_0(T), \quad \forall\, \mathbf{c} \in \mathbb{R}^2 \quad \forall\, T \in \mathcal{T}_h \right\},$$

$$\mathrm{H}_h^{\omega} := \quad \left\{ \theta_h \in \mathrm{C}(\overline{\Omega}) \,:\, \theta_h|_T \in \mathrm{P}_1(T), \quad \forall\, T \in \mathcal{T}_h \right\},$$

$$\mathrm{H}_h^{\varphi} := \quad \Big\{ \psi_h \in \mathrm{L}^2(\Omega) :\; \psi_h|_T \in P_1(T)\ \forall\, T \in \mathcal{T}_h, \quad (\llbracket \psi_h \rrbracket, 1)_e = 0\ \forall\, e \in \mathcal{E}_h(\Omega),$$

$$\text{and} \quad (\psi_h, 1)_e = 0\ \forall\, e \in \mathcal{E}_h(\Gamma) \Big\}.$$

$$\mathbb{H}_{h,0}^{\boldsymbol{\sigma}} := \mathbb{H}_h^{\boldsymbol{\sigma}} \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \quad \text{and} \quad \mathrm{H}_{h,0}^{\omega} := \mathrm{H}_h^{\omega} \cap \widetilde{\mathrm{W}}^{1,4}(\Omega),$$

Discrete variational problem

Find $(\boldsymbol{\sigma}_h, (\omega_h, \varphi_h)) \in \mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \times (\mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_h)$, such that:

$$\begin{aligned}
\mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathbf{b}_h(\boldsymbol{\tau}_h, (\omega_h, \varphi_h)) + \mathbf{c}(\omega_h; \omega_h, \boldsymbol{\tau}_h) &= F(\boldsymbol{\tau}_h) \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}^{\boldsymbol{\sigma}}_{h,0}, \\
\mathbf{b}_h(\boldsymbol{\sigma}_h, (\theta_h, \psi_h)) &= G_h(\theta_h, \psi_h) \quad \forall\, (\theta_h, \psi_h) \in \mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_h
\end{aligned}$$

Discrete bilinear form

$\mathbf{b}_h : \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times (\widetilde{\mathrm{W}}^{1,4}(\Omega) \times [\mathrm{W}^{1,4}_0(\Omega) + \mathrm{H}^{\varphi}_h]) \to \mathbb{R}$

$$\mathbf{b}_h(\boldsymbol{\tau}, (\omega_h, \psi_h)) := (\mathbf{div}\, \boldsymbol{\tau}, \operatorname{curl} \omega_h + \nabla_h \psi_h)_\Omega.$$

Discrete functional

$G_h : \widetilde{\mathrm{W}}^{1,4}(\Omega) \times [\mathrm{W}^{1,4}_0(\Omega) + \mathrm{H}^{\varphi}_h] \to \mathbb{R}$

$$G_h(\omega_h, \psi_h) := -\frac{1}{\nu}(\mathbf{f}, \operatorname{curl} \omega_h + \nabla_h \psi_h)_\Omega.$$

- Discrete Helmholtz decomposition

$$[P_0(\mathcal{T}_h)]^2 = \operatorname{curl} \mathrm{H}^{\omega}_{h,0} \oplus \nabla_h \mathrm{H}^{\varphi}_h,$$

- We equip H^{φ}_h with the following broken norm:

$$\|\psi_h\|_h = \left(\sum_{T \in \mathcal{T}_h} |\psi_h|^4_{1,4;T} \right)^{1/4}.$$

- There exists $C > 0$, independent of h and ν, such that

$$|\theta_h|_{1,4;\Omega} + \|\phi_h\|_h \leq C \|\mathbf{z}_h\|_{0,4;\Omega},$$

for all $\mathbf{z}_h \in [P_0(\mathcal{T}_h)]^2$, $\theta_h \in \mathrm{H}^{\omega}_{h,0}$ and $\phi_h \in \mathrm{H}^{\varphi}_h$, such that $\mathbf{z}_h = \operatorname{curl} \theta_h + \nabla_h \phi_h$.

