Cyclic Homology

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Introduction

In this course, we will study certain invariants of associative algebras, called *cyclic homology*. These invariants were independently introduced by Alain Connes and Boris Tsygan in the 1980s as part of developments in the field of *noncommutative geometry*. The field has grown rapidly ever since to provide deep insights into several areas of Mathematics such as algebraic K-theory, arithmetic geometry, index theory and, p-adic Hodge theory, just to mention a few. In what follows, we shall trace a roadmap for the course and, also the author's understanding of the subject.

Algebra-geometry correspondences

The "invariants" we shall study are those of *associative algebras*, that is, rings that are simultaneously vector spaces over a field. These associative algebras arise in connection with "geometric" objects such as topological spaces, manifolds, and varieties. Let us consider two motivating examples:

$$\left\{\text{Locally compact Hausdorff spaces}\right\} \longleftrightarrow \left\{\text{Commutative } C^* \text{ - algebras}\right\}$$

$$\left\{\text{Affine algebraic varieties}\right\} \longleftrightarrow \left\{\text{Reduced, finitely generated commutative algebras}\right\}$$

In each of these two examples, the corresponding takes the geometric object X on the left hand side to an algebra of *functions* $\mathcal{F}(X)$ on the right hand side. When X is a topological space, $\mathcal{F}(X) = C_0(X)$ is the algebra of continuous functions vanishing at ∞, which is a C^*-algebra with the supremum norm. For an affine algebraic variety, $\mathcal{F}(X) = \mathcal{O}(X)$ - the coordinate ring of X. The other direction of the correspondence is far more nontrivial; it says that a purely algebraic object as on the right hand side can be used to derive geometric information. Indeed, given a commutative C^*-algebra A, its Gelfand spectrum $\hat{A} := \{\xi\colon A \to \mathbb{C} : \xi \text{ nonzero, multiplicative and linear map}\}$ with the topology of pointwise convergence is a locally compact Hausdorff space. This is a bijective correspondence by the Gelfand-Naimark Theorem. Similarly, given a reduced, finitely generated commutative algebra (say over an algebraically closed field k) written as $A = k[x_1, \dots, x_n]/(f_1, \dots, f_m)$, we can define $\mathrm{Spec}(A) = \{x \in k^n : f_i(x) = 0 \text{ for all } i\}$. That this is a bijective correspondence is the *Hilbert Nullstellensatz.*

In the correspondences above, the algebraic objects that appear on the right hand side are manifestly *commutative*. This suggests the possibility that arbitrary associative algebras should in some sense correspond to a broader notion of geometry:

$$\left\{\text{Noncommutative spaces}\right\} \longleftrightarrow \left\{\text{Associative algebras}\right\}.$$

A prototypical example of a noncommutative space is the quotient space X/G of the action of group G on a topological space X. The issue that typically arises is that even if X is a well-behaved topological space (compact and Hausdorff), its quotient space X/G does not inherit these properties. These badly behaved quotient spaces are modelled by *groupoids*, whose corresponding "function spaces" are noncommutative algebras. Examples of groupoids treated in this realm include the noncommutative torus, path algebras of graphs and the holonomy groupoid of a foliated manifold. In conclusion, associative algebras (including commutative algebras, of course) model a wide array of geometric objects and phenomena.

At this point of time, you might wonder what one precisely gains by passing to (associative) algebras, instead of simply working with geometric objects. The answer to this is rather subtle and lies at the philosophical core of the subject. *Associative algebras eliminate the need for coordinates.* Consider a manifold M, whose dataset comprises a specific *choice* of atlas, which in turn consists of charts. These charts are ways of locally viewing a manifold in the Euclidean space, that is, they are coordinates. If one were to work purely geometrically, they would get around this issue by choosing a maximal atlas by considering *all* possible charts. On the other hand, certain atlases appear more natural than others, and in an ideal world, we should not have to choose between coordinate systems! The elegant way of dealing with this is using the algebra of smooth functions $\mathrm{C}^\infty(M)$ on M. Equipping this algebra with a certain Jacobson topology, there is a homeomorphism between M and the space of nonzero algebra homomorphisms $\mathrm{C}^\infty(M) \to \mathbb{C}$. This identification can be used to identify the smooth structure on M uniquely up to diffeomorphism.

Invariants and their role in geometry

An invariant is an assignment of the form

$$\Big\{ \text{ Geometric objects } \Big\} \cong \Big\{ \text{ Algebraic objects } \Big\} \longrightarrow \Big\{ \text{Numbers } \Big\};$$

where the assignment preserves a choice of relations we impose on the left hand side. Let us now try to unravel this definition by way of examples with increasing levels of complexity.

K-theoretic invariants. The most basic and yet powerful invariants in Mathematics have their roots in good old linear algebra. We first start with

The dimension of a vector space.

Example. Let F be a fixed field. The assignment

$$\Big\{ \text{ Finite dimensional } F - \text{vector spaces } \Big\} \longrightarrow \mathbb{N}, \quad V \mapsto \dim(V)$$

is an invariant, where the choice of relations is isomorphisms of vector spaces. That is, isomorphic finite-dimensional vector spaces have the same dimension.

