Banach spaces-based mixed finite element method for nonlinear problems in fluid mechanic

Jessika Camaño

Departamento de Matemática y Física Aplicadas, UCSC and CI^{2}MA, UDEC, Chile.

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Preliminary results

Preliminary results

Definition (Regular domain)

An open set $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$, is said to be a C^1*-regular domain* if Ω is bounded, lies locally on one side of its boundary $\Gamma = \partial\Omega$, and there exist open sets $X_0, X_1, \ldots, X_N \subset \mathbb{R}^n$ such that:

(a) $\overline{X}_0 \subset \Omega$.

(b) $\overline{\Omega} \subset \bigcup_{i=0}^{N} X_i$.

(c) For each $i = 1, \ldots, N$, there exists an invertible mapping

$$\psi_i : X_i \longrightarrow B(0,1),$$

with $\psi_i \in C^1(X_i)$, $\quad \psi_i^{-1} \in C^1(B(0,1))$, such that

$$\psi_i(X_i \cap \Omega) := \left\{x := (x', x_n) \in \mathbb{R}^n : \|x\| < 1,\ x_n > 0\right\} = B(0,1) \cap \mathbb{R}^n_+,$$

and

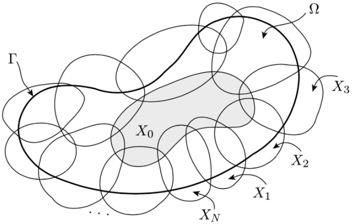
$$\psi_i(X_i \cap \Gamma) := \left\{x := (x', x_n) \in \mathbb{R}^n : \|x\| < 1,\ x_n = 0\right\} = B(0,1) \cap \partial\mathbb{R}^n_+.$$

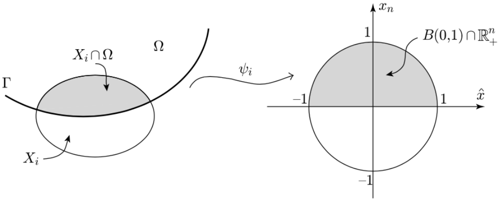
In this case, the collection

$$\{(X_i, \psi_i)\}_{i=1}^{N}$$

is called a **system of local coordinate charts** defining the boundary Γ.

Preliminary results





Preliminary results

- If, in the above definition,
$$\psi_i,\ \psi_i^{-1} \in C^m,$$
then Ω is said to be an m***-regular domain*** (or a C^m ***domain***). Likewise, if
$$\psi_i,\ \psi_i^{-1} \in C^{0,1},$$
that is, if both mappings are Lipschitz continuous, then Ω is said to be a $C^{0,1}$ ***domain*** (or a ***Lipschitz domain***).
- Let $M, N \subset \mathbb{R}^n$, and let $f : M \longrightarrow N$ be a function. We say that f is ***Lipschitz continuous*** if there exists a constant $K > 0$ such that
$$\|f(x) - f(y)\| \leq K\|x - y\|, \qquad \forall\, x, y \in M.$$

Preliminary results

- For $1 \leq p \leq +\infty$, let

$$\mathrm{L}^p(\Omega) := \{f : \Omega \to \mathbb{R} : \|f\|_{0,p,\Omega} \leq +\infty\},$$

  where

$$\|f\|_{0,p,\Omega} := \left(\int_\Omega |f|^p\right)^{\frac{1}{p}}, \quad \text{para } 1 \leq p < +\infty$$

$$\|f\|_{0,\infty,\Omega} = \text{ínf}\{M \geq 0 : |f(x)| \leq M \text{ c.e.t.p. en } \Omega\}$$

- We consider the Sobolev spaces $\mathrm{W}^{r,p}(\Omega)$ with $r \geq 0$, endowed with the norm $\|\cdot\|_{\mathrm{W}^{r,p}(\Omega)}$.
- Note that $\mathrm{W}^{0,p}(\Omega) = \mathrm{L}^p(\Omega)$ and if $p = 2$, we write $\mathrm{H}^r(\Omega)$ in place of $\mathrm{W}^{r,2}(\Omega)$, with the corresponding Lebesgue and Sobolev norms denoted by $\|\cdot\|_{0,\Omega}$ and $\|\cdot\|_{r,\Omega}$, respectively.

Preliminary results

- We write $|\cdot|_{r,\Omega}$ for the H^r-seminorm.
- $\mathrm{H}^{1/2}(\Gamma)$ is the spaces of traces of functions of $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^{-1/2}(\Gamma)$ denotes its dual. With $\langle\cdot,\cdot\rangle$ we denote the corresponding product of duality between $\mathrm{H}^{1/2}(\Gamma)$ and $\mathrm{H}^{-1/2}(\Gamma)$.
- By $\mathbf{S}$ and $\mathbb{S}$ we will denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space S.
- For any vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$ we set the **gradient**, **divergence** and **tensor product operators**, as

$$\nabla \mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}, \quad \operatorname{div} \mathbf{v} := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \mathbf{v} \otimes \mathbf{w} := (v_i w_j)_{i,j=1,n}.$$

- For any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}\ \boldsymbol{\tau}$ be the **divergence operator** div acting along the rows of $\boldsymbol{\tau}$, and define the **transpose**, **the trace**, the **tensor inner product**, and the **deviatoric tensor**, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\boldsymbol{\tau}_{ji})_{i,j=1,n}, \quad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij}, \text{ and } \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\operatorname{tr}(\boldsymbol{\tau})\mathbb{I},$$

where $\mathbb{I}$ is the identity tensor in $\mathbb{R}^{n\times n}$.

Preliminary results

- We denote
$$(v,w)_\Omega := \int_\Omega vw, \quad (\mathbf{v},\mathbf{w})_\Omega := \int_\Omega \mathbf{v}\cdot\mathbf{w}, \quad (\mathbf{v},\mathbf{w})_\Gamma := \int_\Gamma \mathbf{u}\cdot\mathbf{v} \quad \text{and} \quad (\boldsymbol{\tau},\boldsymbol{\zeta})_\Omega := \int_\Omega \boldsymbol{\tau} : \boldsymbol{\zeta}.$$
- We recall that the Hilbert space
$$\mathbf{H}(\mathrm{div};\Omega) := \left\{\tau \in \mathbf{L}^2(\Omega) : \quad \mathrm{div}\,\tau \in \mathrm{L}^2(\Omega)\right\},$$
equipped with the usual norm
$$\|\tau\|^2_{\mathrm{div},\Omega} := \|\tau\|^2_{0,\Omega} + \|\mathrm{div}\,\tau\|^2_{0,\Omega}.$$
- The tensor version of $\mathbf{H}(\mathrm{div};\Omega)$ corresponds to
$$\mathbb{H}(\mathbf{div}\,;\Omega) := \left\{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}\ \boldsymbol{\tau} \in \mathbf{L}^2(\Omega)\right\},$$
whose norm will be denoted $\|\cdot\|_{\mathbf{div}\,,\Omega}$.
- Given $p > 1$ if $n = 2$ and $p \geq 6/5$ if $n = 3$, in what follows we will also employ the non-standard Banach space $\mathbb{H}(\mathbf{div}_p\,;\Omega)$ defined by
$$\mathbb{H}(\mathbf{div}_p\,;\Omega) := \{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) \ : \ \mathbf{div}\ \boldsymbol{\tau} \in \mathbf{L}^p(\Omega)\},$$
endowed with the norm
$$\|\boldsymbol{\tau}\|_{\mathbf{div}_p\,,\Omega} := \left(\|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\mathbf{div}\ \boldsymbol{\tau}\|^2_{\mathbf{L}^p(\Omega)}\right)^{1/2}.$$

Preliminary results

- We define the sign function sgn, given by

$$\mathrm{sgn}(v) := \begin{cases} 1 & \text{if} \quad v \geq 0, \\ -1 & \text{if} \quad v < 0, \end{cases}$$

for any scalar function v. It is quite clear that for a given v, there holds

$$v \,\mathrm{sgn}(v) = |v|.$$

- For a vector function $\mathbf{v} = (v_i)_{i=1,n}$, we extend the sign function as follows: $\mathbf{sgn}(\mathbf{v})_i = \mathrm{sgn}(v_i)$ and observe that $\mathbf{sgn}(\mathbf{v}) \cdot \mathbf{v} = |v_1| + |v_2| + ... + |v_n|$, hence

$$|\mathbf{v}| \leq \mathbf{sgn}(\mathbf{v}) \cdot \mathbf{v} \leq \sqrt{n}|\mathbf{v}|.$$

Note also that, for all $\mathbf{v} \in \mathbb{R}^n$ we have the property $|\mathbf{sgn}(\mathbf{v})| = \sqrt{n}$. Above, $|\cdot|$ denotes the absolute value or the Euclidean norm in $\mathbb{R}^n$

Preliminary results

- **Hölder inequality:**

$$\int_\Omega |fg| \leq \|f\|_{\mathrm{L}^p(\Omega)} \|g\|_{\mathrm{L}^q(\Omega)}, \quad \forall\, f \in \mathrm{L}^p(\Omega),\ \forall\, g \in \mathrm{L}^q(\Omega), \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

- **Classical estimates:**

$$\|w\|_{1,\Omega} \leq C_P |w|_{1,\Omega} \quad \forall w \in \mathrm{H}^1_0(\Omega)$$

and

$$\|w\|_{\mathrm{L}^r(\Omega)} \leq C_{Sob} \|w\|_{1,\Omega} \quad \forall\, w \in \mathrm{H}^1(\Omega)\,, \quad \text{for } r \geq 1 \text{ if } n = 2 \text{ or } r \in [1,6] \text{ if } n = 3,$$

with $C_P > 0$ and $C_{Sob} > 0$ depending only on $|\Omega|$.

JOINT WORK WITH C. GARCÍA AND R. OYARZÚA

Navier–Stokes problem

The steady-state Navier-Stokes problem

- **Equations**:

$$
\begin{aligned}
-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in} \quad \Omega,\\
\operatorname{div}\mathbf{u} &= \mathbf{0} \quad \text{in} \quad \Omega,\\
\mathbf{u} &= \mathbf{u}_D \quad \text{on} \quad \Gamma,\\
\int_\Omega p &= 0.
\end{aligned}
$$

- **Unknowns**: velocity $\mathbf{u}$ and pressure p.
- **Parameters and Data**: $\nu > 0$, viscosity, $\mathbf{f}$ external force on Ω and $\mathbf{u}_D$ velocity on Γ.
- **Compatibility condition on $\mathbf{u}_D$**:

$$\int_\Gamma \mathbf{u}_D \cdot \mathbf{n} = 0.$$

- **Geometry**: $\Omega \subset \mathbb{R}^n$, $n \in \{2,3\}$, a bounded domain with Lipschitz boundary Γ and $\mathbf{n}$ the outward unit normal vector.

The nonlinear pseudostress tensor

- We introduce $\boldsymbol{\sigma} := \nu\nabla\mathbf{u} - p\mathbb{I} - \mathbf{u}\otimes\mathbf{u}$ in Ω.
- The equilibrium equation remains:

$$-\,\mathbf{div}\,\boldsymbol{\sigma} = \mathbf{f} \quad \text{in} \quad \Omega.$$

- From incompressibility condition $\operatorname{div}\mathbf{u} = \mathbf{0}$:

$$\operatorname{tr}(\boldsymbol{\sigma}) = -np - \operatorname{tr}(\mathbf{u}\otimes\mathbf{u}) \quad \text{in} \quad \Omega$$

- Pressure in terms of $\boldsymbol{\sigma}$ and $\mathbf{u}$:

$$p = -\frac{1}{n}(\operatorname{tr}(\boldsymbol{\sigma}) + \operatorname{tr}(\mathbf{u}\otimes\mathbf{u})) \quad \text{in} \quad \Omega.$$

- We rewrite Navier–Stokes equations as:

$$\begin{aligned} \boldsymbol{\sigma}^d &= \nu\nabla\mathbf{u} - (\mathbf{u}\otimes\mathbf{u})^d \quad \text{in} \quad \Omega, & -\,\mathbf{div}\,\boldsymbol{\sigma} &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \mathbf{u} &= \mathbf{u}_D \quad \text{on} \quad \Gamma, & (\operatorname{tr}(\boldsymbol{\sigma}), 1)_\Omega &= -(\operatorname{tr}(\mathbf{u}\otimes\mathbf{u}), 1)_\Omega. \end{aligned}$$

where

$$\boldsymbol{\sigma}^d := \boldsymbol{\sigma} - \frac{1}{n}(\operatorname{tr}\boldsymbol{\sigma})\mathbb{I}, \quad \text{and} \quad \mathbf{div}\,\boldsymbol{\sigma} = \big(\operatorname{div}(\sigma_{i1}, \ldots, \sigma_{in})\big)_{1\le i\le n}.$$

Further variables of interest

- Pressure;

$$p = -\frac{1}{n}(\mathrm{tr}\,(\boldsymbol{\sigma}) + \mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u}))$$

- Vorticity $\boldsymbol{\omega} := \frac{1}{2}(\nabla\mathbf{u} - \nabla\mathbf{u}^t)$;

$$\boldsymbol{\omega} = \frac{1}{2\nu}(\boldsymbol{\sigma} - \boldsymbol{\sigma}^t).$$

- Stress $\tilde{\boldsymbol{\sigma}} := \nu(\nabla\mathbf{u} + (\nabla\mathbf{u})^t) - p\mathbb{I}$;

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^d + (\mathbf{u}\otimes\mathbf{u})^d + \boldsymbol{\sigma}^t + \mathbf{u}\otimes\mathbf{u}$$

- Velocity gradient $\mathbf{G} = \nabla\mathbf{u}$;

$$\mathbf{G} = \frac{1}{\nu}(\boldsymbol{\sigma}^d + (\mathbf{u}\otimes\mathbf{u})^d)$$

A conservative mixed variational formulation

- Find $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{X} \times \mathbf{M}$, such that $(\operatorname{tr}(\boldsymbol{\sigma}), 1)_\Omega = -(\operatorname{tr}(\mathbf{u} \otimes \mathbf{u}), 1)_\Omega$, and:

$$\begin{aligned}
\frac{1}{\nu}(\boldsymbol{\sigma}^d, \boldsymbol{\tau}^d)_\Omega + (\mathbf{div}\ \boldsymbol{\tau}, \mathbf{u})_\Omega + \frac{1}{\nu}(\mathbf{u} \otimes \mathbf{u}, \boldsymbol{\tau}^d)_\Omega &= \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D \rangle, \\
(\mathbf{div}\ \boldsymbol{\sigma}, \mathbf{v})_\Omega &= -(\mathbf{f}, \mathbf{v})_\Omega,
\end{aligned}$$

  for all $(\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{X} \times \mathbf{M}$.
- Trial space for velocity $\mathbf{u}$[1]

$$\mathbf{M} := \mathbf{L}^4(\Omega).$$

- Trial space for nonlinear-pseudostress $\boldsymbol{\sigma}$

$$\mathbb{X} := \mathbb{H}(\mathbf{div}_{4/3}; \Omega) := \{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) \ : \ \mathbf{div}\ \boldsymbol{\tau} \in \mathbf{L}^{4/3}(\Omega)\}$$

[1] Camaño, Muñoz, Oyarzúa. *Numerical analysis of a dual-mixed problem in non-standard Banach spaces.* Electronic Transactions on Numerical Analysis **vol. 48** (2018), pp. 114–130.

