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Exercise sheet 1.

Name

Exercise	1	2	3	4	Σ
Points					

Deadline: **Friday, 03.09.2022, 16:00**.

Please use this page as a cover sheet and enter your name in the appropriate fields. It is best if you do this sheet in pairs of two. Please staple your solutions to this cover sheet.

This exercise sheet is about ideas introduced in the first lecture week. This includes algebraic K-theory of unital rings, categories and functors.

Exercise 1. Recall that a monoid M is a set together with an associative, binary operation with a unit element. For a commutative monoid M, consider the set $\mathcal{G}(M)$ of equivalence classes of elements $(x, y) \in M^2$ under the relation $(x_1, y_1) \sim (x_2, y_2)$ if and only if there are z_1 and z_2 such that $(x_1 + z_1, x_2 + z_1) = (y_1 + z_2, y_2 + z_2)$.

1. Find a binary operation that turns $\mathcal{G}(M)$ into an abelian group.
2. Show that there is a monoid homomorphism $M \to \mathcal{G}(M)$ which satisfies the property that any monoid homomorphism $M \to A$ into an abelian group factorises uniquely through $\mathcal{G}(M)$.
3. Compute $\mathcal{G}(M)$ when M is $(\mathbb{Z} \setminus \{0\}, \cdot)$ and $(\mathbb{N}, \cdot)$.
4. Is the map $M \to \mathcal{G}(M)$ injective?
5. Show that the assignment $M \mapsto \mathcal{G}(M)$ is a functor from the category of monoids to the category of abelian groups.

Exercise 2. Let $\mathfrak{Nor}$ be the category of normed spaces with bounded linear maps as morphisms. Let $f \colon X \to Y$ be a morphism in $\mathfrak{Nor}$. Show that f is a monomorphism if and only if it is injective, and an epimorphism if and only if its range is dense. You may use the following without proof: if $Z \subseteq Y$ is a closed subspace, then the quotient space Y/Z with the quotient norm is again a normed space. Find a morphism f in $\mathfrak{Nor}$ that is both monic and epic, but not an isomorphism.

Exercise 3. Let R be a commutative, unital ring. We say that an R-module P is *projective* if the representable functor $\mathrm{Hom}_R(P, -) \colon \mathsf{Mod}_R \to \mathsf{Mod}_R$ preserves surjective R-module maps. Show that an R-module is projective if and only if is a direct summand of a free module, if and only if any surjective R-module map onto it splits by an R-module map. Show that for every R-module M, there is a surjective R-module map $P \to M$, where P is a projective R-module.

Exercise 4. Let R be a commutative, unital ring. Let $\mathbf{V}(R)$ denote monoid of of finitely generated projective R-modules. Show that the assignment $R \mapsto \mathbf{V}(R)$ is a functor from the category of rings to the category of commutative monoids. Show that this functor is *additive* in the sense that $\mathbf{V}(R_1 \oplus R_2) \cong \mathbf{V}(R_1) \times \mathbf{V}(R_2)$. Deduce that $K_0 \colon R \mapsto K_0(R)$ is an additive functor.