Example. Let us consider the same assignment as above, but this time, we divide out the isomorphism relation. That is, two finite-dimensional vector spaces are in the same equivalence class if and only if they are isomorphic. The resulting collection $\mathbf{V}(F) = \{[F^n] : n \in \mathbb{N}\}$ actually has more structure - it has a zero object, namely, the zero dimensional F-vector space, and an associative, commutative addition law $[V] + [W] := [V \oplus W]$. That is, $\mathbf{V}(F)$ is a commutative *monoid.* Of course, $\mathbb{N}$ with its usual addition is also a commutative monoid. Furthermore, it is trivial to see that the dimension map is an isomorphism $\mathbf{V}(F) \cong \mathbb{N}$ of commutative monoids. By performing an operation called *group completion*, we can turn any commutative monoid $\mathbf{M}$ into a group $\mathcal{G}(\mathbf{M})$. As a consequence, we have an induced isomorphism

$$K_0(F) := \mathcal{G}(\mathbf{V}(F)) \cong \mathcal{G}(\mathbb{N}) \cong \mathbb{Z}$$

of groups.

Example. The examples above really only depend on a choice of field F. Once such a field has been chosen, for each power of a field, we get a number - namely, the power itself. In other words, what we have is a “function" that associates to a field F, the group $K_0(F) \cong \mathbb{Z}$. The domain of this “function" is now the collection of all fields, that is, we now have an assignment

$$\{\text{ Fields }\} \longrightarrow \{\text{ Abelian groups }\}, \quad F \mapsto K_0(F) \cong \mathbb{Z} \ni \text{numbers.}$$

Example. We can now generalise from the collection of fields to the collection of all unital rings. The intermediate collection of finite dimensional vector spaces over a field is then replaced by the collection of finitely generated *projective* left (or right) modules over a ring R. This is still a commutative monoid $\mathbf{V}(R)$ under direct sum of modules. The dimension map is replaced by the *rank* of a finitely generated module, which descends to an assignment

$$\{\text{ Unital Rings }\} \longrightarrow \{\text{ Abelian groups }\}, \quad R \mapsto K_0(R).$$

Notice that we have now replaced "Numbers" in the original definition of an invariant with Abelian groups, but we will just have to learn to live with this new definition of a number. Finally, by a standard argument, we can extend the above invariant to include all rings (that is, even non-unital rings). Since rings are $\mathbb{Z}$-algebras, we now have an invariant

$$\{\mathbb{Z} - \text{Algebras }\} \longrightarrow \{\text{ Abelian groups }\}, R \mapsto K_0(R),$$

called the zeroth *algebraic K-theory* group of a ring R.

The determinant of a matrix. Let us again start with a field F and let $GL_n(F)$ denote the ring of invertible $n \times n$-matrices. Let $GL_\infty(F)$ denote the ring obtained by taking the union of each $GL_n(F)$.

Example. The assignment $GL_\infty(F) \to F^*$ mapping $A \mapsto \det(A)$ is a group homomorphism into the group of units of the field F. Let $E_\infty(F)$ denote the subgroup of elementary matrices; these have unit determinant. As a consequence, the determinant map descends to a group homomorphism

$$\det: K_1(F) := GL_\infty(F)/E_\infty(F) \to F^*,$$

which is, in fact, an isomorphism.

Example. In general, for any ring R, we can define $K_1(R) := GL_\infty(R)/E_\infty(R)$. This gives us an invariant

$$\{\mathbb{Z} - \text{Algebras }\} \longrightarrow \{\text{ Abelian groups }\}, R \mapsto K_1(R),$$

called the *first algebraic K-theory* of a ring.

Remark. We could actually go on and define *higher algebraic K-groups* $K_n(R)$ of a ring R for each integer n. Their definition requires some tools we do not yet have. For the moment, these are just a collection of abelian groups.

So what does algebraic K-theory have to do with geometry? The origins of K_0 go back to Grothendieck's generalisation of the Hirzebruch-Riemann-Roch Theorem, which relates the *Euler characteristic* of say a compact, complex manifold M with certain invariants of vector bundles over M (namely, the *Chern class* of a given vector bundle and the *Todd class* of its tangent bundle). The link between vector bundles and K_0 is established by the Serre-Swan Theorem, which provides a bijective correspondence between finitely generated projective modules over $\mathrm{C}^\infty(M)$ and smooth vector bundles over M.

To see geometric applications of K_1, we turn to *index theory*, which studies the extent to which a differential operator fails to be bijective. A particularly important such class of differential operators is that of Fredholm operators, which we describe through a toy example. Let F be a field and let $F^\infty = \bigoplus_{n=0}^\infty F$ - these are infinite tuples, where all but finitely many entries are zero. An operator $D: F^\infty \to F^\infty$ is called *Fredholm* if its kernel and cokernel are both finite-dimensional. The *index* of D is the difference $\mathrm{ind}(D) = \dim(\ker(D)) - \dim(\mathrm{coker}(D))$. Let E be the endomorphism ring on F^∞. It is not hard to see that an operator is Fredholm if and only if its image under the quotient map $E \twoheadrightarrow E/M_\infty(F)$ is invertible. The index of D is then the image of the "canonical" map $K_1(E/M_\infty(F)) \to K_0(M_\infty(F)) \cong K_0(F) \cong \mathbb{Z}$. In this course, we will make precise how such maps arise.