Consistency with the classical velocity-pressure approach

- Find $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega)$ such that $\mathbf{u}|_\Gamma = \mathbf{u}_D$ and

$$\begin{aligned} \nu(\nabla\mathbf{u}, \nabla\mathbf{v})_\Omega \,+\, ((\nabla\mathbf{u})\mathbf{u}, \mathbf{v})_\Omega \,-\, (p, \operatorname{div}\mathbf{v})_\Omega \quad &= \,(\mathbf{f}, \mathbf{v})_\Omega \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \\ (q, \operatorname{div}\mathbf{u})_\Omega \quad &= 0 \quad \forall q \in L^2_0(\Omega). \end{aligned}$$

- For a solution $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega)$ then $\mathbf{u} \in \mathbf{L}^4(\Omega)$ and

$$\boldsymbol{\sigma} = \nu\nabla\mathbf{u} - p\mathbb{I} - \mathbf{u}\otimes\mathbf{u} \in \mathbb{L}^2(\Omega).$$

- In addition

$$\begin{aligned} (\nu\nabla\mathbf{u} - p\mathbb{I}, \nabla\mathbf{v})_\Omega &= -\big((\nabla\mathbf{u})\mathbf{u} \,-\, \mathbf{f}, \mathbf{v}\big)_\Omega \quad \forall\, \mathbf{v} \in [C_0^\infty(\Omega)]^n, \\ \Rightarrow \qquad \nu\Delta\mathbf{u} - \nabla p &= (\nabla\mathbf{u})\mathbf{u} \,-\, \mathbf{f} \text{ in the distributional sense.} \end{aligned}$$

- Therefore $(\nabla\mathbf{u})\mathbf{u} \in \mathbf{L}^{4/3}(\Omega)$, which implies

$$\mathbf{div}\,\boldsymbol{\sigma} = \nu\Delta\mathbf{u} - (\nabla\mathbf{u})\mathbf{u} - \nabla p \in \mathbf{L}^{4/3}(\Omega)$$

- Definition of the space $\mathbb{H}(\mathbf{div}_{4/3}; \Omega)$ for the unknown $\boldsymbol{\sigma}$ is coherent with the classical velocity-pressure formulation!

Continuous Formulation

Variational problem (First approach)

- **Variational problem (First approach)**:
  Find $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{X} \times \mathbf{M}$, such that $(\mathrm{tr}\,(\boldsymbol{\sigma}), 1)_\Omega = -(\mathrm{tr}\,(\mathbf{u} \otimes \mathbf{u}), 1)_\Omega$, and

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau}, \mathbf{u}) + \mathbf{c}(\mathbf{u}; \mathbf{u}, \boldsymbol{\tau}) &= F(\boldsymbol{\tau}), \quad \forall\, \boldsymbol{\tau} \in \mathbb{X} \\ \mathbf{b}(\boldsymbol{\sigma}, \mathbf{v}) &= G(\mathbf{v}), \quad \forall\, \mathbf{v} \in \mathbf{M}\,. \end{aligned}$$

- **Bilinear forms**:

$$\mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) := \frac{1}{\nu}(\boldsymbol{\sigma}^d, \boldsymbol{\tau}^d)_\Omega \quad \text{and} \quad \mathbf{b}(\boldsymbol{\tau}, \mathbf{v}) := (\mathbf{div}\ \boldsymbol{\tau}, \mathbf{v})_\Omega.$$

- **Trilinear form**:

$$\mathbf{c}(\mathbf{w}; \mathbf{v}, \boldsymbol{\tau}) := \frac{1}{\nu}(\mathbf{w} \otimes \mathbf{v}, \boldsymbol{\tau}^d)_\Omega.$$

- **Functionals**:

$$F(\boldsymbol{\tau}) = \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D \rangle \quad \text{and} \quad G(\mathbf{v}) = -(\mathbf{f}, \mathbf{v})_\Omega.$$

Decomposition of $\mathbb{X}$

$$\mathbb{X} \,=\, \mathbb{X}_0 \,\oplus\, P_0(\Omega)\,\mathbb{I}\,,$$

where

$$\mathbb{X}_0 := \{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}\,_{4/3};\Omega) \;:\; (\mathrm{tr}\,(\boldsymbol{\tau}), 1)_\Omega = 0\}.$$

$P_0(\Omega)\,\mathbb{I}$ is a topological supplement for $\mathbb{X}_0$.

More precisely, each $\boldsymbol{\tau} \in \mathbb{X}$ can be decomposed uniquely as:

$$\boldsymbol{\tau} \,=\, \boldsymbol{\tau}_0 \,+\, c\,\mathbb{I}\,, \quad \text{with} \quad \boldsymbol{\tau}_0 \,\in\, \mathbb{X}_0 \quad \text{and} \quad c \,:=\, \frac{1}{n\,|\Omega|}\int_\Omega \mathrm{tr}\,\boldsymbol{\tau} \,\in\, \mathbb{R}\,.$$

Variational problem (Second approach)

Find $(\boldsymbol{\sigma}_0, \mathbf{u}) \in \mathbb{X}_0 \times \mathbf{M}$, such that

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma}_0, \boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau}, \mathbf{u}) \; + \; \mathbf{c}(\mathbf{u}; \mathbf{u}, \boldsymbol{\tau}) \quad &= F(\boldsymbol{\tau}) \quad \forall \boldsymbol{\tau} \in \mathbb{X}_0, \\ \mathbf{b}(\boldsymbol{\sigma}_0, \mathbf{v}) \quad &= G(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{M}. \end{aligned}$$

Lemma

If $(\boldsymbol{\sigma}, \mathbf{u})$ is a solution of the first approach, then

$$(\boldsymbol{\sigma}_0, \mathbf{u}) = (\boldsymbol{\sigma} + \frac{1}{n|\Omega|}(\mathrm{tr}\,(\mathbf{u} \otimes \mathbf{u}), 1)_\Omega \mathbb{I}, \mathbf{u})$$

is a solution of the second approach. Conversely, if $(\boldsymbol{\sigma}_0, \mathbf{u})$ is a solution of the second approach then $(\boldsymbol{\sigma}, \mathbf{u}) = (\boldsymbol{\sigma}_0 - \frac{1}{n|\Omega|}(\mathrm{tr}\,(\mathbf{u} \otimes \mathbf{u}), 1)_\Omega \mathbb{I}, \mathbf{u})$ is a solution of the first approach.

$$\boldsymbol{\sigma}_0 := \boldsymbol{\sigma} + \frac{1}{n|\Omega|}(\mathrm{tr}\,(\mathbf{u} \otimes \mathbf{u}), 1)_\Omega \mathbb{I}.$$

The fixed-point operator

$$\mathbf{K} := \left\{ \mathbf{v} \in \mathbf{M} \,:\, \|\mathbf{v}\|_{\mathbf{M}} \,\leq\, \frac{2}{\gamma}\left(C_F \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{\mathbf{L}^{4/3}(\Omega)} \right) \right\},$$

with γ and C_F being the positive constant defined below.

The fixed-point operator

$$\mathcal{J} : \mathbf{K} \to \mathbf{K}, \quad \mathbf{w} \to \mathcal{J}(\mathbf{w}) = \mathbf{u},$$

where, given $\mathbf{w} \in \mathbf{K}$, $\mathbf{u}$ is part of the solution of the problem:
Find $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{X}_0 \times \mathbf{M}$, such that

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) \,+\, \mathbf{b}(\boldsymbol{\tau}, \mathbf{u}) \,+\, \mathbf{c}(\mathbf{w}; \mathbf{u}, \boldsymbol{\tau}) \quad &= \, F(\boldsymbol{\tau}), \quad \forall\, \boldsymbol{\tau} \in \mathbb{X} \\ \mathbf{b}(\boldsymbol{\sigma}, \mathbf{v}) \quad &= \, G(\mathbf{v}), \quad \forall\, \mathbf{v} \in \mathbf{M}. \end{aligned}$$

$\mathcal{J}(\mathbf{u}) = \mathbf{u} \quad \Leftrightarrow \quad (\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{X}_0 \times \mathbf{M}$ satisfies the second approach.

$\Leftrightarrow \quad \left(\boldsymbol{\sigma}_0 - \dfrac{1}{n|\Omega|}(\operatorname{tr}(\mathbf{u} \otimes \mathbf{u}), 1)_\Omega \mathbb{I}, \mathbf{u}\right) \in \mathbb{X} \times \mathbf{M}$

satisfies the first approach.

Preliminary results

- **Hölder inequality:**

$$\int_\Omega |fg| \leq \|f\|_{\mathrm{L}^p(\Omega)} \|g\|_{\mathrm{L}^q(\Omega)}, \quad \forall\, f \in \mathrm{L}^p(\Omega),\ \forall\, g \in \mathrm{L}^q(\Omega), \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

- **Classical estimates:**

$$\|w\|_{1,\Omega} \leq C_P |w|_{1,\Omega} \quad \forall w \in \mathrm{H}_0^1(\Omega)$$

and

$$\|w\|_{\mathrm{L}^r(\Omega)} \leq C_{Sob} \|w\|_{1,\Omega} \quad \forall\, w \in \mathrm{H}^1(\Omega)\,, \quad \text{for } r \geq 1 \text{ if } n = 2 \text{ or } r \in [1,6] \text{ if } n = 3,$$

with $C_P > 0$ and $C_{Sob} > 0$ depending only on $|\Omega|$.

Preliminary results

Lemma

There exists $C_d > 0$*, such that*

$$C_d \|\boldsymbol{\tau}\|_{0,\Omega}^2 \leq \|\boldsymbol{\tau}^d\|_{0,\Omega}^2 + \|\mathbf{div}\ \boldsymbol{\tau}\|_{\mathbf{L}^{4/3}(\Omega)}^2 \qquad \forall \boldsymbol{\tau} \in \mathbb{X}_0.$$

Proof:

Given $\boldsymbol{\tau} \in \mathbb{X}_0$, and denoting by $\mathbf{z}$ the unique element in $\mathbf{W}^\perp$ such that

$$\operatorname{div} \mathbf{z} = \operatorname{tr}(\boldsymbol{\tau}) \quad \text{and} \quad \|\mathbf{z}\|_{1,\Omega} \leq C \|\operatorname{tr}(\boldsymbol{\tau})\|_{0,\Omega},$$

where $\mathbf{W} := \{\mathbf{w} \in \mathbf{H}_0^1(\Omega) : \operatorname{div} \mathbf{w} = 0 \quad \text{in} \quad \Omega\}$ and $\mathbf{H}_0^1(\Omega) = \mathbf{W} \oplus \mathbf{W}^\perp$, using the Hölder inequality, the previous Sobolvev inclusion with $p = r = 4$ and $q = 4/3$, the previous result and following the same steps employed in the proof of Lemma 2.3 [2] it can be readily seen that

$$\|\operatorname{tr}(\boldsymbol{\tau})\|_{0,\Omega}^2 \leq nC \|\mathbf{z}\|_{1,\Omega} \left\{ \|\mathbf{div}\ \boldsymbol{\tau}\|_{\mathbf{L}^{4/3}(\Omega)}^2 + \|\boldsymbol{\tau}^d\|_{0,\Omega}^2 \right\}^{1/2},$$

which together with the definition of $\boldsymbol{\tau}^d$ implies the result.

[2] G. N. Gatica *A Simple Introduction to the Mixed Finite Element Method. Theory and Applications*, Springer Briefs in Mathematics, Springer, Cham Heidelberg New York Dordrecht London, (2014).