The discrete fixed-point operator

- The discrete fixed-point operator is

$$\mathcal{J}_h : \mathrm{H}^{\omega}_{h,0} \to \mathrm{H}^{\omega}_{h,0}, \quad \widehat{\omega}_h \to \mathcal{J}_h(\widehat{\omega}_h) = \omega_h,$$

  where, for a given $\widehat{\omega}_h \in \mathrm{H}^{\omega}_{h,0}$, the function ω_h denotes the second component of the solution to the following problem:
  Find $(\boldsymbol{\sigma}_h, (\omega_h, \varphi_h)) \in \mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \times (\mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_h)$ such that

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathbf{b}_h(\boldsymbol{\tau}_h, (\omega_h, \varphi_h)) + \mathbf{c}(\widehat{\omega}_h; \omega_h, \boldsymbol{\tau}_h) &= F(\boldsymbol{\tau}_h) \qquad && \forall\, \boldsymbol{\tau}_h \in \mathbb{H}^{\boldsymbol{\sigma}}_{h,0}, \\ \mathbf{b}_h(\boldsymbol{\sigma}_h, (\theta_h, \psi_h)) &= G_h(\theta_h, \psi_h) && \forall\, (\theta_h, \psi_h) \in \mathrm{H}^{\omega}_{h,0} \end{aligned}$$

- $(\boldsymbol{\sigma}_h, (\omega_h, \varphi_h)) \in \mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \times (\mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_h)$ is a solution of (3) if and only if ω_h is a fixed point of $\mathcal{J}_h$, that is

$$\mathcal{J}_h(\omega_h) = \omega_h.$$

Well-definiteness of $\mathcal{J}_h$

- $$|\mathbf{b}_h(\boldsymbol{\tau}_h, (\omega_h, \psi_h))| \leq \|\boldsymbol{\tau}_h\|_{\mathbf{div}\,_{4/3};\Omega}\, \|(\omega_h, \psi_h)\| \quad \text{and} \quad |G_h(\omega_h, \psi_h)| \leq \frac{1}{\nu} \|\mathbf{f}\|_{0,4/3;\Omega}\, \|(\omega_h, \psi_h$$

- Discrete kernel of $\mathbf{b}_h$

$$\mathbb{V}_h := \left\{ \boldsymbol{\tau}_h \in \mathbb{H}^{\boldsymbol{\sigma}}_{h,0} :\ \mathbf{b}_h(\boldsymbol{\tau}_h, (\omega_h, \psi_h)) = 0, \quad \forall\, (\omega_h, \psi_h) \in \mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_h \right\}.$$

- Since $\mathbf{div}\,(\mathbb{H}^{\boldsymbol{\sigma}}_{h,0}) \subseteq [P_0(\mathcal{T}_h)]^2$, then $\mathbb{V}_h$ can be characterized as

$$\mathbb{V}_h = \{ \boldsymbol{\tau}_h \in \mathbb{H}^{\boldsymbol{\sigma}}_{h,0} :\ \mathbf{div}\ \boldsymbol{\tau}_h = \mathbf{0} \quad \text{in} \quad \Omega \}.$$

- Ellipticity of $\mathbf{a}$ on $\mathbb{V}_h$

$$\mathbf{a}(\boldsymbol{\tau}_h, \boldsymbol{\tau}_h) \gtrsim \|\boldsymbol{\tau}_h\|^2_{\mathbf{div}\,_{4/3};\Omega} \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{V}_h.$$

- Discrete inf–sup condition on $\mathbf{b}_h$

$$\sup_{\mathbf{0} \neq \boldsymbol{\tau}_h \in \mathbb{H}^{\boldsymbol{\sigma}}_{h,0}} \frac{\mathbf{b}_h(\boldsymbol{\tau}_h, (\omega_h, \psi_h))}{\|\boldsymbol{\tau}_h\|_{4/3,\mathbf{div}\,,\Omega}} \gtrsim \|(\omega_h, \psi_h)\| \qquad \forall\, (\omega_h, \psi_h) \in \mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_h.$$

Lemma

Let $\mathrm{K}_h \subseteq \mathrm{H}^{\omega}_{h,0}$ *be the convex and bounded set given by*

$$\mathrm{K}_h := \left\{ \omega_h \in \mathrm{H}^{\omega}_{h,0} \;:\quad \|\omega_h\|_{1,4;\Omega} \leq \frac{2}{\tilde{\gamma}} \left(C_F \|\mathbf{u}_D\|_{1/2,\Gamma} + \frac{1}{\nu} \|\mathbf{f}\|_{0,4/3;\Omega} \right) \right\}.$$

Assume that

$$\frac{4}{\nu \tilde{\gamma}^2} \left(C_F \|\mathbf{u}_D\|_{1/2,\Gamma} + \frac{1}{\nu} \|\mathbf{f}\|_{0,4/3;\Omega} \right) \leq 1.$$