Cohomological invariants. The K-theoretic invariants defined above are in some sense the most fundamental invariants of geometric objects (that is, algebras) one can conceive. More precisely, these invariants satisfy a minimal set of properties that are meant to serve as a role model for any other invariant. But given that the list of properties that K-theory ought to satisfy is rather small, these groups are extremely hard to compute. The invariants one runs into more commonly while in geometric situations are very concrete and easy to compute. For instance, the Euler characteristic of a compact n-dimensional manifold is an alternating sum

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \dim(\mathsf{hdR}^i(M))$$

of the dimensions *de Rham cohomology groups*.

Example. The shadow of the Euler characteristic mentioned is the de Rham cohomology

$$\{\ \text{Manifolds}\ \} \longrightarrow \{\ \text{Abelian groups}\ \}, \quad M \mapsto \mathsf{hdR}^i(M),$$

of a manifold, where the groups $\mathsf{hdR}^i(M)$ are computed using differential forms on the manifold.

Example. In a similar vein as above, for an affine algebraic variety (or affine scheme) over a field of characteristic zero, a useful invariant is its *algebraic de Rham cohomology*

$$\{\ \text{Affine schemes}\ \} \cong \{\ \text{Commutative rings}\ \} \longrightarrow \{\ \text{Abelian groups}\ \}, \quad A \mapsto \mathsf{hdR}^i(A),$$

which is computed using *algebraic differential forms* or *Kähler differentials*. One could again talk about Euler characteristics in this context to get actual numbers, but it is best we get used to abelian groups as being the codomain of invariants.

The two types of de Rham cohomology theories mentioned above are examples of "homotopy invariant" assignments. This means that if we continuously deform a manifold or a variety, the groups hdR^i should remain the same. In particular, if a manifold or a variety is contractible as a topological space, then its de Rham cohomology should vanish, as we would like to think of points as being uninteresting. A manifestation of this property should be familiar to you from the following theorem from calculus:

THEOREM (Poincare Lemma). *Every closed form on* $\mathbb{R}^n$ *is exact.*

Notice that in the second example above, we have the condition that the field over which the varieties are defined be of characteristic zero. This condition is there to prove the homotopy invariance of de Rham cohomology, which entails integrating differential forms, thereby introducing denominators - something illegal in positive characteristic. However, sometime in the 1960s, there arose a desire to construct an invariant similar to de Rham cohomology, which was postulated to have striking consequences in arithmetic geometry. This led to the *Weil conjectures*, and ultimately a proof of the Riemann hypothesis for certain classes of varieties over finite fields.

Reconcilliation between different invariants. We now have two different classes of invariants - algebraic K-theory and de Rham cohomology. As already mentioned, algebraic K-theory is hard to compute, but is rather easy to define, using only generalisations of concepts from linear algebra. On the other hand, de Rham cohomology is somewhat harder to define as it uses more geometric data (such as differential forms), but is often far easier to compute. Finally, both classes of invariants have fundamental relevance in different areas of geometry, as I have hopefully convinced the reader. At this point of time, the following natural question arises:

- How are K-theory and de Rham type invariants related?

Before we try to answer this question, we first notice that de Rham type invariants only make sense (apriori) for commutative algebras such as smooth functions on a manifold $\mathrm{C}^\infty(M)$ or the coordinate ring $\mathcal{O}(V)$ of a variety. On the other hand, the study of K-theory very quickly takes us away from realm of ordinary commutative algebras, even if we are only interested in, say, affine schemes. The reconcilliation between algebraic K-theory and de Rham cohomology (in its various forms) is performed by *cyclic homology*, which is the subject of this course.

CHAPTER 1

Basics of category theory

1. Categories, functors and natural transformations

1.1. Categories.

Definition 1.1. A *category* $\mathcal{C}$ consists of a collection of objects $\mathcal{C}^0$ and a collection of morphisms $\mathcal{C}^1$ such that:

- each morphism has a specified *domain* and *codomain*; denoted $f: X \to Y$;
- each object $X \in \mathcal{C}^0$ has an *identity* morphism $1_X: X \to X$;
- For any two morphisms $f: X \to Y$, $g: Y \to Z \in \mathcal{C}^1$, there exists a composite morphism $g \circ f: X \to Z$.

This data is subject to the following requirements:

- For $f: X \to Y$ in $\mathcal{C}^1$, $1_Y \circ f = f$ and $f \circ 1_X = f$;
- For morphisms $f: X \to Y$, $g: Y \to Z$ and $h: Z \to W$, $h \circ (g \circ f) = (h \circ g) \circ f$. We shall unambiguously denote the composition by hgf.

1.1.1. *Examples of categories.*

Example 1.2. The collection Set of all sets and set-theoretic functions between them is a category.

Example 1.3. The collections Ab and Group of abelian groups and groups, with group homomorphisms as morphisms, is a category.

Example 1.4. Let R be a ring. Then the collection Mod_R of R-modules with R-module homomorphisms is a category.

Example 1.5. The collection $\mathsf{TVS}_{\mathbb{C}}$ of topological $\mathbb{C}$-vector spaces with continuous linear maps between them is a category.

A slightly different example is as follows:

Example 1.6. Let G be a group. We denote by BG the collection whose objects $BG^0 = \{\bullet\}$ and whose morphisms $BG^1 = G$. Composition between two morphisms $g: \bullet \to \bullet$, $h: \bullet \to \bullet$ is given by the group G's composition law: $g \circ h: \bullet \to \bullet$ and the identity is given by the identity of the group. The axioms of a group ensure that the axioms of a category are satisfied.