Well-definiteness of $\mathcal{J}$

- **Continuity of the forms $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$:**

$$|\mathbf{a}(\boldsymbol{\sigma},\boldsymbol{\tau})| \leq \frac{1}{\nu}\|\boldsymbol{\sigma}\|_{\mathbb{X}}\|\boldsymbol{\tau}\|_{\mathbb{X}} \quad \forall\, \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathbb{X},$$

$$|\mathbf{b}(\boldsymbol{\tau},\mathbf{v})| \leq \|\boldsymbol{\tau}\|_{\mathbb{X}}\|\mathbf{v}\|_{\mathbf{M}} \quad \forall\, \boldsymbol{\tau} \in \mathbb{X}, \forall\, \mathbf{v} \in \mathbf{M},$$

and

$$|\mathbf{c}(\mathbf{w};\mathbf{v},\boldsymbol{\tau})| \leq \frac{1}{\nu}\|\mathbf{w}\|_{\mathbf{M}}\|\mathbf{v}\|_{\mathbf{M}}\|\boldsymbol{\tau}\|_{\mathbb{X}} \quad \forall\, \boldsymbol{\tau} \in \mathbb{X}, \forall\, \mathbf{w},\mathbf{v} \in \mathbf{M}.$$

- **Kernel of $\mathbf{b}$:**

$$\mathbb{V} := \{\boldsymbol{\tau} \in \mathbb{X}_0 : \mathbf{b}(\boldsymbol{\tau},\mathbf{v}) = 0,\ \forall\, \mathbf{v} \in \mathbf{M}\} = \{\boldsymbol{\tau} \in \mathbb{X}_0 : \quad \mathbf{div}\ \boldsymbol{\tau} = \mathbf{0} \quad \text{in } \Omega\}\,.$$

- **Ellipticity of $\mathbf{a}$ on $\mathbb{V}$:** There exists $\alpha := C_d/\nu > 0$, such that

$$\mathbf{a}(\boldsymbol{\tau},\boldsymbol{\tau}) \;\geq\; \alpha\|\boldsymbol{\tau}\|^2_{\mathbb{X}} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{V}.$$

$$(C_d\|\boldsymbol{\tau}\|^2_{0,\Omega} \;\leq\; \|\boldsymbol{\tau}^d\|^2_{0,\Omega} \,+\, \|\mathbf{div}\ \boldsymbol{\tau}\|^2_{\mathbf{L}^{4/3}(\Omega)} \qquad \forall\, \boldsymbol{\tau} \,\in\, \mathbb{X}_0.)$$

Well-definiteness of $\mathcal{J}$

Lemma

There holds,

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}\in\mathbb{X}_0} \frac{\mathbf{b}(\boldsymbol{\tau},\mathbf{v})}{\|\boldsymbol{\tau}\|_{\mathbb{X}}} \geq \beta\|\mathbf{v}\|_{\mathbf{M}} \qquad \forall\,\mathbf{v}\,\in\,\mathbf{M},$$

with

$$\beta := (n + nC_P^2 C_{Sob}^2)^{-1/2}.$$

Proof:

Given $\mathbf{v}\in\mathbf{M}$, we let $\mathbf{h}(\mathbf{v}) := |\mathbf{v}|^3\mathbf{sgn}(\mathbf{v})$ and observe that

$$(|\mathbf{h}(\mathbf{v})|^{4/3},1)_\Omega = (|\mathbf{v}|^4|\mathbf{sgn}(\mathbf{v})|^{4/3},1)_\Omega = n^{2/3}(|\mathbf{v}|^4,1)_\Omega < +\infty,$$

which implies that $\mathbf{h}(\mathbf{v})\in\mathbf{L}^{4/3}(\Omega)$.

Then, defining $\tilde{\boldsymbol{\tau}} = -\nabla\mathbf{z} + \dfrac{1}{n|\Omega|}(\operatorname{div}\mathbf{z},1)\mathbb{I}\in\mathbb{L}^2(\Omega)$, with $\mathbf{z}\in\mathbf{H}_0^1(\Omega)$ being the unique solution of the variational problem

$$(\nabla\,\mathbf{z},\nabla\,\mathbf{w})_\Omega = (\mathbf{h}(\mathbf{v}),\mathbf{w})_\Omega \qquad \forall\,\mathbf{w}\in\mathbf{H}_0^1(\Omega), \tag{1}$$

it readily follows that

$$\mathbf{div}\,\tilde{\boldsymbol{\tau}} = \mathbf{h}(\mathbf{v})\in\mathbf{L}^{4/3}(\Omega), \qquad (\operatorname{tr}(\tilde{\boldsymbol{\tau}}),1)_\Omega = 0, \tag{2}$$

and, consequently, $\tilde{\boldsymbol{\tau}}\in\mathbb{X}_0$.

Well-definiteness of $\mathcal{J}$

Moreover, we have

$$\|\mathbf{div}\,\tilde{\boldsymbol{\tau}}\|_{\mathbf{L}^{4/3}(\Omega)} = \|\mathbf{h}(\mathbf{v})\|_{\mathbf{L}^{4/3}(\Omega)} = \sqrt{n}\|\mathbf{v}\|_{\mathbf{M}}^3. \tag{3}$$

On the other hand, from (1) with $\mathbf{w} = \mathbf{z}$, (3) and the Hölder inequality, we obtain

$$|\mathbf{z}|_{1,\Omega}^2 = (\nabla\,\mathbf{z}, \nabla\,\mathbf{z})_\Omega = (\mathbf{h}(\mathbf{v}), \mathbf{z})_\Omega \le \|\mathbf{h}(\mathbf{v})\|_{\mathbf{L}^{4/3}(\Omega)}\|\mathbf{z}\|_{\mathbf{L}^4(\Omega)} = \sqrt{n}\|\mathbf{v}\|_{\mathbf{M}}^3\|\mathbf{z}\|_{\mathbf{M}},$$

which, implies

$$\|\mathbf{z}\|_{1,\Omega} \le \sqrt{n} C_{sob} C_P \|\mathbf{v}\|_{\mathbf{M}}^3.$$

From the latter, it readily follows that

$$\|\tilde{\boldsymbol{\tau}}\|_{0,\Omega} = \left\{ |\mathbf{z}|_{1,\Omega}^2 - \frac{1}{n|\Omega|}(\mathrm{div}\,\mathbf{z}, 1)_\Omega^2 \right\}^{1/2} \le \|\mathbf{z}\|_{1,\Omega} \le \sqrt{n} C_{sob} C_P \|\mathbf{v}\|_{\mathbf{M}}^3,$$

which combined with (3), yields

$$\|\tilde{\boldsymbol{\tau}}\|_{\mathbb{X}} \le \beta^{-1}\|\mathbf{v}\|_{\mathbf{M}}^3, \tag{4}$$

with $\beta := (n + nC_P^2 C_{Sob}^2)^{-1/2}$. In this way, it follows that

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}\in\mathbb{X}_0} \frac{\mathbf{b}(\boldsymbol{\tau}, \mathbf{v})}{\|\boldsymbol{\tau}\|_{\mathbb{X}}} \ge \frac{\mathbf{b}(\tilde{\boldsymbol{\tau}}, \mathbf{v})}{\|\tilde{\boldsymbol{\tau}}\|_{\mathbb{X}}} \ge \beta \frac{(\mathbf{v}, \mathbf{h}(\mathbf{v}))_\Omega}{\|\mathbf{v}\|_{\mathbf{M}}^3} = \beta \frac{(|\mathbf{v}|^3 \mathbf{sgn}(\mathbf{v}), \mathbf{v})_\Omega}{\|\mathbf{v}\|_{\mathbf{M}}^3} \ge \beta\|\mathbf{v}\|_{\mathbf{M}},$$

which concludes the proof.

Well-definiteness of $\mathcal{J}$

Remark:
In the proof above, another feasible choice for $\mathbf{h}(\mathbf{v}) \in \mathbf{L}^{4/3}(\Omega)$ is $\mathbf{h}(\mathbf{v}) = |\mathbf{v}|^2\mathbf{v}$. In general, if $\mathbf{v} \in \mathbf{L}^q(\Omega)$, with $q > 1$, then $\mathbf{h}(\mathbf{v}) = |\mathbf{v}|^{q-2}\mathbf{v} \in \mathbf{L}^p(\Omega)$ with $p = \frac{q}{q-1}$, since

$$\int_\Omega |\mathbf{h}(\mathbf{v})|^p = \int_\Omega |\mathbf{v}|^{p(q-1)} = \int_\Omega |\mathbf{v}|^q < +\infty.$$

Well-definiteness of $\mathcal{J}$

Lemma

There hold:

$$|F(\boldsymbol{\tau})| = |\langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D \rangle| \leq C_F \|\mathbf{u}_D\|_{1/2,\Gamma} \|\boldsymbol{\tau}\|_{\mathbb{X}} \tag{5}$$

and

$$|G(\mathbf{v})| = |(\mathbf{f}, \mathbf{v})_\Omega| \leq \|\mathbf{f}\|_{\mathbf{L}^{4/3}(\Omega)} \|\mathbf{v}\|_{\mathbf{M}}, \tag{6}$$

where C_F is a positive constant depending on C_{Sob} (cf. (23)).

Proof:

- The proof of (6) is straightforward.
- Now, for (5) let us first recall that given $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}\,; \Omega)$, the normal trace $\boldsymbol{\tau}\mathbf{n}$ is defined as the functional in $\mathbf{H}^{-1/2}(\Gamma)$ given by
$$\langle \boldsymbol{\tau}\mathbf{n}, \boldsymbol{\xi} \rangle = (\boldsymbol{\tau}, \nabla \tilde{\gamma}_0^{-1}(\boldsymbol{\xi}))_\Omega + (\tilde{\gamma}_0^{-1}(\boldsymbol{\xi}), \mathbf{div}\ \boldsymbol{\tau})_\Omega \qquad \forall\, \boldsymbol{\xi} \in \mathbf{H}^{1/2}(\Gamma), \tag{7}$$
where $\tilde{\gamma}_0^{-1} : \mathbf{H}^{1/2}(\Gamma) \to [\mathbf{H}_0^1(\Omega)]^\perp$ is the right inverse of the well known trace operator $\gamma_0 : \mathbf{H}^1(\Omega) \to \mathbf{H}^{1/2}(\Gamma)$.
- Then, since $\tilde{\gamma}_0^{-1}(\boldsymbol{\xi}) \in \mathbf{H}^1(\Omega)$ and $\mathbf{H}^1(\Omega) \subset \mathbf{L}^4(\Omega)$, the last term in (7) is still well defined if $\mathbf{div}\ \boldsymbol{\tau} \in \mathbf{L}^{4/3}(\Omega)$. This implies that $\boldsymbol{\tau}\mathbf{n} \in \mathbf{H}^{-1/2}(\Gamma)$ for all $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3}\,; \Omega)$, and then F is well defined.
- Moreover, it readily follows that there exists $C_F > 0$, such that
$$|\langle \boldsymbol{\tau}\mathbf{n}, \boldsymbol{\xi} \rangle| \leq C_F \|\boldsymbol{\tau}\|_{\mathbb{X}} \|\boldsymbol{\xi}\|_{1/2,\Gamma}, \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3}\,; \Omega),\ \forall\, \boldsymbol{\xi} \in \mathbf{H}^{1/2}(\Gamma),$$
which implies (5).

Remark

In general, owing to the Sobolev embedding $\mathbf{H}^1(\Omega) \subset \mathbf{L}^q(\Omega)$, with $1 \leq q < +\infty$ if $n = 2$ and $q \in [1, 6]$ if $n = 3$, we have that for a given $\boldsymbol{\xi} \in \mathbf{H}^{1/2}(\Gamma)$ the estimate

$$|(\tilde{\gamma}_0^{-1}(\boldsymbol{\xi}), \mathbf{div}\ \boldsymbol{\tau})_\Omega| \leq C\|\tilde{\gamma}_0^{-1}(\boldsymbol{\xi})\|_{\mathbf{L}^q(\Omega)}\|\mathbf{div}\ \boldsymbol{\tau}\|_{\mathbf{L}^p(\Omega)},$$

holds only for $p > 1$ if $n = 2$ and $p \geq 6/5$ if $n = 3$, satisfying with $\frac{1}{p} + \frac{1}{q} = 1$.