Then, given $\widehat{\omega}_h \in \mathrm{K}_h$*, there exists a unique* $\omega_h \in \mathrm{K}_h$ *such that* $\mathcal{J}_h(\widehat{\omega}_h) = \omega_h$*.*

Theorem

Let $\mathbf{f} \in \mathbf{L}^{4/3}(\Omega)$ *and* $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$ *such that*

$$\frac{4}{\nu \tilde{\tilde{\gamma}}^2} \left(C_F \|\mathbf{u}_D\|_{1/2,\Gamma} + \frac{1}{\nu} \|\mathbf{f}\|_{0,4/3;\Omega} \right) < 1.$$

Then, the operator $\mathcal{J}_h$ *has a unique fixed-point* $\omega_h \in \mathrm{K}_h$*. Equivalently, the discrete problem has a unique solution* $(\boldsymbol{\sigma}_h, (\omega_h, \varphi_h)) \in \mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \times (\mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_{h})$*. Moreover, there hold*

$$\|(\boldsymbol{\sigma}_h, (\omega_h, \varphi_h))\| \leq \frac{2}{\tilde{\tilde{\gamma}}} \left(C_F \|\mathbf{u}_D\|_{1/2,\Gamma} + \frac{1}{\nu} \|\mathbf{f}\|_{0,4/3;\Omega} \right).$$

Conservativity

- From the second equation of our problem, we have

$$\left(\mathbf{div}\ \boldsymbol{\sigma}_h + \nu^{-1}\mathbf{f}, \operatorname{curl}\theta_h + \nabla_h\psi_h\right)_\Omega = 0 \quad \forall\,(\theta_h,\psi_h) \in \mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_h$$

  which implies

$$\mathbf{div}\ \boldsymbol{\sigma}_h = -\nu^{-1}\mathbf{P}_h(\mathbf{f}).$$

  Consequently, the method satisfies the discrete equilibrium equation exactly, and hence it is **momentum-conservative** whenever $\mathbf{f} \in [P_0(\mathcal{T}_h)]^2$.
  If $\mathbf{f} \notin [P_0(\mathcal{T}_h)]^2$, then, under the additional regularity assumption $\mathbf{f} \in \mathbf{W}^{1,4/3}(\Omega)$, we obtain

$$\|\nu^{-1}\mathbf{f} - \mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,4/3;\Omega} = \nu^{-1}\|\mathbf{f} - \mathbf{P}_h(\mathbf{f})\|_{0,4/3;\Omega} \le C\nu^{-1}h|\mathbf{f}|_{1,4/3;\Omega},$$

- Recalling that

$$\operatorname{curl}\mathrm{H}^{\omega}_{h,0} = \{\mathbf{v}_h \in \mathbf{H}(\mathrm{div};\Omega) : \mathbf{v}_h|_T \in \mathbf{RT}_0(T), \quad \forall\, T \in \mathcal{T}_h\} \cap \mathbf{H}(\mathrm{div}^0;\Omega),$$

  by setting

$$\mathbf{u}_h = \operatorname{curl}\omega_h \quad \text{in} \quad \Omega,$$

  we obtain an exactly divergence-free approximation of the velocity $\mathbf{u}$, and hence the method is **mass-conservative**.

Convergence

Key point:

$$\mathbf{b}_h(\boldsymbol{\tau}, (\theta, \psi)) = \mathbf{b}(\boldsymbol{\tau}, (\theta, \psi)), \quad \forall \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega), \ \forall (\theta, \psi) \in \widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}_0^{1,4}(\Omega).$$

Theorem

Assume that

$$\frac{8}{\nu \min\{\gamma, \tilde{\gamma}\}^2} \left(C_F \|\mathbf{u}_D\|_{1/2,\Gamma} + \frac{1}{\nu} \|\mathbf{f}\|_{0,4/3;\Omega} \right) \leq 1. \tag{4}$$

Let $(\boldsymbol{\sigma}, (\omega, 0)) \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times (\widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}_0^{1,4}(\Omega))$ *and* $(\boldsymbol{\sigma}_h, (\omega_h, \varphi_h)) \in \mathbb{H}_{h,0}^{\boldsymbol{\sigma}} \times (\mathrm{H}_{h,0}^{\omega} \times \mathrm{H}_h^{\varphi})$ *be the unique solutions of the continuous and discrete problems, respectively. Then, there holds*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega} + \|\omega - \omega_h\|_{1,4;\Omega} + \|\varphi_h\|_h \lesssim \inf_{(\boldsymbol{\tau}_h, \theta_h) \in \mathbb{H}_{h,0}^{\boldsymbol{\sigma}} \times \mathrm{H}_{h,0}^{\omega}} \left\{ \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega} + \|\omega - \theta_h\|_{1,4;\Omega} \right\}.$$