The categories in Examples 1.2-1.5 are examples of *concrete categories.* Roughly speaking, this means that their underlying objects have an underlying set and the morphisms are functions between these underlying sets, preserving further structure. On the other hand, categories of the type in Example 1.6 is an example of an *abstract category.*

1.1.2. *Some special morphisms in a category.*

Definition 1.7. A category $\mathcal{C}$ is said to be *small* if the collection of all morphisms $\mathcal{C}^1$ is a set. It is called *locally small* if for any two objects X, $Y \in \mathcal{C}^0$, the collection $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms between them is a set.

Example 1.8. The category BG introduced in Example 1.6 is a small category. The category Set of all sets is not small, but it is *locally small*.

Example 1.9. Let $\mathcal{P} = (P, \leq)$ be a set with a preorder - that is, a non-empty set with a binary relation that is reflexive and transitive. We can view this as a category as follows: the objects of $\mathcal{P}$ are the elements of P. Given two objects x, $y \in P$, there is a unique morphism between x and y if and only if $x \leq y$. It is instructive to check that this is really a category.

Remark 1.10. Unless specifically needed or allowed, all categories in this course will be assumed to be locally small.

Definition 1.11. Let $\mathcal{C}$ be a category and $f\colon X \to Y$ a morphism. We call f

- an *epimorphism* if whenever there are morphisms $g, h\colon Y \rightrightarrows Z$ such that $g \circ f = h \circ f$, we have $g = h$;
- a *split epimorphism* if there is a morphism $g\colon Y \to X$ such that $f \circ g = 1_Y$;
- a *monomorphism* if whenever $g, h\colon Z \rightrightarrows X$ satisfy $f \circ g = f \circ h$, we have $g = h$;
- a *split monomorphism* if there is a morphism $g\colon Y \to X$ such that $g \circ f = 1_X$;
- an *isomorphism* if there exists a morphism $g\colon Y \to X$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$.

Example 1.12. In the category of sets, an isomorphism is precisely a set-theoretic bijection. A monomorphism is an injective function and an epimorphism is a surjective function of sets. Notice that in Set, any monomorphism/epimorphism is split by the Axiom of Choice.

Example 1.13. There are two interesting categories we can talk about from topology. The first is the category Top of topological spaces and continuous maps between them. In this category, an isomorphism is a *homeomorphism* between topological spaces. But this notion of an isomorphism is too strong. For instance, consider the real numbers $\mathbb{R}$ with its usual topology. This is a contractible space, that is, it is homotopy equivalent to a point. We would therefore like to enlarge our class of isomorphisms to include even homotopy equivalent spaces. This is achieved by the category hTop whose objects are topological spaces and whose morphisms are *homotopy classes* of continuous maps. In this category, isomorphisms are homotopy equivalences. And in particular, $\mathbb{R}$ is isomorphic to a point.

Exercise 1.14. Let $\mathfrak{Nor}$ be the category of normed spaces with bounded linear maps as morphisms. What are monomorphisms, epimorphism and isomorphisms in this category?

1.1.3. *Duality.* Often correspondences between algebraic and geometric objects reverse directions between constituent morphisms. For instance, in algebraic geometry, suppose $f\colon X \to Y$ is a morphism between two affine varieties over an algebraically closed field k, then there is an induced k-algebra homomorphism $f^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ between rings of regular functions, defined by taking ‘pullbacks’.

In the world of C*-algebras, suppose $f: X \to Y$ is a continuous map between locally compact Hausdorff topological spaces, then there is an induced *-homomorphism between C*-algebras $f^*: C^*(Y) \to C^*(X)$. The reason why such relations exist is because the categories of affine varieties with polynomial maps (in the example from algebraic geometry) and locally compact spaces with continuous maps (in the example from non-commutative geometry), can be identified with the *opposites* of the categories of reduced, finitely generated commutative rings, and commutative C*-algebras, respectively.

Definition 1.15. Let $\mathcal{C}$ be a category. Its *opposite* category $\mathcal{C}^{\mathrm{op}}$ is defined as the category whose objects $(\mathcal{C}^{\mathrm{op}})^0 = \mathcal{C}^0$ and whose morphisms are morphisms of $\mathcal{C}$ with their directions reversed. That is, if $f: X \to Y$ is a morphism in $\mathcal{C}$, then $f^{\mathrm{op}}: Y \to X$ is a morphism in $\mathcal{C}^{\mathrm{op}}$.

Lemma 1.16. *Let $\mathcal{C}$ be a category. If f is a monomorphism/epimorphism in $\mathcal{C}$, then f^{op} is an epimorphism/monomorphism in $\mathcal{C}^{\mathrm{op}}$.*

Lemma 1.17. *A morphism $f: X \to Y$ in a category $\mathcal{C}$ is a monomorphism if and only if for all objects $Z \in \mathcal{C}^0$, the induced morphism $f_*: \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y)$, $g \mapsto f \circ g$ is an injection. Dually, f is an epimorphism if and only if $f^*: \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ is an injection.*

Exercise 1.18. Show that the canonical inclusion $\mathbb{Z} \to \mathbb{Q}$ is both a monomorphism as well as an epimorphism in the category of unital rings (with ring homomorphisms as morphisms). This says that a morphism that is both a monomorphism as well as an epimorphism, need not be an isomorphism.