Babuška–Brezzi theory in Banach spaces

Let X and M be two reflexive Banach spaces and let $a \in \mathcal{L}(X \times X; \mathbb{R})$ and $b \in \mathcal{L}(X \times M; \mathbb{R})$ be two bilinear forms. Then, given $f \in X'$ and $g \in M'$, we are interested in the following problem: Seek $u \in X$ and $p \in M$ such that

$$\begin{aligned} a(u, v) + b(v, p) &= f(v), \quad \forall\, v \in X, \\ b(u, q) &= g(q), \quad \forall\, q \in M. \end{aligned}$$

The previous problem is well-posed if and only if

$$\inf_{u \in \mathrm{Ker}(B)} \sup_{v \in \mathrm{Ker}(B)} \frac{a(u, v)}{\|u\|_X \, \|v\|_X} \geq \alpha > 0,$$

$$\forall\, v \in \mathrm{Ker}(B), \qquad [\forall\, u \in \mathrm{Ker}(B),\ a(u, v) = 0] \implies v = 0,$$

and

$$\inf_{q \in M} \sup_{v \in X} \frac{b(v, q)}{\|v\|_X \, \|q\|_M} \geq \beta > 0.$$

Furthermore, the following *a priori* estimates hold:

$$\begin{cases} \|u\|_X \leq c_1 \|f\|_{X'} + c_2 \|g\|_{M'}, \\ \|p\|_M \leq c_3 \|f\|_{X'} + c_4 \|g\|_{M'}, \end{cases}$$

with

$$c_1 = \frac{1}{\alpha}, \qquad c_2 = \frac{1}{\beta}\left(1 + \frac{\|a\|}{\alpha}\right), \qquad c_3 = \frac{1}{\beta}\left(1 + \frac{\|a\|}{\alpha}\right), \qquad c_4 = \frac{\|a\|}{\beta^2}\left(1 + \frac{\|a\|}{\alpha}\right).$$

Well-definiteness of $\mathcal{J}$

$$\mathbf{A} : (\mathbb{X}_0 \times \mathbf{M}) \times (\mathbb{X}_0 \times \mathbf{M}) \to \mathbb{R}$$
$$\mathbf{A}((\boldsymbol{\sigma}, \mathbf{u}), (\boldsymbol{\tau}, \mathbf{v})) := \mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau}, \mathbf{u}) + \mathbf{b}(\boldsymbol{\sigma}, \mathbf{v}).$$

$\mathbf{A}$ is bounded. In addition, $\mathbf{A}$ satisfies:

$$\sup_{\mathbf{0} \neq (\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{X}_0 \times \mathbf{M}} \frac{\mathbf{A}((\boldsymbol{\zeta}, \mathbf{z}), (\boldsymbol{\tau}, \mathbf{v}))}{\|(\boldsymbol{\tau}, \mathbf{v})\|} \geq \gamma \|(\boldsymbol{\zeta}, \mathbf{z})\| \qquad \forall\, (\boldsymbol{\zeta}, \mathbf{z}) \in \mathbb{X}_0 \times \mathbf{M},$$

with

$$\gamma := \frac{C_d \beta \min\{1, \nu\beta\}}{2(C_d + 1)(\nu\beta + 1)}.$$

Banach–Nečas–Babuška Theorem

Theorem

Let W be a Banach space and let V be a reflexive Banach space. Let $a \in \mathcal{L}(W \times V; \mathbb{R})$ and $f \in V'$. Then, the following problem:

$$\text{Find } u \in W \text{ such that } a(u,v) = f(v), \quad \forall\, v \in V.$$

is well-posed if and only if:

1. *There exists $\alpha > 0$ such that*
$$\inf_{w \in W} \sup_{v \in V} \frac{a(w,v)}{\|w\|_W \, \|v\|_V} \geq \alpha.$$
2. *For every $v \in V$,*
$$\big(\forall w \in W,\ a(w,v) = 0\big) \Longrightarrow v = 0.$$

Moreover, the following a priori estimate holds:

$$\forall f \in V', \qquad \|u\|_W \leq \frac{1}{\alpha} \|f\|_{V'}.$$

Well-definiteness of $\mathcal{J}$

Theorem

Assume that
$$\frac{4}{\nu\gamma^2}\left(C_F\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{\mathbf{L}^{4/3}(\Omega)}\right) \leq 1.$$
Then, given $\mathbf{w} \in \mathbf{K}$*, there exists a unique* $\mathbf{u} \in \mathbf{K}$ *such that* $\mathcal{J}(\mathbf{w}) = \mathbf{u}$*.*

Proof:

- Given $\mathbf{w} \in \mathbf{K}$, we define:
$$\mathbf{A}_{\mathbf{w}}((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v})) := \mathbf{A}((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v})) + \mathbf{c}(\mathbf{w};\mathbf{u},\boldsymbol{\tau}).$$
- Problem: Find $(\boldsymbol{\sigma},\mathbf{u}) \in \mathbb{X}_0 \times \mathbf{M}$, such that
$$\mathbf{A}_{\mathbf{w}}((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v})) = F(\boldsymbol{\tau}) + G(\mathbf{v}) \qquad \forall\,(\boldsymbol{\tau},\mathbf{v}) \in \mathbb{X}_0 \times \mathbf{M}.$$

Well-definiteness of $\mathcal{J}$

- Given $(\boldsymbol{\zeta}, \mathbf{z}), (\hat{\boldsymbol{\tau}}, \hat{\mathbf{v}}) \in \mathbb{X}_0 \times \mathbf{M}$ with $(\hat{\boldsymbol{\tau}}, \hat{\mathbf{v}}) \neq \mathbf{0}$, we observe that

$$\sup_{\mathbf{0} \neq (\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{X}_0 \times \mathbf{M}} \frac{\mathbf{A}_{\mathbf{w}}((\boldsymbol{\zeta}, \mathbf{z}), (\boldsymbol{\tau}, \mathbf{v}))}{\|(\boldsymbol{\tau}, \mathbf{v})\|} \geq \frac{|\mathbf{A}((\boldsymbol{\zeta}, \mathbf{z}), (\hat{\boldsymbol{\tau}}, \hat{\mathbf{v}}))|}{\|(\hat{\boldsymbol{\tau}}, \hat{\mathbf{v}})\|} - \frac{|\mathbf{c}(\mathbf{w}; \mathbf{z}, \hat{\boldsymbol{\tau}})|}{\|(\hat{\boldsymbol{\tau}}, \hat{\mathbf{v}})\|}$$

$$\geq \frac{|\mathbf{A}((\boldsymbol{\zeta}, \mathbf{z}), (\hat{\boldsymbol{\tau}}, \hat{\mathbf{v}}))|}{\|(\hat{\boldsymbol{\tau}}, \hat{\mathbf{v}})\|} - \frac{1}{\nu} \|\mathbf{w}\|_{\mathbf{M}} \|(\boldsymbol{\zeta}, \mathbf{z})\|,$$

  which implies

$$\sup_{\mathbf{0} \neq (\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{X}_0 \times \mathbf{M}} \frac{\mathbf{A}_{\mathbf{w}}((\boldsymbol{\zeta}, \mathbf{z}), (\boldsymbol{\tau}, \mathbf{v}))}{\|(\boldsymbol{\tau}, \mathbf{v})\|} \geq \left(\gamma - \frac{1}{\nu} \|\mathbf{w}\|_{\mathbf{M}} \right) \|(\boldsymbol{\zeta}, \mathbf{z})\|.$$

- From the definition of set $\mathbf{K}$, and the previous assumption, we easily get

$$\frac{1}{\nu} \|\mathbf{w}\|_{\mathbf{M}} \leq \frac{2}{\nu\gamma} \left(C_F \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{\mathbf{L}^{4/3}(\Omega)} \right) \leq \frac{\gamma}{2}.$$

Well-definiteness of $\mathcal{J}$

- Hence,

$$\sup_{\mathbf{0}\neq(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}} \frac{\mathbf{A}_{\mathbf{w}}((\boldsymbol{\zeta},\mathbf{z}),(\boldsymbol{\tau},\mathbf{v}))}{\|(\boldsymbol{\tau},\mathbf{v})\|} \geq \frac{\gamma}{2}\|(\boldsymbol{\zeta},\mathbf{z})\| \quad \forall\,(\boldsymbol{\zeta},\mathbf{z})\in\mathbb{X}_0\times\mathbf{M}.$$

- On the other hand, for a given $(\boldsymbol{\zeta},\mathbf{z})\in\mathbb{X}_0\times\mathbf{M}$, we observe that

$$\begin{aligned}\sup_{(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}} \mathbf{A}_{\mathbf{w}}((\boldsymbol{\tau},\mathbf{v}),(\boldsymbol{\zeta},\mathbf{z})) &\geq \sup_{\mathbf{0}\neq(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}} \frac{\mathbf{A}_{\mathbf{w}}((\boldsymbol{\tau},\mathbf{v}),(\boldsymbol{\zeta},\mathbf{z}))}{\|(\boldsymbol{\tau},\mathbf{v})\|}\\ &= \sup_{\mathbf{0}\neq(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}} \frac{\mathbf{A}((\boldsymbol{\tau},\mathbf{v}),(\boldsymbol{\zeta},\mathbf{z}))+\mathbf{c}(\mathbf{w};\mathbf{v},\boldsymbol{\zeta})}{\|(\boldsymbol{\tau},\mathbf{v})\|},\end{aligned}$$

from which,

$$\begin{aligned}\sup_{(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}} \mathbf{A}_{\mathbf{w}}((\boldsymbol{\tau},\mathbf{v}),(\boldsymbol{\zeta},\mathbf{z})) &\geq \frac{|\mathbf{A}((\boldsymbol{\tau},\mathbf{v}),(\boldsymbol{\zeta},\mathbf{z}))+\mathbf{c}(\mathbf{w};\mathbf{v},\boldsymbol{\zeta})|}{\|(\boldsymbol{\tau},\mathbf{v})\|}\\ &\geq \frac{|\mathbf{A}((\boldsymbol{\tau},\mathbf{v}),(\boldsymbol{\zeta},\mathbf{z}))|}{\|(\boldsymbol{\tau},\mathbf{v})\|} - \frac{|\mathbf{c}(\mathbf{w};\mathbf{v},\boldsymbol{\zeta})|}{\|(\boldsymbol{\tau},\mathbf{v})\|},\end{aligned}$$

for all $(\boldsymbol{\tau},\mathbf{v})\in(\mathbb{X}_0\times\mathbf{M})\backslash\{\mathbf{0}\}$, which implies

$$\sup_{(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}} \mathbf{A}_{\mathbf{w}}((\boldsymbol{\tau},\mathbf{v}),(\boldsymbol{\zeta},\mathbf{z})) \geq \sup_{\mathbf{0}\neq(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}} \frac{\mathbf{A}((\boldsymbol{\tau},\mathbf{v}),(\boldsymbol{\zeta},\mathbf{z}))}{\|(\boldsymbol{\tau},\mathbf{v})\|} - \frac{1}{\nu}\|\mathbf{w}\|_{\mathbf{M}}\|(\boldsymbol{\zeta},\mathbf{z})\|,$$

- Therefore, using the fact that $\mathbf{A}(\cdot,\cdot)$ is symmetric, we obtain

$$\sup_{(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}} \mathbf{A}_{\mathbf{w}}((\boldsymbol{\tau},\mathbf{v}),(\boldsymbol{\zeta},\mathbf{z})) \,\geq\, \gamma\|(\boldsymbol{\zeta},\mathbf{z})\| - \frac{1}{\nu}\|\mathbf{w}\|_{\mathrm{M}}\|(\boldsymbol{\zeta},\mathbf{z})\|,$$

  which yields

$$\sup_{(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}} \mathbf{A}_{\mathbf{w}}((\boldsymbol{\tau},\mathbf{v}),(\boldsymbol{\zeta},\mathbf{z})) \,\geq\, \frac{\gamma}{2}\|(\boldsymbol{\zeta},\mathbf{z})\| > 0 \quad \forall\,(\boldsymbol{\zeta},\mathbf{z})\in\mathbb{X}_0\times\mathbf{M},\ (\boldsymbol{\zeta},\mathbf{z})\neq\mathbf{0}.$$

- We apply the the Banach–Nečas–Babuška theorem (Extension to Banach spaces of the Babuška–Brezzi theorem) to $\mathbf{A}_{\mathbf{w}}(\cdot,\cdot)$.
- $\|\mathbf{u}\|_{\mathrm{M}} \,\leq\, \|(\boldsymbol{\sigma},\mathbf{u})\| \leq \frac{2}{\gamma}\left(C_F\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{\mathbf{L}^{4/3}(\Omega)}\right).$

Well-posedness of the continuous problem

Theorem

Let $\mathbf{f} \in \mathbf{L}^{4/3}(\Omega)$ *and* $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$ *such that*

$$\frac{4}{\nu\gamma^2}\left(C_F\|\mathbf{u}_D\|_{\mathbf{H}^{1/2}(\Gamma)} + \|\mathbf{f}\|_{\mathbf{L}^{4/3}(\Omega)}\right) < 1.$$

Then, there exists a unique $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{X}_0 \times \mathbf{M}$ *solution of the second approach. In addition, there exists* $C > 0$ *such that*

$$\|\mathbf{u}\|_{\mathbf{M}} + \|\boldsymbol{\sigma}\|_{\mathbb{X}} \leq C\left(\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{\mathbf{L}^{4/3}(\Omega)}\right).$$

Proof:

- Let $\mathbf{w}_1,\, \mathbf{w}_2,\, \mathbf{u}_1,\, \mathbf{u}_2 \in \mathbf{K}$, such that $\mathbf{u}_1 = \mathcal{J}(\mathbf{w}_1)$ and $\mathbf{u}_2 = \mathcal{J}(\mathbf{w}_2)$.
- From the definition of $\mathcal{J}$, it follows that there exist unique $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in \mathbb{X}_0$, such that for all $(\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{X}_0 \times \mathbf{M}$, there hold

$$\mathbf{A}_{\mathbf{w}_1}((\boldsymbol{\sigma}_1, \mathbf{u}_1), (\boldsymbol{\tau}, \mathbf{v})) = F(\boldsymbol{\tau}) + G(\mathbf{v}), \quad \text{and} \quad \mathbf{A}_{\mathbf{w}_2}((\boldsymbol{\sigma}_2, \mathbf{u}_2), (\boldsymbol{\tau}, \mathbf{v})) = F(\boldsymbol{\tau}) + G(\mathbf{v}).$$