Theorem

Let $(\boldsymbol{\sigma}, (\omega, 0)) \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times (\widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}_0^{1,4}(\Omega))$ *and* $(\boldsymbol{\sigma}_h, (\omega_h, \varphi_h)) \in \mathbb{H}_{h,0}^{\boldsymbol{\sigma}} \times (\mathrm{H}_{h,0}^{\omega} \times \mathrm{H}_h^{\varphi})$ *be the unique solutions of the continuous and discrete problems, respectively, with* $\mathbf{f}$ *and* $\mathbf{u}_D$ *satisfying* (4), *and assume that* $\boldsymbol{\sigma} \in \mathbb{H}^1(\Omega)$, $\mathbf{div}\, \boldsymbol{\sigma} \in \mathbf{W}^{1,4/3}(\Omega)$ *and* $\omega \in \mathrm{W}^{2,4}(\Omega)$. *Then, the following convergence estimate holds:*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega} + \|\omega - \omega_h\|_{1,4;\Omega} + \|\varphi_h\|_h \lesssim h \left\{ |\boldsymbol{\sigma}|_{1,\Omega} + |\mathbf{div}\, \boldsymbol{\sigma}|_{1,4/3;\Omega} + \|\omega\|_{2,4;\Omega} \right\}.$$

Computing further variables of interest

Recalling that the velocity can be recovered from the stream function as $\mathbf{u} = \operatorname{curl}\omega$, we can recover the pressure p, the velocity gradient $\mathbf{G} := \nabla\mathbf{u}$, the vorticity $\boldsymbol{\gamma} := \frac{1}{2}\big(\nabla\mathbf{u} - \nabla\mathbf{u}^t\big)$, and the stress tensor $\widetilde{\boldsymbol{\sigma}} := \nu(\nabla\mathbf{u} + \nabla\mathbf{u}^t) - p\mathbb{I}$ as follows:

$$
\begin{aligned}
p &= -\frac{1}{2}\left(\nu\,\mathrm{tr}\,(\boldsymbol{\sigma}) + \mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u}) - \frac{1}{|\Omega|}\big(\mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u}),1\big)_\Omega\right),\\
\mathbf{G} &= \boldsymbol{\sigma}^{\mathrm{d}} + \frac{1}{\nu}(\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}},\\
\boldsymbol{\gamma} &= \frac{1}{2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}^t),\\
\widetilde{\boldsymbol{\sigma}} &= \nu(\boldsymbol{\sigma}^{\mathrm{d}} + \boldsymbol{\sigma}^t) - \frac{1}{2|\Omega|}\big(\mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u}),1\big)_\Omega\,\mathbb{I} + 2(\mathbf{u}\otimes\mathbf{u}) - \frac{1}{2}\mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u})\mathbb{I}.
\end{aligned}
$$

Corollary

Let $(\boldsymbol{\sigma},(\omega,0)) \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times (\widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}_0^{1,4}(\Omega))$ *and* $(\boldsymbol{\sigma}_h,(\omega_h,\varphi_h)) \in \mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \times (\mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_h)$ *be the unique solutions of the continuous and discrete problems, respectively, with* $\mathbf{f}$ *and* $\mathbf{u}_D$ *satisfying* (4), *let* $\mathbf{u} = \operatorname{curl}\omega$ *and* $\mathbf{u}_h = \operatorname{curl}\omega_h$ *and assume that* $\boldsymbol{\sigma} \in \mathbb{H}^1(\Omega)$, $\mathbf{div}\,\boldsymbol{\sigma} \in \mathbf{W}^{1,4/3}(\Omega)$ *and* $\omega \in \mathrm{W}^{2,4}(\Omega)$. *Then, there holds*

$$
\begin{aligned}
&\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega} + \|p-p_h\|_{0,\Omega} + \|\widetilde{\boldsymbol{\sigma}}-\widetilde{\boldsymbol{\sigma}}_h\|_{0,\Omega} + \|\mathbf{G}-\mathbf{G}_h\|_{0,\Omega} + \|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_{0,\Omega}\\
&\qquad \lesssim h\big\{|\boldsymbol{\sigma}|_{1,\Omega} + |\mathbf{div}\,\boldsymbol{\sigma}|_{1,4/3;\Omega} + \|\omega\|_{2,4;\Omega}\big\}.
\end{aligned}
$$