Exercise 1.19. A morphism that is a monomorphism as well as a *split* epimorphism is an isomorphism. Dually, a morphism that is a *split* monomorphism and an epimorphism is an isomorphism.

Proof. Let $f: X \to Y$ be a monomorphism and a split epimorphism. Then there exists a morphism $g: Y \to X$ such that $f \circ g = 1_Y$. Therefore, $f \circ g \circ f = 1_Y \circ f = f = f \circ 1_X$. Now using that f is a monomorphism, we get $g \circ f = 1_X$, as required. □

1.2. Functors. We would now like to understand what the right 'morphisms' are between two categories.

Definition 1.20. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *covariant functor* $F: \mathcal{C} \to \mathcal{D}$ consists of the following data:

- an object $Fc \in \mathcal{D}^0$ for every object $c \in \mathcal{C}^0$;
- a morphism $Ff: Fc \to Fd \in \mathcal{D}^1$, for every morphism $f: c \to d \in \mathcal{C}^1$.

This data is subject to the following conditions:

- for any two composable morphisms f and g, $F(g \circ f) = F(g) \circ F(f)$;
- for every object $c \in \mathcal{C}$, $F(1_c) = 1_{Fc}$.

Definition 1.21. A *contravariant functor* $F: \mathcal{C} \to \mathcal{D}$ is a covariant functor $F: \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

Remark 1.22. Throughout this course, the term 'functor' by default refers to a covariant functor.

1.2.1. *Examples of functors.* Let us now see a few examples of functors, with a view towards what follows in the course.

Example 1.23.
- The assignment $\mathsf{Group} \to \mathsf{Set}$ that forgets the group structure;
- The assignment $\mathsf{Mod}_R \to \mathsf{Set}$ that forgets the module structure;
- Similarly, there are structure-forgetting assignments from $\mathsf{Top} \to \mathsf{Set}$.

The functors above that forget some part of the structure on say a group, ring, field, module, or a topological space will henceforth be called the *forgetful functor*. In particular, it will be made clear, or will be clear from the context, what part of the structure is being forgotten. One can often also 'stack' up structure on a set, and obtain a topological space, a group, a ring, a field, etc.

Example 1.24.
- Assigning to a set, the discrete topology yields a topological space. This is a functor $\mathsf{Set} \to \mathsf{Top}$;
- Let S be a set. We can canonically associate to S an R-module by taking the *free* R-module generated by it. As an R-module, this is given by the direct sum $\bigoplus_S R$. For instance, if R is a field, then this construction yields the unique vector space (up to isomorphism) with S as its basis. If $R = \mathbb{Z}$, we obtain the free abelian group generated by S.

Example 1.25. Of a slightly different nature is the functor which assigns to a topological space, a topological space with a distinguished point. This is a functor from the category $\mathsf{Top} \to \mathsf{Top}_*$, where the latter is the category whose objects are pairs (X, x_0), where $x_0 \in X$ is a distinguished point. Its morphisms are continuous maps that preserve distinguished points. There is a forgetful functor $\mathsf{Top}_* \to \mathsf{Top}$ which forgets the distinguished point of a pointed topological space.

Example 1.26. Many examples in practice come from embedding a category into a larger category. For instance, the embedding $\mathsf{Ab} \to \mathsf{Group}$ is a functor, as can be trivially checked. A more complicated example is the embedding $\mathsf{Sh}(X) \hookrightarrow \mathsf{PSh}(X)$ of sheaves into presheaves on a set. Here $\mathsf{PSh}(X)$ denotes the category of set-valued contravariant functors on the category $\mathsf{Op}(X)$ of open subsets of X with inclusions of subsets as morphisms. The category $\mathsf{Sh}(X)$ consists of presheaves on X that satisfy a certain descent condition. There is also a canonical functor $\mathsf{PSh}(X) \to \mathsf{Sh}(X)$ called the *sheafification* of X.

Example 1.27. Let $c \in \mathcal{C}$ be an object in an arbitrary locally small category. We can define the functor

$$\mathrm{Hom}(c, -)\colon \mathcal{C} \to \mathsf{Set}, \quad d \mapsto \mathrm{Hom}_{\mathcal{C}}(c, d).$$

This is called the *covariant representable functor at c*. Similarly, we can define the *contravariant representable functor at c* by using $\mathrm{Hom}(-, c)\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$.

Example 1.28. Let I be a small category, and let $\mathcal{C}$ be any category. We call a functor $F\colon I \to \mathcal{C}$ a *diagram* in $\mathcal{C}$ of *shape* I, or a diagram *indexed* by I. Such functors will be important in later sections, when we discuss 'limits' and 'colimits' of diagrams. Of particular relevance will be diagrams indexed by a preorder as in Example 1.9. Can you describe such functors explicitly?

1.2.2. *Basic terminology about functors.* We have so far defined categories and functors between categories. These assemble into a category - the category CAT of all categories, with functors as morphisms. In what follows, we define some special classes of functors. In particular, we shall see what a reasonable notion of an isomorphism should be in this category.

DEFINITION 1.29. A functor $F: \mathcal{C} \to \mathcal{D}$ is called

- *full* if $\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(Fx, Fy)$ is surjective for each x, $y \in \mathcal{C}$;
- *faithful* if $\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(Fx, Fy)$ is injective for each x, $y \in \mathcal{C}$;
- *essentially surjective* if for each $d \in \mathcal{D}^0$, there is a $c \in \mathcal{C}^0$ such that $Fc \cong d$.