- Then, subtracting both equations, adding and subtracting suitable terms, and recalling the definition of $\mathbf{A}_{\mathbf{w}}$, we easily arrive at

$$\mathbf{A}_{\mathbf{w}_1}((\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \mathbf{u}_1 - \mathbf{u}_2), (\boldsymbol{\tau}, \mathbf{v})) = -\mathbf{c}(\mathbf{w}_1 - \mathbf{w}_2; \mathbf{u}_2, \boldsymbol{\tau}) \qquad \forall\, (\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{X}_0 \times \mathbf{M}.$$

- Therefore, recalling that $\mathbf{w}_1 \in \mathbf{K}$, from the latter identity , we obtain

$$\begin{aligned} \frac{\gamma}{2}\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{M}} &\leq \sup_{\mathbf{0}\neq(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}} \frac{\mathbf{A}_{\mathbf{w}_1}((\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2,\mathbf{u}_1-\mathbf{u}_2),(\boldsymbol{\tau},\mathbf{v}))}{\|(\boldsymbol{\tau},\mathbf{v})\|} \\ &= \sup_{\mathbf{0}\neq(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}} \frac{-\mathbf{c}(\mathbf{w}_1-\mathbf{w}_2;\mathbf{u}_2,\boldsymbol{\tau})}{\|(\boldsymbol{\tau},\mathbf{v})\|} \\ &\leq \frac{1}{\nu}\|\mathbf{w}_1-\mathbf{w}_2\|_{\mathbf{M}}\|\mathbf{u}_2\|_{\mathbf{M}}, \end{aligned}$$

  which together with the fact that $\mathbf{u}_2 \in \mathbf{K}$, implies

$$\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbf{M}} \leq \frac{4}{\nu\gamma^2}\left(C_F\|\mathbf{u}_D\|_{\mathbf{H}^{1/2}(\Gamma)} + \|\mathbf{f}\|_{\mathbf{L}^{4/3}(\Omega)}\right)\|\mathbf{w}_1-\mathbf{w}_2\|_{\mathbf{M}}.$$

- The latter and the previous assumption readily imply that $\mathcal{J}$ is a **contraction mapping**.
- Now, let $\mathbf{u} \in \mathbf{K}$ be the unique fixed-point of $\mathcal{J}$ and let $\boldsymbol{\sigma} \in \mathbb{X}_0$ be such that $(\boldsymbol{\sigma},\mathbf{u}) \in \mathbb{X}_0 \times \mathbf{M}$ is the unique solution of the problem.
- According to the definition of $\mathbf{K}$, evidently $\mathbf{u}$ satisfies

$$\|\mathbf{u}\|_{\mathbf{M}} \leq \frac{2}{\gamma}\left(C_F\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{\mathbf{L}^{4/3}(\Omega)}\right).$$

- In turn,

$$\|\boldsymbol{\sigma}\|_{\mathbb{X}} \leq \|(\boldsymbol{\sigma},\mathbf{u})\| \leq \frac{2}{\gamma}\sup_{\mathbf{0}\neq(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}} \frac{\mathbf{A}_{\mathbf{u}}((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v}))}{\|(\boldsymbol{\tau},\mathbf{v})\|} = \frac{2}{\gamma}\sup_{\mathbf{0}\neq(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}} \frac{F(\boldsymbol{\tau})+G(\mathbf{v})}{\|(\boldsymbol{\tau},\mathbf{v})\|},$$

  thus

$$\|\boldsymbol{\sigma}\|_{\mathbb{X}} \leq \frac{2}{\gamma}\left(C_F\|\mathbf{u}_D\|_{\mathbf{H}^{1/2}(\Gamma)} + \|\mathbf{f}\|_{\mathbf{L}^{4/3}(\Omega)}\right).$$

Galerkin scheme

Discretes spaces

- We denote by $P_l(S)$ the space of polynomials of total degree at most l defined on S.
- For each integer $k \geq 0$ and for each $T \in \mathcal{T}_h$, we define the local Raviart–Thomas space of order k as:
$$\mathbf{RT}_k(T) := [P_k(T)]^n \oplus \widetilde{P}_k(T)\mathbf{x},$$
where $\mathbf{x} := (x_1, \ldots, x_n)^t$ is a generic vector of $\mathbb{R}^n$ and $\widetilde{P}_k(T)$ is the space of polynomials of total degree equal to k defined on T.

Discretes spaces and formulation

- **Discrete spaces:**

$$\mathbb{X}_h := \left\{ \boldsymbol{\tau}_h \in \mathbb{X} : \quad \mathbf{c}^{\mathrm{t}}\, \boldsymbol{\tau}_h|_K \in \mathbf{RT}_k(K) \quad \forall\, \mathbf{c} \in \mathbb{R}^n \quad \forall\, K \in \mathcal{T}_h \right\} \subseteq \mathbb{X},$$

$$\mathbf{M}_h := \{ \mathbf{v}_h \in \mathbf{M} \,:\, \mathbf{v}_h|_K \in [P_k(K)]^n, \quad \forall\, K \in \mathcal{T}_h \} \subseteq \mathbf{M}.$$

- Notice that

$$\mathbb{X}_h = \mathbb{X}_{h,0} \oplus P_0(\Omega)\mathbb{I} \qquad \text{with} \qquad \mathbb{X}_{h,0} = \mathbb{X}_h \cap \mathbb{X}_0,$$

- The Galerkin scheme associated to the second approach reads: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{X}_{h,0} \times \mathbf{M}_h$, such that

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{u}_h) + \mathbf{c}(\mathbf{u}_h; \mathbf{u}_h, \boldsymbol{\tau}_h) &= F(\boldsymbol{\tau}_h) \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_{h,0}, \\ \mathbf{b}(\boldsymbol{\sigma}_h, \mathbf{v}_h) &= G(\mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in \mathbf{M}_h. \end{aligned}$$

Preliminary results

- Given $r > 1$, let us define the space

$$\mathbf{Z}_r := \{\tau \in \mathbf{H}(\mathrm{div}_r;\Omega) \ : \ \tau|_T \in \mathbf{W}^{1,r}(T), \quad \forall\, T \in \mathcal{T}_h\},$$

and let

$$\Pi_h^k : \mathbf{Z}_r \to \mathbf{X}_h := \{\tau \in \mathbf{H}(\mathrm{div};\Omega) \ : \ \tau|_T \in \mathbf{RT}_k(T), \quad \forall\, T \in \mathcal{T}_h\},$$

be the **Raviart–Thomas interpolator operator**, which is characterized by the identities

$$\int_e (\Pi_h^k(\tau)\cdot\nu)\xi = \int_e (\tau\cdot\nu)\xi \qquad \forall\ \xi \in P_k(e),\ \forall \text{ edge or face } e \text{ of } \mathcal{T}_h,$$

and

$$\int_T \Pi_h^k(\tau)\cdot\psi = \int_T \tau\cdot\psi \qquad \forall\ \psi \in [P_{k-1}(T)]^n,\ \forall\ T \in \mathcal{T}_h \text{ (if } k \geq 1) .$$

Preliminary results

- Notice that for $r > 1$ the trace operator is well-defined from $\mathrm{W}^{1,r}(K)$ to $\mathrm{W}^{1-\frac{1}{r},r}(\partial K) \subseteq \mathrm{L}^1(\partial K)$, thus Π_h^k is well defined in $\mathbf{Z}_r$.
- In turn, it is well known that the following identity holds

$$\mathrm{div}(\Pi_h^k(\tau)) = \mathcal{P}_h^k(\mathrm{div}\tau) \quad \forall\, \tau \in \mathbf{Z}_r,$$

  where $\mathcal{P}_h^k : L^r(\Omega) \to M_h := \{v \in \mathrm{L}^r(\Omega) \,:\, v|_K \in P_k(K) \quad \forall\, K \in \mathcal{T}_h\}$ is the operator satisfying

$$\int_\Omega (\mathcal{P}_h^k(v) - v) z_h = 0 \quad \forall\, z_h \in M_h,$$

  and the following error estimate: For each $0 \le t \le k+1$ and for each $w \in \mathrm{W}^{t,r}(\Omega)$, with $1 \le r \le \infty$, there holds

$$\|w - \mathcal{P}_h^k(w)\|_{\mathrm{L}^r(\Omega)} \le C h^t |w|_{\mathrm{W}^{t,r}(\Omega)}.$$

- Notice that for $r = 2$, $\mathcal{P}_h^k$ coincides with the usual orthogonal projection.
- Notice also that with $t = 0$ and the triangle inequality, it can be easily obtained that $\mathcal{P}_h^k$ is a continuous operator.

Preliminary results

Lemma

Let $r > 1$. Then, there exists $C_1 > 0$, independent of h, such that for each $\tau \in \mathbf{W}^{l+1,r}(T)$ with $0 \leq l \leq k$, and for each $0 \leq m \leq l+1$, there holds

$$|\tau - \Pi_h^k(\tau)|_{\mathbf{W}^{m,r}(T)} \leq C_1 \frac{h_T^{l+2}}{\rho_T^{m+1}} |\tau|_{\mathbf{W}^{l+1,r}(T)},$$

where h_T is the diameter of T, that is $h_T = \max_{x,y \in T} \|x - y\|$, and ρ_T is the diameter of the largest sphere contained in T. Moreover, there exists $C_2 > 0$, independent of h, such that for each $\tau \in \mathbf{W}^{1,r}(T)$, with $\operatorname{div} \tau \in \mathrm{W}^{l+1,r}(T)$ and $0 \leq l \leq k$, and for each $0 \leq m \leq l+1$, there holds

$$|\operatorname{div} \tau - \operatorname{div}(\Pi_h^k(\tau))|_{\mathrm{W}^{m,r}(T)} \leq C_2 \frac{h_T^{l+1}}{\rho_T^{m}} |\operatorname{div} \tau|_{\mathrm{W}^{l+1,r}(T)}.$$

Preliminary results

- Owing to the regularity of the mesh, it is not difficult see that the following global estimate holds

$$\|\tau - \Pi_h^k(\tau)\|_{0,\Omega} + \|\mathrm{div}\,\tau - \mathrm{div}\,(\Pi_h(\tau))\|_{\mathrm{L}^r(\Omega)} \leq c h^{l+1} \Big\{ |\tau|_{\mathbf{H}^{l+1}(\Omega)} + |\mathrm{div}\,\tau|_{\mathrm{W}^{l+1,r}(\Omega)} \Big\},$$

  for all $0 \leq l \leq k+1$, and for all $\tau \in \mathbf{H}^{l+1}(\Omega)$ with $\mathrm{div}\,\tau \in \mathrm{W}^{l+1,r}(\Omega)$.
- It can be proved that

$$\|\tau - \Pi_h^k(\tau)\|_{0,T} \leq C h_T^{1-\frac{n(2-r)}{2r}} |\tau|_{\mathbf{W}^{1,r}(T)} \quad \forall\, \tau \in \mathbf{W}^{1,r}(T),$$

  with $r > 1$ if $n = 2$, and $r \in (1, 6]$ if $n = 3$, and in particular for $r = 4/3$, there holds

$$\|\tau - \Pi_h^k(\tau)\|_{0,T} \leq C h_T^{1-\frac{n}{4}} |\tau|_{\mathbf{W}^{1,4/3}(T)} \quad \forall\, \tau \in \mathbf{W}^{1,4/3}(T).$$

The discrete fixed-point operator

$$\mathbf{K}_h := \left\{ \mathbf{v} \in \mathbf{M}_h \ : \ \|\mathbf{v}\|_{\mathbf{M}} \ \le \ \frac{2}{\widehat{\gamma}} \left(C_F \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{\mathbf{L}^{4/3}(\Omega)} \right) \right\},$$

with $\widehat{\gamma}$ a positive constant defined below.

The discrete fixed-point operator

$$\mathcal{J}_h : \mathbf{K}_h \to \mathbf{K}_h, \quad \mathbf{w}_h \to \mathcal{J}_h(\mathbf{w}_h) \ = \ \mathbf{u}_h,$$

where, given $\mathbf{w}_h \in \mathbf{K}_h$, $\mathbf{u}_h$ is part of the solution of the problem:
Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{X}_{h,0} \times \mathbf{M}_h$,

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{u}_h) + \mathbf{c}(\mathbf{w}_h; \mathbf{u}_h, \boldsymbol{\tau}_h) \quad &= F(\boldsymbol{\tau}_h), \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_{h,0}, \\ \mathbf{b}(\boldsymbol{\sigma}_h, \mathbf{v}_h) \quad &= G(\mathbf{v}_h), \quad \forall\, \mathbf{v}_h \in \mathbf{M}_h. \end{aligned}$$

$\mathcal{J}_h(\mathbf{u}_h) = \mathbf{u}_h \quad \Leftrightarrow \quad ((\boldsymbol{\sigma}_h, \mathbf{u}_h)) \in \mathbb{X}_{h,0} \times \mathbf{M}_h$satisfies

the Galerkin scheme associated to the second approach.