Numerical result

Numerical example 1: Manufactured exact solution in $\Omega = (0,1)^2$:

$$\mathbf{u}(x,y) := \begin{pmatrix} \pi \exp(x)\cos(\pi y) \\ -\exp(x)\sin(\pi y) \end{pmatrix}, \qquad p(x,y) := x^3 + y^3 - \frac{1}{2},$$

Errors and rates of convergence for the mixed $\mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \times \mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_{h}$ approximations

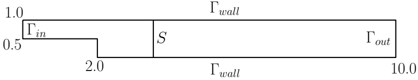
h	DOF	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e(ω)	r(ω)	e(φ)	r(φ)	Iter
0.373	213	11.1e+01	–	1.37e-00	–	8.15e-01	–	4
0.196	825	4.93e-00	1.203	6.16e-01	1.175	3.72e-01	1.158	4
0.098	3149	2.51e-00	1.012	3.02e-01	1.067	1.82e-01	1.069	4
0.048	12337	1.26e-00	1.011	1.50e-01	1.022	9.01e-02	1.028	4
0.028	48883	6.27e-01	1.009	7.56e-02	0.997	4.62e-02	0.971	4
0.014	196755	3.07e-01	1.025	3.65e-02	1.045	2.25e-02	1.036	3
0.007	775189	1.53e-01	1.015	1.86e-02	0.982	1.11e-02	1.029	3

Postprocessed variables

e($\mathbf{u}$)	r($\mathbf{u}$)	e(p)	r(p)	e($\mathbf{G}$)	r($\mathbf{G}$)	e($\boldsymbol{\gamma}$)	r($\boldsymbol{\gamma}$)	e($\widetilde{\boldsymbol{\sigma}}$)	r($\widetilde{\boldsymbol{\sigma}}$)
1.43e-00	–	1.92e-00	–	5.57e-00	–	3.04e-00	–	9.95e-00	–
6.54e-01	1.155	8.07e-01	1.281	2.46e-00	1.205	1.61e-00	0.940	4.01e-00	1.341
3.27e-01	1.037	3.48e-01	1.257	1.22e-00	1.051	8.13e-01	1.016	1.97e-00	1.060
1.60e-01	1.042	1.55e-01	1.179	6.28e-01	0.972	4.34e-01	0.920	9.81e-01	1.024
8.17e-02	0.979	7.74e-02	1.014	3.11e-01	1.022	2.12e-01	1.037	4.91e-01	1.005
3.97e-02	1.036	3.65e-02	1.081	1.51e-01	1.039	1.04e-01	1.032	2.37e-01	1.046
1.99e-02	1.011	1.75e-02	1.067	7.59e-02	0.998	5.33e-02	0.970	1.17e-01	1.036

Table: EXAMPLE 1: Mesh sizes, degrees of freedom, errors, rates of convergence, and number of iterations for the mixed $\mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \times \mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_{h}$ approximations of the Navier–Stokes equations.

Numerical Example 2: Backward-facing step flow (mass loss):
Domain: we consider the domain $\Omega = [0,10] \times [0,1]$ with a re-entrant corner at $(2, 0.5)$.



Data: We let $\nu = 1.0$ and $\mathbf{f}(x_1, x_2) = \mathbf{0}$ in Ω. The boundary conditions are prescribed as follows

$$\mathbf{u}_D = (8(x_2 - 0.5)(1 - x_2), 0)^t \quad \text{on} \quad \Gamma_{in}, \quad \mathbf{u}_D = \mathbf{0} \quad \text{on} \quad \Gamma_{wall}, \quad \boldsymbol{\sigma}\mathbf{n} = \mathbf{0} \quad \text{on} \quad \Gamma_{out}.$$

Percentage mass loss across the surface S:

$$\% m_{loss} := 100 \left| \int_{\Gamma_{in}} \mathbf{u}_h \cdot \mathbf{n} - \int_S \mathbf{u}_h \cdot \mathbf{n}_S \right| \Big/ \left| \int_{\Gamma_{in}} \mathbf{u}_h \cdot \mathbf{n} \right|.$$

- We use a computational grid consisting of 189.350 triangles with a mesh size of $h = 0.0185$, leading to a total of $N = 951.151$ degrees of freedom for both schemes.
- We compare the mass losses obtained with the proposed pseudostress-stream-function formulation (solid line) with those obtained using the formulation of the the standar discrete pseudostress-based scheme (dashed line). The comparison is performed along 100 lines S uniformly distributed throughout Ω.