DEFINITION 1.30. A functor $F: \mathcal{C} \to \mathcal{D}$ is called an *equivalence of categories* if it is full, faithful and essentially surjective.

1.3. Natural transformations. Just like functors are relations between categories, natural transformations are relations between functors. More concretely,

DEFINITION 1.31. Let $F, G: \mathcal{C} \rightrightarrows \mathcal{D}$ be a parallel pair of functors between two categories. The data of a *natural transformation* $\alpha: F \Rightarrow G$ is a collection of morphisms $\alpha_c: Fc \to Gc$ called its *components*, for each $c \in \mathcal{C}^0$, which satisfy the following:

$$\begin{array}{ccc} Fc & \xrightarrow{\alpha_c} & Gc \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ Fd & \xrightarrow{\alpha_d} & Gd \end{array}$$

for every morphism $f: c \to d$ in $\mathcal{C}$.

A *natural isomorphism* is a natural transformation $\alpha: F \Rightarrow G$ where each component $\alpha_c: Fc \to Gc$ is an isomorphism in $\mathcal{D}$.

1.3.1. *Examples of natural transformations.*

EXAMPLE 1.32. Consider the category of finite-dimensional vector spaces Vect_k over a field k, and the functor $DD: \mathsf{Vect}_k \to \mathsf{Vect}_k$, which assigns to a vector space V, its double-dual $V^{**} := \mathrm{Hom}(\mathrm{Hom}(V, k), k)$. You should try to check that this really is a functor, that is, a linear map $V \to W$ maps functorially to a linear map $V^{**} \to W^{**}$. Now, it is well-known that a finite-dimensional vector space is isomorphic to its dual, and hence its double-dual. But more is true - the isomorphism is *natural*: the components of the natural transformation are given by $\mathrm{ev}: V \to V^{**}$, $v \mapsto (f \mapsto f(v))$.

EXERCISE 1.33. Show that the dual space functor $D: \mathsf{Vect}_k \to \mathsf{Vect}_k$, $V \mapsto \mathrm{Hom}(V, k)$ is **not** a natural isomorphism, although a finite-dimensional vector space is isomorphic to its dual space.

EXAMPLE 1.34. Let G be a group and $\mathcal{C}$ be any category. Consider the classifying space category from Example 1.6. A functor $F: BG \to \mathcal{C}$ is a fixed object $X \in \mathcal{C}^0$ with a G-action on it. Note that any functor is of this form. By abuse of notation, we simply denote such a functor by the unique object $X \in \mathcal{C}^0$ in its image. What is a natural transformation between two such functors $\alpha: X \Rightarrow Y$? There is obviously nothing interesting going on at the level of objects. So the components of α are simply morphisms $X \to Y$ in $\mathcal{C}$. Now if $g \in G$ is a morphism in BG, then the naturality assumption says that for any morphism $f: X \to Y$, we must have $g \cdot f = f \cdot g$. That is, a natural transformation corresponds to a G-equivariant morphism $X \to Y$ in $\mathcal{C}$. In the category of Set, this the familiar notion of a G-equivariant map of sets, that is, maps $f: X \to Y$ satisfying $f(g \cdot x) = g \cdot f(x)$.

1.4. The Yoneda Lemma. The Yoneda Lemma is one of the most important results in category theory. It classifies natural transformations between a representable functor and an arbitrary, set-valued functor on a category. We first take a closer look at the representable functors introduced in Example 1.27, enriched with the vocabulary of natural transformations.

Definition 1.35. A set-valued functor $F: \mathcal{C} \to \mathsf{Set}$ on a (locally small) category is called *representable* if there exists a $c \in \mathcal{C}^0$ and a natural isomorphism between F and $\mathcal{C}(c, -)$, if F is covariant, and $\mathcal{C}(-, c)$ if F is contravariant. The object c is called a representing object of F.

Example 1.36. The forgetful functor $U: \mathsf{Group} \to \mathsf{Set}$ from the category of groups to sets, as in Example 1.23, is represented by the group $\mathbb{Z}$. Indeed, we have $\mathrm{Hom}(\mathbb{Z}, G) \cong UG$ as any group homomorphism $f: \mathbb{Z} \to G$ is determined by its image on the generator 1. Conversely, if $g \in G$, then $1 \mapsto g$ extends to a group homomorphism. It is easy to see that the two constructions are inverse to each other. I leave it to you to check that this construction is natural.

Exercise 1.37. Show that the forgetful functor $U: \mathsf{Top} \to \mathsf{Set}$ is representable.

Proof. Consider a singleton set $\{\bullet\}$, and let X be an arbitrary topological space. Then for any $x \in X$, there is a unique continuous map $f_x: \{\bullet\} \to X$, mapping $\bullet \mapsto x$. Furthermore, any continuous map $\{\bullet\} \to X$ determines a unique element of X, prescribed by its image. Therefore, there is a natural bijection of sets $\mathrm{Hom}(\{\bullet\}, X) \cong X$, as required. □

Example 1.38. An important source of examples of representable functors comes from algebraic topology. Let A be an abelian group and $n \geq 0$. The singular cohomology groups with coefficients in A are functors $H^n(-, A): \mathsf{hTop}^{\mathrm{op}} \to \mathsf{Set}$ that satisfy the so-called Eilenberg-Steenrod axioms. If we restrict to the subcategory of homotopy classes of CW-complexes, then for each n, $H^n(-, A)$ is representable by the so-called Eilenberg-Maclane space $K(A, n)$. That is, we have a natural isomorphism $[X, K(A, n)] \cong H^n(X, A)$ between homotopy classes of continuous maps and singular cohomology groups.