Well-definiteness of $\mathcal{J}_h$

- Since $\mathbf{div}\ \mathbb{X}_h \subseteq \mathbf{M}_h$, **the discrete kernel of** $\mathbf{b}$ corresponds to

$$\mathbb{V}_h := \{\boldsymbol{\tau}_h \in \mathbb{X}_{h,0} :\ \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{v}_h) = 0,\ \forall\, \mathbf{v}_h \in \mathbf{M}_h\} = \{\boldsymbol{\tau}_h \in \mathbb{X}_{h,0} :\ \mathbf{div}\ \boldsymbol{\tau}_h = \mathbf{0} \text{ in } \Omega\}\,.$$

- **Ellipticity of $\mathbf{a}$ on $\mathbb{V}_h$:** There exists $\alpha = C_d/\nu > 0$ such that

$$\mathbf{a}(\boldsymbol{\tau}_h, \boldsymbol{\tau}_h) \ \geq\ \alpha \|\boldsymbol{\tau}_h\|_{\mathbb{X}}^2 \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{V}_h.$$

- $\mathbf{A}$ satisfies the following discrete inf-sup condition

$$\sup_{\mathbf{0} \neq (\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathbb{X}_{h,0} \times \mathbf{M}_h} \frac{\mathbf{A}((\boldsymbol{\sigma}_h, \mathbf{u}_h), (\boldsymbol{\tau}_h, \mathbf{v}_h))}{\|(\boldsymbol{\tau}_h, \mathbf{v}_h)\|} \geq \widehat{\gamma} \|(\boldsymbol{\sigma}_h, \mathbf{u}_h)\|$$

for all $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{X}_{h,0} \times \mathbf{M}_h$, with $\widehat{\gamma} := \dfrac{C_d \widehat{\beta} \min\{1, \nu\widehat{\beta}\}}{2(C_d + 1)(\nu\widehat{\beta} + 1)}$.

Well-definiteness of $\mathcal{J}_h$

Lemma

There exists $\widehat{\beta} > 0$, independent of h, such that

$$\sup_{\mathbf{0} \neq \boldsymbol{\tau}_h \in \mathbb{X}_{h,0}} \frac{\mathbf{b}(\boldsymbol{\tau}_h, \mathbf{v}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbb{X}}} \geq \widehat{\beta} \|\mathbf{v}_h\|_{\mathbf{M}} \qquad \forall\, \mathbf{v}_h \in \mathbf{M}_h.$$

Proof:

- Given $\mathbf{v}_h \in \mathbf{M}_h$, we set

$$\mathbf{g}(\mathbf{v}_h) := \begin{cases} \mathbf{sgn}(\mathbf{v}_h)|\mathbf{v}_h|^3 & \text{in} \quad \Omega, \\ \mathbf{0} & \text{in} \quad B \backslash \overline{\Omega}, \end{cases}$$

  where $B \subseteq \mathbb{R}^n$ is an open and bounded convex set containing $\overline{\Omega}$.
- Since $\mathbf{g}(\mathbf{v}_h) \in \mathbf{L}^{4/3}(\Omega)$, a well known result on regularity of elliptic problems implies that there exists a unique weak solution $\mathbf{z} \in \mathbf{W}^{2,4/3}(B) \cap \mathbf{W}_0^{1,4/3}(B)$ of the boundary value problem

$$-\Delta \mathbf{z} = \mathbf{g}(\mathbf{v}_h) \quad \text{in} \quad B \quad \text{and} \quad \mathbf{z} = 0 \quad \text{on} \quad \partial B,$$

  which satisfies

$$\|\mathbf{z}\|_{\mathbf{W}^{2,4/3}(\Omega)} \leq C \|\mathbf{g}(\mathbf{v}_h)\|_{\mathbf{L}^{4/3}(B)} = C \| |\mathbf{v}_h|^3 \|_{\mathbf{L}^{4/3}(\Omega)} = C \|\mathbf{v}_h\|_{\mathbf{M}}^3,$$

  with $C > 0$.

Well-definiteness of $\mathcal{J}_h$

- Hence, we set $\hat{\boldsymbol{\tau}} = -\nabla \mathbf{z}|_\Omega \in \mathbb{W}^{1,4/3}(\Omega)$, and observe that

$$\|\hat{\boldsymbol{\tau}}\|_{\mathbb{W}^{1,4/3}(\Omega)} \leq C \|\mathbf{v}_h\|_{\mathbf{M}}^3,$$

which together with the continuity of the embedding from $\mathrm{W}^{1,4/3}(\Omega)$ into $\mathrm{L}^2(\Omega)$, implies

$$\|\hat{\boldsymbol{\tau}}\|_{0,\Omega} \leq C \|\mathbf{v}_h\|_{\mathbf{M}}^3.$$

- Then, we define $\hat{\boldsymbol{\tau}}_h = \boldsymbol{\Pi}_h^k(\hat{\boldsymbol{\tau}}) - \dfrac{1}{n|\Omega|}\Big(\mathrm{tr}\,(\boldsymbol{\Pi}_h^k(\hat{\boldsymbol{\tau}})), 1\Big)_\Omega \mathbb{I} \in \mathbb{X}_{h,0}$ and observe that

$$\mathbf{div}\,\hat{\boldsymbol{\tau}}_h = \mathbf{P}_h^k(\mathbf{div}\,\hat{\boldsymbol{\tau}}) = \mathbf{P}_h^k(\mathbf{sgn}(\mathbf{v}_h)|\mathbf{v}_h|^3).$$

Well-definiteness of $\mathcal{J}_h$

- In turn, we obtain

$$\begin{aligned}
\|\hat{\boldsymbol{\tau}}_h\|_{0,\Omega} &\leq \left\| \hat{\boldsymbol{\tau}} - \frac{1}{n|\Omega|} \left(\operatorname{tr}(\hat{\boldsymbol{\tau}}), 1\right)_\Omega \mathbb{I} - \hat{\boldsymbol{\tau}}_h \right\|_{0,\Omega} + \left\| \hat{\boldsymbol{\tau}} - \frac{1}{n|\Omega|} \left(\operatorname{tr}(\hat{\boldsymbol{\tau}}), 1\right)_\Omega \mathbb{I} \right\|_{0,\Omega} \\
&= \left\| \hat{\boldsymbol{\tau}} - \boldsymbol{\Pi}_h^k(\hat{\boldsymbol{\tau}}) - \frac{1}{n|\Omega|} \left(\operatorname{tr}\left(\hat{\boldsymbol{\tau}} - \boldsymbol{\Pi}_h^k(\hat{\boldsymbol{\tau}})\right), 1\right)_\Omega \mathbb{I} \right\|_{0,\Omega} + \left\| \hat{\boldsymbol{\tau}} - \frac{1}{n|\Omega|} \left(\operatorname{tr}(\hat{\boldsymbol{\tau}}), 1\right)_\Omega \mathbb{I} \right\| \\
&\leq \left\| \hat{\boldsymbol{\tau}} - \boldsymbol{\Pi}_h^k(\hat{\boldsymbol{\tau}}) \right\|_{0,\Omega} + \|\hat{\boldsymbol{\tau}}\|_{0,\Omega} \\
&\leq c_1 \left\{ \sum_{T \in \mathcal{T}_h} h_T^{2(1-\frac{n}{4})} |\hat{\boldsymbol{\tau}}|^2_{\mathbb{W}^{1,4/3}(T)} \right\}^{1/2} + c_2 \|\mathbf{v}_h\|^3_{\mathbf{M}}, \\
&\leq c_1 h^{1-\frac{n}{4}} |\hat{\boldsymbol{\tau}}|_{\mathbb{W}^{1,4/3}(\Omega)} + c_2 \|\mathbf{v}_h\|^3_{\mathbf{M}},
\end{aligned}$$

which, imply

$$\|\hat{\boldsymbol{\tau}}_h\|_{0,\Omega} \leq C \|\mathbf{v}_h\|^3_{\mathbf{M}}.$$

Well-definiteness of $\mathcal{J}_h$

- Hence, using the fact that $\mathbf{P}_h^k$ is a continuous operator, we easily obtain

$$\|\hat{\boldsymbol{\tau}}_h\|_{\mathbb{X}} = \left\{ \|\hat{\boldsymbol{\tau}}_h\|_{0,\Omega}^2 + \|\mathbf{div}\,(\hat{\boldsymbol{\tau}}_h)\|_{\mathbf{L}^{4/3}(\Omega)}^2 \right\}^{1/2} \leq \hat{c}\|\mathbf{v}_h\|_{\mathbf{M}}^3,$$

  with $\hat{c} > 0$ independent of h.

- Therefore, we obtain

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathbb{X}_{h,0} \\ \boldsymbol{\tau}_h \neq \theta}} \frac{\mathbf{b}(\boldsymbol{\tau}_h, \mathbf{v}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbb{X}}} \geq \frac{\mathbf{b}(\hat{\boldsymbol{\tau}}_h, \mathbf{v}_h)}{\|\hat{\boldsymbol{\tau}}_h\|_{\mathbb{X}}} \geq \frac{1}{\hat{c}} \frac{(|\mathbf{v}_h|^3 \mathbf{sgn}(\mathbf{v}_h), \mathbf{v}_h)_\Omega}{\|\mathbf{v}_h\|_{\mathbf{M}}^3} \geq \frac{1}{\hat{c}} \frac{\|\mathbf{v}_h\|_{\mathbf{M}}^4}{\|\mathbf{v}_h\|_{\mathbf{M}}^3} = \frac{1}{\hat{c}}\|\mathbf{v}_h\|_{\mathbf{M}},$$

  which concludes the proof with $\widehat{\beta} = \dfrac{1}{\hat{c}}$.

Well-definiteness of $\mathcal{J}_h$

- $\mathbf{A}$ satisfies the following discrete inf-sup condition

$$\sup_{\mathbf{0}\neq(\boldsymbol{\tau}_h,\mathbf{v}_h)\in\mathbb{X}_{h,0}\times\mathbf{M}_h} \frac{\mathbf{A}((\boldsymbol{\sigma}_h,\mathbf{u}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h))}{\|(\boldsymbol{\tau}_h,\mathbf{v}_h)\|} \geq \widehat{\gamma}\|(\boldsymbol{\sigma}_h,\mathbf{u}_h)\|$$

for all $(\boldsymbol{\sigma}_h,\mathbf{u}_h) \in \mathbb{X}_{h,0}\times\mathbf{M}_h$, with $\widehat{\gamma} := \dfrac{C_d\widehat{\beta}\min\{1,\nu\widehat{\beta}\}}{2(C_d+1)(\nu\widehat{\beta}+1)}$.

Well-definiteness of $\mathcal{J}_h$ and discrete problem

Theorem

Assume that

$$\frac{4}{\nu\widehat{\gamma}^2}\left(C_F\|\mathbf{u}_D\|_{1/2,\Gamma}+\|\mathbf{f}\|_{\mathbf{L}^{4/3}(\Omega)}\right)\leq 1.$$

Then, given $\mathbf{w}_h\in\mathbf{K}_h$, *there exists a unique* $\mathbf{u}_h\in\mathbf{K}_h$ *such that* $\mathcal{J}_h(\mathbf{w}_h)=\mathbf{u}_h$.

Theorem

Let $\mathbf{f}\in\mathbf{L}^{4/3}(\Omega)$ *and* $\mathbf{u}_D\in\mathbf{H}^{1/2}(\Gamma)$ *such that*

$$\frac{4}{\nu\widehat{\gamma}^2}(C_F\|\mathbf{u}_D\|_{1/2,\Gamma}+\|\mathbf{f}\|_{\mathbf{L}^{4/3}(\Omega)})<1.$$

Then, there exists a unique $(\boldsymbol{\sigma}_h,\mathbf{u}_h)\in\mathbb{X}_{h,0}\times\mathbf{M}_h$ *solution to discrete version of the second approach. In addition, there exists* $C>0$, *independent of* h, *such that*

$$\|\mathbf{u}_h\|_{\mathbf{M}}+\|\boldsymbol{\sigma}_h\|_{\mathbb{X}}\leq C\left(\|\mathbf{u}_D\|_{1/2,\Gamma}+\|\mathbf{f}\|_{\mathbf{L}^{4/3}(\Omega)}\right).$$

Conservativity

- From the second equation of our problem we have that

$$(\mathbf{div}\,\boldsymbol{\sigma}_h - \mathbf{f},\, \mathbf{v}_h)_\Omega \;=\; 0 \qquad \forall\, \mathbf{v}_h \in \mathbf{M}_h\,,$$

- which implies

$$\mathbf{div}\,\boldsymbol{\sigma}_h \;=\; \mathbf{P}_h^k(\mathbf{f})$$

  with $\mathbf{P}_h^k$ the $L^2(\Omega)-$orthogonal projector onto the piecewise polynomials vectors on $\mathbf{M}_h$.
- Therefore, if $\mathbf{f} \in \mathbf{M}_h$ then

$$\mathbf{div}\,\boldsymbol{\sigma}_h \;= \mathbf{f}$$

- The method is momentum conservative

Conservativity

- If $\mathbf{f}$ is not piecewise polynomial, the latter also implies that

$$\|\mathbf{div}\,\boldsymbol{\sigma}_h + \mathbf{f}\|_{\mathbf{L}^{4/3}(\Omega)} = \|\mathbf{f} - \mathbf{P}_h^k(\mathbf{f})\|_{\mathbf{L}^{4/3}(\Omega)}.$$

In fact, if $\mathbf{f} \in \mathbf{W}^{k+1,4/3}(\Omega)$, then we have that

$$\|\mathbf{div}\,\boldsymbol{\sigma}_h + \mathbf{f}\|_{\mathbf{L}^{4/3}(\Omega)} \leq C h^{k+1} \|\mathbf{f}\|_{\mathbf{W}^{k+1,4/3}(\Omega)},$$

that is, $\|\mathbf{div}\,\boldsymbol{\sigma}_h + \mathbf{f}\|_{\mathbf{L}^{4/3}(\Omega)} = O(h^{k+1})$.