The celebrated Brown Representability Theorem from algebraic topology actually tells us precisely when a functor from a reasonable category of topological spaces to sets is representable. It states that if $F: \mathsf{hCW}^{\mathrm{op}} \to \mathsf{Set}$ is a functor from the homotopy category of pointed compact spaces, or CW-complexes, that maps coproducts of spaces to products to sets, and satisfies 'excision' (often called the Mayer-Vietoris property), it is representable.

Now that we have seen some examples of representable functors, we can finally state the Yoneda Lemma:

Theorem 1.39. *Let $F: \mathcal{C} \to \mathsf{Set}$ be any convariant functor on a locally small category. Then for any $c \in \mathcal{C}^0$, there is a natural bijection of sets:*

$$\mathrm{Hom}(\mathrm{Hom}_{\mathcal{C}}(c, -), F) \cong Fc$$

that associates to a natural transformation $\alpha: \mathrm{Hom}_{\mathcal{C}}(c, -) \Rightarrow F$, the element $\alpha_c(1_c) \in Fc$. Moreover, the bijection is natural in both F and c.

PROOF. We already have one direction of the bijection in the statement of the Theorem. In the other direction, take $x \in Fc$. We need to find a natural transformation $\Phi(x): \mathcal{C}(c,-) \Rightarrow F$. Let $d \in \mathcal{C}^1$ and $f: c \to d \in \mathcal{C}^1$. Then $\Phi(x)_d(f) := Ff(x)$ does the job - that is, it is natural, and it is inverse the other assignment. □

The following implication of the Yoneda Lemma has deep implications in algebraic and analytic geometry, as we shall later unravel.

COROLLARY 1.40. *The functors*

$$y: \mathcal{C} \to \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}), \quad c \mapsto \mathrm{Hom}_{\mathcal{C}}(-, c)$$

and

$$y^{\mathrm{op}}: \mathcal{C}^{\mathrm{op}} \to \mathrm{Hom}(\mathcal{C}, \mathsf{Set}), \quad c \mapsto \mathrm{Hom}_{\mathcal{C}}(c, -)$$

are full and faithful.

EXERCISE 1.41. Use the Yoneda embedding from Corollary 1.40 above to show that if two representable functors are naturally isomorphic, then the representing objects for them are isomorphic. Deduce in particular that if two objects represent the same functor, then they are isomorphic.

PROOF. Use that fully-faithful functors *reflect* isomorphisms: that is, if $F: \mathcal{C} \to \mathcal{D}$ is a fully-faithful functor, and $f: x \to y$ is a morphism in $\mathcal{C}$ such that $Fd: Fx \to Fy$ is an isomorphism in $\mathcal{D}$, then so is f. Using this, and the fact that the Yoneda embedding is fully-faithful, the result follows. □

The embeddings y and y^{op} are called the Yoneda embeddings. To tantalise those who have seen algebraic geometry previously, consider the following high-brow perspective of scheme theory. Let Sch_k be the category of schemes over a field k. Then Corollary 1.40 says that we can embed this category into the functor category $\mathrm{Hom}(\mathsf{Sch}^{\mathrm{op}}, \mathsf{Set})$. But actually, more is true: we can in fact embed Sch into the functor category $\mathrm{Hom}(\mathsf{Aff}_k^{\mathrm{op}}, \mathsf{Set})$, where Aff_k is the category of *affine* schemes over k. Now, we know that the Spec functor and the global sections functor are inverse to each, yielding an equivalence of categories $\mathsf{Aff}_k^{\mathrm{op}} \cong \mathsf{CAlg}_k$, where the latter is the category of commutative k-algebras. Consequently, the category of schemes embeds into the functor category on CAlg_k. The essential image of this embedding therefore yields an identification between schemes over k and certain functors $\mathsf{CAlg}_k \to \mathsf{Set}$, which we understand in a more rudimentary sense. As an upshot, if you previously knew nothing about schemes (even its definition), then you now know everything about it from a completely different perspective. The Yoneda functor $\mathrm{Hom}_{\mathsf{Sch}_k}(-, X)$ applied to a scheme X is called the *functor of points* of the scheme.

After that aside from algebraic geometry, we move on to a more immediate (but nonetheless important) application of the Yoneda Lemma.

EXAMPLE 1.42. Let V and W be finite-dimensional k-vector spaces. Consider the functor $\mathsf{Bilin}(V, W; -): \mathsf{Vect}_k \to \mathsf{Set}$ that assigns to a k-vector space U, the set of k-bilinear maps $V \times W \to U$. We know that the tensor product $V \otimes_k W \in \mathsf{Vect}_k$ is a representing object for the functor $\mathsf{Bilin}(V, W; -)$. That is, we have a natural isomorphism

$$\mathrm{Hom}_k(V \otimes_k W, U) \cong \mathsf{Bilin}(V, W; U) \tag{1.43}$$

for any k-vector space U. At this point of time, let us assume that this is all we know about the object $V \otimes_k W$.