Cea's Estimate

Theorem

Assume that

$$\frac{4}{\nu\gamma\widehat{\gamma}}\left(C_F\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{\mathbf{L}^{4/3}(\Omega)}\right) \le \frac{1}{2},$$

with γ *and* $\widehat{\gamma}$ *being the positive constants in continuous and discrete inf-sup conditions, respectively. Let* $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{X}_0 \times \mathbf{M}$ *and* $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{X}_{h,0} \times \mathbf{M}_h$ *be the unique solutions of continuous and discrete versions of the second approach, respectively. Then, there exists* $C_{cea} > 0$*, independent of* h*, such that*

$$\|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \mathbf{u} - \mathbf{u}_h)\| \le C_{cea} \inf_{(\boldsymbol{\tau}_h, \mathbf{v}_h)\in\mathbb{X}_{h,0}\times\mathbf{M}_h} \|(\boldsymbol{\sigma} - \boldsymbol{\tau}_h, \mathbf{u} - \mathbf{v}_h)\|.$$

Proof:

We define $\mathbf{e}_{\boldsymbol{\sigma}} := \boldsymbol{\sigma} - \boldsymbol{\sigma}_h$ and $\mathbf{e}_{\mathbf{u}} := \mathbf{u} - \mathbf{u}_h$, and for any $(\boldsymbol{\zeta}_h, \mathbf{z}_h) \in \mathbb{X}_{h,0} \times \mathbf{M}_h$, we write

$$\mathbf{e}_{\boldsymbol{\sigma}} = \boldsymbol{\xi}_{\boldsymbol{\sigma}} + \boldsymbol{\chi}_{\boldsymbol{\sigma}} := (\boldsymbol{\sigma} - \boldsymbol{\zeta}_h) + (\boldsymbol{\zeta}_h - \boldsymbol{\sigma}_h), \qquad \mathbf{e}_{\mathbf{u}} = \boldsymbol{\xi}_{\mathbf{u}} + \boldsymbol{\chi}_{\mathbf{u}} := (\mathbf{u} - \mathbf{z}_h) + (\mathbf{z}_h - \mathbf{u}_h).$$

Cea's Estimate

- Recalling the definition of the bilinear form $\mathbf{A}$ we have that the following identities hold

$$\mathbf{A}((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v}))+\mathbf{c}(\mathbf{u};\mathbf{u},\boldsymbol{\tau})=F(\boldsymbol{\tau})+G(\mathbf{v})\qquad\forall\,(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}$$

  and

$$\mathbf{A}((\boldsymbol{\sigma}_h,\mathbf{u}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h))+\mathbf{c}(\mathbf{u}_h;\mathbf{u}_h,\boldsymbol{\tau}_h)=F(\boldsymbol{\tau}_h)+G(\mathbf{v}_h)\qquad\forall\,(\boldsymbol{\tau}_h,\mathbf{v}_h)\in\mathbb{X}_{h,0}\times\mathbf{M}_h,$$

  from which we deduce the Galerkin orthogonality property

$$\mathbf{A}((\mathbf{e}_{\boldsymbol{\sigma}},\mathbf{e}_{\mathbf{u}}),(\boldsymbol{\tau}_h,\mathbf{v}_h))+[\mathbf{c}(\mathbf{u};\mathbf{u},\boldsymbol{\tau}_h)-\mathbf{c}(\mathbf{u}_h;\mathbf{u}_h,\boldsymbol{\tau}_h)]\;=\;0\qquad\forall\,(\boldsymbol{\tau}_h,\mathbf{v}_h)\in\mathbb{X}_{h,0}\times\mathbf{M}_h.$$

- Then, using the identity

$$\mathbf{c}(\mathbf{u};\mathbf{u},\boldsymbol{\tau}_h)=\mathbf{c}(\mathbf{u}-\mathbf{u}_h;\mathbf{u},\boldsymbol{\tau}_h)+\mathbf{c}(\mathbf{u}_h;\mathbf{u},\boldsymbol{\tau}_h),$$

  we obtain that for all $(\boldsymbol{\tau}_h,\mathbf{v}_h)\in\mathbb{X}_{h,0}\times\mathbf{M}_h$, there holds

$$\begin{aligned}\mathbf{A}_{\mathbf{u}_h}((\boldsymbol{\chi}_{\boldsymbol{\sigma}},\boldsymbol{\chi}_{\mathbf{u}}),(\boldsymbol{\tau}_h,\mathbf{v}_h))\;=\;&-\mathbf{A}((\boldsymbol{\xi}_{\boldsymbol{\sigma}},\boldsymbol{\xi}_{\mathbf{u}}),(\boldsymbol{\tau}_h,\mathbf{v}_h))\;-\;\mathbf{c}(\boldsymbol{\xi}_{\mathbf{u}};\mathbf{u},\boldsymbol{\tau}_h)\\&-\mathbf{c}(\boldsymbol{\chi}_{\mathbf{u}};\mathbf{u},\boldsymbol{\tau}_h)\;-\;\mathbf{c}(\mathbf{u}_h;\boldsymbol{\xi}_{\mathbf{u}},\boldsymbol{\tau}_h),\end{aligned}$$

Cea's Estimate

- which together with the definition of $\mathbf{A}$ implies

$$\begin{aligned}\mathbf{A}_{\mathbf{u}_h}((\boldsymbol{\chi}_{\boldsymbol{\sigma}},\boldsymbol{\chi}_{\mathbf{u}}),(\boldsymbol{\tau}_h,\mathbf{v}_h)) = \quad & -\mathbf{a}(\boldsymbol{\xi}_{\boldsymbol{\sigma}},\boldsymbol{\tau}_h) - \mathbf{b}(\boldsymbol{\xi}_{\boldsymbol{\sigma}},\mathbf{v}_h) - \mathbf{b}(\boldsymbol{\tau}_h,\boldsymbol{\xi}_{\mathbf{u}}) - \mathbf{c}(\boldsymbol{\xi}_{\mathbf{u}};\mathbf{u},\boldsymbol{\tau}_h) \\ & -\mathbf{c}(\boldsymbol{\chi}_{\mathbf{u}};\mathbf{u},\boldsymbol{\tau}_h) - \mathbf{c}(\mathbf{u}_h;\boldsymbol{\xi}_{\mathbf{u}},\boldsymbol{\tau}_h),\end{aligned}$$

  for all $(\boldsymbol{\tau}_h,\mathbf{v}_h) \in \mathbb{X}_{h,0} \times \mathbf{M}_h$.

- Then, since $\mathbf{u}_h \in \mathbf{K}_h$, we apply the discrete inf-sup condition of $\mathbf{A}_{\mathbf{u}_h}$ at the left hand side, the continuity properties of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ at the right hand side, to obtain

$$\|\boldsymbol{\chi}_{\boldsymbol{\sigma}}\|_{\mathbb{X}} + \|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{M}} \le \frac{2}{\widehat{\gamma}}\left(\frac{(1+\nu)}{\nu}\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathbb{X}} + \left\{1 + \frac{1}{\nu}\|\mathbf{u}_h\|_{\mathbf{M}} + \frac{1}{\nu}\|\mathbf{u}\|_{\mathbf{M}}\right\}\|\boldsymbol{\xi}_{\mathbf{u}}\|_{\mathbf{M}} + \frac{1}{\nu}\|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{M}}\|\mathbf{u}\|_{\mathbf{M}}\right),$$

  from which

$$\|\boldsymbol{\chi}_{\boldsymbol{\sigma}}\|_{\mathbb{X}} + \left(1 - \frac{2}{\nu\widehat{\gamma}}\|\mathbf{u}\|_{\mathbf{M}}\right)\|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{M}} \le \frac{2}{\widehat{\gamma}}\left(\frac{(1+\nu)}{\nu}\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathbb{X}} + \left\{1 + \frac{1}{\nu}\|\mathbf{u}_h\|_{\mathbf{M}} + \frac{1}{\nu}\|\mathbf{u}\|_{\mathbf{M}}\right\}\|\boldsymbol{\xi}_{\mathbf{u}}\|_{\mathbf{M}}\right).$$

- Hence, using the fact that $\mathbf{u} \in \mathbf{K}$ and $\mathbf{u}_h \in \mathbf{K}_h$, we obtain

$$\|\boldsymbol{\chi}_{\boldsymbol{\sigma}}\|_{\mathbb{X}} + \|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{M}} \le C\left(\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathbb{X}} + \|\boldsymbol{\xi}_{\mathbf{u}}\|_{\mathbf{M}}\right),$$

  with $C > 0$ independent of h.

- In this way,

$$\|(\mathbf{e}_{\boldsymbol{\sigma}},\mathbf{e}_{\mathbf{u}})\| \le \|(\boldsymbol{\chi}_{\boldsymbol{\sigma}},\boldsymbol{\chi}_{\mathbf{u}})\| + \|(\boldsymbol{\xi}_{\boldsymbol{\sigma}},\boldsymbol{\xi}_{\mathbf{u}})\| \le (1+C)\|(\boldsymbol{\xi}_{\boldsymbol{\sigma}},\boldsymbol{\xi}_{\mathbf{u}})\|,$$

  which combined to the fact that $(\boldsymbol{\zeta}_h,\mathbf{v}_h) \in \mathbb{X}_{h,0} \times \mathbf{M}_h$ is arbitrary, concludes the proof.

Rates of convergence

Theorem

Let $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{X}_0 \times \mathbf{M}$ *and* $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{X}_{h,0} \times \mathbf{M}_h$ *be the unique solutions of continuous and discrete versions of the second approach, respectively with* $\mathbf{f}$ *and* $\mathbf{u}_D$ *satisfying the previous inequality. Assume further that* $\boldsymbol{\sigma} \in \mathbb{H}^{l+1}(\Omega)$, $\mathbf{div}\,\boldsymbol{\sigma} \in \mathbf{W}^{l+1,4/3}(\Omega)$ *and* $\mathbf{u} \in \mathbf{W}^{l+1,4}(\Omega)$, *for* $0 \le l \le k+1$. *Then there exists* $C_{rate} > 0$, *independent of* h, *such that*

$$\|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \mathbf{u} - \mathbf{u}_h)\| \;\le\; C_{rate} h^{l+1} \left\{ |\boldsymbol{\sigma}|_{\mathbb{H}^{l+1}(\Omega)} + |\mathbf{div}\,\boldsymbol{\sigma}|_{\mathbf{W}^{l+1,4/3}} \;+\; |\mathbf{u}|_{\mathbf{W}^{l+1,4}(\Omega)} \right\}.$$

Proof: The result is a straightforward application of the previous theorem and estimates:

- For each $0 \le t \le k+1$ and for each $w \in \mathrm{W}^{t,r}(\Omega)$, with $1 \le r \le \infty$, there holds

$$\|w - \mathcal{P}_h^k(w)\|_{\mathrm{L}^r(\Omega)} \le C h^t |w|_{\mathrm{W}^{t,r}(\Omega)}.$$

- For all $0 \le l \le k+1$, and for all $\tau \in \mathbf{H}^{l+1}(\Omega)$ with $\mathrm{div}\,\tau \in \mathrm{W}^{l+1,r}(\Omega)$, there holds

$$\|\tau - \Pi_h^k(\tau)\|_{0,\Omega} + \|\mathrm{div}\,\tau - \mathrm{div}\,(\Pi_h(\tau))\|_{\mathrm{L}^r(\Omega)} \le c h^{l+1} \Big\{ |\tau|_{\mathbf{H}^{l+1}(\Omega)} + |\mathrm{div}\,\tau|_{\mathrm{W}^{l+1,r}(\Omega)} \Big\}$$

Computing further variables of interest

If $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{X}_{h,0} \times \mathbf{M}_h$ is the solution of the discrete version of our second approach, we propose the following approximations for the aforementioned variables:

$$p_h = -\frac{1}{n}\left(\operatorname{tr}(\boldsymbol{\sigma}_h) + \operatorname{tr}(\mathbf{u}_h \otimes \mathbf{u}_h) - \frac{1}{|\Omega|}(\operatorname{tr}(\mathbf{u}_h \otimes \mathbf{u}_h), 1)_\Omega\right),$$

$$\mathbf{G}_h = \frac{1}{\nu}(\boldsymbol{\sigma}_h^d + (\mathbf{u}_h \otimes \mathbf{u}_h)^d),$$

$$\tilde{\boldsymbol{\sigma}}_h = \boldsymbol{\sigma}_h^d + \boldsymbol{\sigma}_h^t - \frac{1}{n|\Omega|}(\operatorname{tr}(\mathbf{u}_h \otimes \mathbf{u}_h), 1)_\Omega \mathbb{I} + \mathbf{u}_h \otimes \mathbf{u}_h + (\mathbf{u}_h \otimes \mathbf{u}_h)^d$$

and

$$\boldsymbol{\omega}_h = \frac{1}{2\nu}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^t).$$

Rates of convergence

Corollary

Let $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{X}_0 \times \mathbf{M}$ and $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{X}_{h,0} \times \mathbf{M}_h$ be the unique solutions of the continuous and discrete versions of the second approach, respectively, with $\mathbf{f}$ and $\mathbf{u}_D$ satisfying the previous inequality. Assume further that $\boldsymbol{\sigma} \in \mathbb{H}^{l+1}(\Omega)$, $\mathbf{div}\,\boldsymbol{\sigma} \in \mathbf{W}^{l+1,4/3}(\Omega)$ and $\mathbf{u} \in \mathbf{W}^{l+1,4}(\Omega)$, for $0 \leq l \leq k+1$. Finally, let p_h, $\tilde{\boldsymbol{\sigma}}_h$, $\mathbf{G}_h$ and $\boldsymbol{\omega}_h$ given by the previous equalities. Then there exists $\tilde{C} > 0$, independent of h, such that

$$\begin{aligned}
&\|p - p_h\|_{0,\Omega} + \|\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}_h\|_{0,\Omega} + \|\mathbf{G} - \mathbf{G}_h\|_{0,\Omega} + \|\boldsymbol{\omega} - \boldsymbol{\omega}_h\|_{0,\Omega} \\
&\quad \leq \tilde{C} h^{l+1} \left\{ |\boldsymbol{\sigma}|_{\mathbb{H}^{l+1}(\Omega)} + |\mathbf{div}\,\boldsymbol{\sigma}|_{\mathbf{W}^{l+1,4/3}} + |\mathbf{u}|_{\mathbf{W}^{l+1,4}(\Omega)} \right\}.
\end{aligned}$$

Numerical results

Numerical aspects

- We impose the condition $(\mathrm{tr}\,(\boldsymbol{\sigma}_h), 1)_\Omega = 0$ via a penalization strategy. More precisely, we simply replace the discrete problem by the system: Find $((\boldsymbol{\sigma}_h, \mathbf{u}_h), \lambda_h) \in (\mathbb{X}_h \times \mathbf{M}_h) \times \mathbb{R}$, such that

$$
\begin{aligned}
\mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{u}_h) + \mathbf{c}(\mathbf{u}_h; \mathbf{u}_h, \boldsymbol{\tau}_h) + \lambda_h \int_\Omega \mathrm{tr}\,\boldsymbol{\tau}_h &= F(\boldsymbol{\tau}_h), \\
\mathbf{b}(\boldsymbol{\sigma}_h, \mathbf{v}_h) &= G(\mathbf{v}_h), \\
\eta_h \int_\Omega \mathrm{tr}\,\boldsymbol{\sigma}_h &= 0,
\end{aligned}
\tag{8}
$$

$\forall\, ((\boldsymbol{\tau}_h, \mathbf{v}_h), \eta_h) \in (\mathbb{X}_h \times \mathbf{M}_h) \times \mathbb{R}$.

- It is easy to see from the system above, taking $\boldsymbol{\tau}_h = \mathbb{I}$ in the first equation, that λ_h is actually an artificial scalar unknown whose exact value is 0.

- Moreover, from the third equation we easily obtain that $\boldsymbol{\sigma}_h \in \mathbb{X}_{h,0}$. As a consequence, $(\boldsymbol{\sigma}_h, \mathbf{u}_h)$ is a solution to the discrete problem if and only if $((\boldsymbol{\sigma}_h, \mathbf{u}_h), 0)$ is a solution to (8).

Numerical aspects

- FreeFem++ in conjunction with the direct linear solver UMFPACK on structured meshes.
- We use the Newtons method with a fixed tolerance tol= 1E-6, and the iterations are terminated once the relative error of the entire coefficient vectors between two consecutive iterates is sufficiently small, i.e.,

$$\frac{|\mathbf{coeff}^{m+1} - \mathbf{coeff}^{m}|}{|\mathbf{coeff}^{m+1}|} \leq tol,$$

  where $|\cdot|$ is the standard euclidean norm $\mathbb{R}^N$, with N denoting the total number of degrees of freedom defining the finite element subspaces $\mathbb{X}_h$ and $\mathbf{M}_h$.
- We take $(\boldsymbol{\sigma}_h^0, \mathbf{u}_h^0) = (\mathbf{0}, \mathbf{0})$ as initial guess.

Numerical aspects

- We now introduce some additional notations. The individual errors are denoted by:

$$\mathsf{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbb{X}}, \quad \mathsf{e}(\mathbf{u}) := \|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{M}}, \quad \mathsf{e}(p) := \|p-p_h\|_{0,\Omega},$$

$$\mathsf{e}(\boldsymbol{\omega}) := \|\boldsymbol{\omega}-\boldsymbol{\omega}_h\|_{0,\Omega}, \quad \mathsf{e}(\nabla\mathbf{u}) := \|\nabla\mathbf{u}-\mathbf{G}_h\|_{0,\Omega}, \quad \mathsf{e}(\tilde{\boldsymbol{\sigma}}) := \|\tilde{\boldsymbol{\sigma}}-\tilde{\boldsymbol{\sigma}}_h\|_{0,\Omega}.$$

  where p_h, $\boldsymbol{\omega}_h$, $\mathbf{G}_h$, and $\tilde{\boldsymbol{\sigma}}_h$ are the variables computed through the post-processing formulas.

- In addition, we let $r(\boldsymbol{\sigma})$, $r(\mathbf{u})$, $r(p)$, $r(\boldsymbol{\omega})$, $r(\nabla\mathbf{u})$, and $r(\tilde{\boldsymbol{\sigma}})$ be the experimental rates of convergence given by

$$r(\boldsymbol{\sigma}) := \frac{\log(\mathsf{e}(\boldsymbol{\sigma})/\mathsf{e}'(\boldsymbol{\sigma}))}{\log(h/h')}, \quad r(\mathbf{u}) := \frac{\log(\mathsf{e}(\mathbf{u})/\mathsf{e}'(\mathbf{u}))}{\log(h/h')}, \quad r(p) := \frac{\log(\mathsf{e}(p)/\mathsf{e}'(p))}{\log(h/h')},$$

$$r(\boldsymbol{\omega}) := \frac{\log(\mathsf{e}(\boldsymbol{\omega})/\mathsf{e}'(\boldsymbol{\omega}))}{\log(h/h')}, \quad r(\nabla\mathbf{u}) := \frac{\log(\mathsf{e}(\nabla\mathbf{u})/\mathsf{e}'(\nabla\mathbf{u}))}{\log(h/h')}, \quad r(\tilde{\boldsymbol{\sigma}}) := \frac{\log(\mathsf{e}(\tilde{\boldsymbol{\sigma}})/\mathsf{e}'(\tilde{\boldsymbol{\sigma}}))}{\log(h/h')},$$

  where h and h' denote two consecutive meshsizes with errors e and e'.

Numerical Example

Example 1 (analytical solution obtained by Kovasznay)

Let $\Omega := (-1/2, 3/2) \times (0, 2)$. For a given ν, the exact solution is given by

$$\mathbf{u}(x_1, x_2) = \begin{pmatrix} 1 - e^{\lambda x_1} \cos(2\pi x_2) \\ \\ \frac{\lambda}{2\pi} e^{\lambda x_1} \sin(2\pi x_2) \end{pmatrix},$$

$$p(x_1, x_2) = -\frac{1}{2} e^{2\lambda x_1} + \bar{p},$$

where

$$\lambda := \frac{-8\pi^2}{\nu^{-1} + \sqrt{\nu^{-2} + 16\pi^2}}$$

and $\bar{p}$ is such that $\int_\Omega p = 0$.

Number of iterations for the Newton's method

ν	$h = 0.1905$	$h = 0.0978$	$h = 0.0517$	$h = 0.0316$	$h = 0.0156$
1	4	4	4	4	3
0.1	–	5	5	5	5
0.01	–	–	–	6	6

Table: EXAMPLE 1: Convergence behavior of the Newton's method with respect to the parameter ν, considering $\mathrm{RT}_0 - \mathrm{P}_0$.

Errors and rates of convergence for the mixed $\mathrm{RT}_0 - \mathrm{P}_0$ approximation

N	h	$\mathsf{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\mathsf{e}(\mathbf{u})$	$r(\mathbf{u})$	$\Vert\mathbf{div}\,\boldsymbol{\sigma}_h\Vert_{l^\infty}$	Iterations
3035	0.1905	45.2555	–	2.2882	–	9.0949e-13	4
12199	0.0978	22.5468	1.0457	1.1136	1.0809	1.8190e-12	4
48697	0.0517	11.6288	1.0393	0.5645	1.0665	5.4570e-12	4
196483	0.0316	5.6438	1.4634	0.2752	1.4542	1.4552e-11	4
774345	0.0156	2.8305	0.9815	0.1390	0.9720	2.9104e-11	3

$\mathsf{e}(p)$	$r(p)$	$\mathsf{e}(\boldsymbol{\omega})$	$r(\boldsymbol{\omega})$	$\mathsf{e}(\nabla\mathbf{u})$	$r(\nabla\mathbf{u})$	$\mathsf{e}(\tilde{\boldsymbol{\sigma}})$	$r(\tilde{\boldsymbol{\sigma}})$
21.5953	–	24.0425	–	24.1994	–	46.0334	–
10.7272	1.0502	14.3573	0.7738	12.8855	0.9459	21.9551	1.1112
5.4725	1.0565	7.7320	0.9715	6.7505	1.0148	11.0729	1.0745
2.6113	1.4978	3.8602	1.4062	3.3198	1.4367	5.2839	1.4977
1.2954	0.9970	1.9803	0.9493	1.6791	0.9694	2.6059	1.0053

Table: EXAMPLE 1: Degrees of freedom, meshsizes, errors, rates of convergence, L^∞–norm of $\mathbf{div}\,\boldsymbol{\sigma}_h$ and number of iterations for the mixed $\mathrm{RT}_0 - \mathrm{P}_0$ approximation of the Navier-Stokes problem, with $\nu = 1$.

Errors and rates of convergence for the mixed $\mathrm{RT}_1 - \mathrm{P}_1$ approximation

N	h	$\mathsf{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\mathsf{e}(\mathbf{u})$	$r(\mathbf{u})$	$\Vert\mathbf{div}\,\boldsymbol{\sigma}_h\Vert_{l^\infty}$	Iterations
9633	0.1905	9.3862	–	0.2653	–	1.0687e-11	4
38881	0.0978	2.2385	2.1515	0.0613	2.1985	2.9559e-11	4
155521	0.0517	0.5926	2.0863	0.0161	2.1012	6.8212e-11	4
628129	0.0316	0.1396	2.9268	0.0038	2.9436	1.6735e-10	3
2476673	0.0156	0.0354	1.9503	0.0010	1.9315	3.6744e-10	3

$\mathsf{e}(p)$	$r(p)$	$\mathsf{e}(\boldsymbol{\omega})$	$r(\boldsymbol{\omega})$	$\mathsf{e}(\nabla\mathbf{u})$	$r(\nabla\mathbf{u})$	$\mathsf{e}(\tilde{\boldsymbol{\sigma}})$	$r(\tilde{\boldsymbol{\sigma}})$
4.2620	–	4.6641	–	5.2331	–	10.1175	–
0.9890	2.1925	1.2550	1.9703	1.2983	2.0922	2.3556	2.1875
0.2594	2.1008	0.3445	2.0295	0.3491	2.0619	0.6202	2.0948
0.0589	2.9996	0.0882	2.7589	0.0851	2.8562	0.1428	2.9724
0.0146	1.9873	0.0234	1.8837	0.0220	1.9259	0.0355	1.9812

Table: EXAMPLE 1: Degrees of freedom, meshsizes, errors, rates of convergence, L^∞–norm of $\mathbf{div}\,\boldsymbol{\sigma}_h$ and number of iterations for the mixed $\mathrm{RT}_1 - \mathrm{P}_1$ approximation of the Navier-Stokes problem, with $\nu = 1$.

Numerical Example

Example 2

$\Omega := (0,1) \times (0,0.5) \times (0,0.5)$ the viscosity $\nu = 1$ and take $\mathbf{f}$ and $\mathbf{u}_D$ so that the exact solution is given by

$$\mathbf{u}(x_1,x_2,x_3) \; = \begin{pmatrix} -x_1(x_2-x_3)(x_2+x_3) \\ \\ x_2(x_1-x_3)(x_1+x_3) \\ \\ -x_3(x_1-x_2)(x_1+x_2) \end{pmatrix},$$

$$p(x_1,x_2,x_3) \; = x_2 x_3 (x_1 - 0.5)\,.$$

N	h	e($\boldsymbol{\sigma}$)	$r(\boldsymbol{\sigma})$	e($\mathbf{u}$)	$r(\mathbf{u})$	Iterations
7393	0.1768	0.0616	–	0.0210	–	4
57217	0.0884	0.0312	0.9799	0.0105	0.9919	3
190945	0.0589	0.0208	1.0000	0.0070	0.9976	3
450049	0.0442	0.0156	1.0037	0.0053	0.9988	3
876001	0.0354	0.0125	1.0043	0.0042	0.9993	3

e(p)	$r(p)$	e($\boldsymbol{\omega}$)	$r(\boldsymbol{\omega})$	e($\nabla\mathbf{u}$)	$r(\nabla\mathbf{u})$	e($\tilde{\boldsymbol{\sigma}}$)	$r(\tilde{\boldsymbol{\sigma}})$
0.0088	–	0.0550	–	0.0730	–	0.0730	–
0.0041	1.0990	0.0263	1.0677	0.0366	0.9948	0.0366	0.9093
0.0025	1.2391	0.0173	1.0332	0.0245	0.9915	0.0245	0.9508
0.0017	1.2618	0.0129	1.0197	0.0184	0.9917	0.0184	0.9662
0.0013	1.2526	0.0103	1.0130	0.0148	0.9924	0.0148	0.9741

Table: EXAMPLE 2: Degrees of freedom, meshsizes, errors, rates of convergence and number of iterations for the mixed $\mathrm{RT}_0 - \mathrm{P}_0$ approximations of the three–dimensional Navier-Stokes problem, with $\nu = 1$.

Thanks for your attention!!