By Theorem 1.39, we know that the natural isomorphism in (1.43) is uniquely determined by an element of $\mathsf{Bilin}(V, W; V \otimes_k W)$, that is, a bilinear map $\otimes: V \times W \to V \otimes_k W$. This means that if there is a bilinear map $f: V \times W \to U$ for some vector space U, then there exists a unique linear map $f': V \otimes_k W \to U$. To see this identification, consider the following commuting diagram:

$$\begin{array}{ccc} \mathrm{Hom}_k(V \otimes_k W, V \otimes_k W) & \xrightarrow{\cong} & \mathsf{Bilin}(V, W; V \otimes_k W) \\ {\scriptstyle f'_*}\downarrow & & \downarrow{\scriptstyle f'_*} \\ \mathrm{Hom}_k(V \otimes_k W, U) & \xrightarrow{\cong} & \mathsf{Bilin}(V, W; U), \end{array}$$

induced by the map $f': V \otimes_k W \to U$. Tracing out what the diagram yields when we plug in $1_{V \otimes_k W}$ in the top left corner, we obtain a commuting triangle

$$\begin{array}{ccc} V \times W & \xrightarrow{\otimes} & V \otimes_k W \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f'} \\ & & U. \end{array}$$

So in other words, any map $f: V \times W \to U$ is built precisely by composing the universal map $V \times W \to V \otimes_k W$ with the map $f': V \otimes_k W \to U$. And of course, any map $V \otimes_k W \to U$ induces a map $V \times W \to U$.

So far we have only used and assumed that there *exists* a k-vector space $V \otimes_k W$ that implements the natural isomorphism in Equation (1.43). The Yoneda Lemma also provides a concrete description for it, as we now see. Consider the map $V \otimes_k W \to V \otimes_k W/\mathsf{span}\{v \otimes w : v \in V, w \in W\}$ which takes the quotient of $V \otimes_k W$ by the linear span of the image of the universal bilinear map $\otimes: V \times W \to V \otimes_k W$. Now the restriction of the quotient map to the image of $- \otimes -$ yields the zero bilinear map, by definition of a quotient vector space. But the zero map $V \otimes_k W \to V \otimes_k W/\mathsf{span}\{v \otimes w : v \in V, w \in W\}$ also has this property. So by the uniqueness statement in the commuting triangle above, we must have that the quotient map $V \otimes_k W \to V \otimes_k W/\mathsf{span}\{v \otimes w : v \in V, w \in W\}$ is the zero map. Finally, since the quotient map is surjective, by the First Isomorphism Theorem, we must have that $V \otimes_k W$ is isomorphic to the k-vector space $\mathsf{span}\{v \otimes w : v \in V, w \in W\}$, where $\otimes$ implicitly has the rules of bilinearity as part of its definition.

As a concluding remark, we show that the Yoneda Lemma can be used to prove certain natural properties without invoking untidy arguments that uses concrete bases. In this spirit, we have the following:

LEMMA 1.44. *For any k-vector spaces V and W, show that $V \otimes_k W \cong W \otimes_k V$.*

PROOF. By Exercise 1.41, it suffices to show that $\mathsf{Bilin}(V, W, -)$ is naturally isomorphic to $\mathsf{Bilin}(W, V, -)$. If U is any vector space, then $\mathsf{Bilin}(V, W; U) \cong \mathsf{Bilin}(W, V; U)$, via $f \mapsto ((w, v) \mapsto f(v, w))$. It is trivial to check that this isomorphism is actually natural. Finally, by the preceding contents of this example, we have the following natural isomorphisms:

$$\mathsf{Vect}_k(V \otimes_k W, -) \cong \mathsf{Bilin}(V, W; -) \cong \mathsf{Bilin}(W, V; -) \cong \mathsf{Vect}_k(W \otimes_k V, -),$$

showing that $V \otimes_k W \cong W \otimes_k V$. $\square$

Exercise 1.45. For any k-vector space V, show that $V \otimes_k k \cong k \otimes_k V \cong V$ using the proof technique in Lemma 1.44. Similarly, show that $V \otimes_k (W \otimes_k U) \cong (V \otimes_k W) \otimes_k U$.

Proof. We prove the associativity claim, for which we claim that $V \otimes_k (W \otimes_k U)$ and $(V \otimes_k W) \otimes_k U$ both represent the functor

$$\mathsf{Trilin}(V, W, U; -): \mathsf{Vect}_k \to \mathsf{Set}$$

of trilinear maps $V \times W \times U \to Z$ to a vector space Z. More concretely, fix a $v \in V$. Then a trilinear map $f: V \times W \times U \to Z$ induces a bilinear map $f_v: W \times U \to Z$. Furthermore, the assignment $v \mapsto f_v$ is linear in V. Now bilinear maps $W \times U \to Z$ are in bijection with linear maps $W \otimes_k U \to Z$. So f_v induces a linear map on $W \otimes U$, and hence a bilinear map $V \times (W \otimes_k U) \to Z$. This yields a linear map $V \otimes_k (W \otimes_k U) \to Z$. But likewise, f induces a bilinear map $f_u: V \times W \to Z$ for each $u \in U$, which ultimately induces a linear map $(V \otimes_k W) \otimes_k U \to Z$. The only thing to really prove is that

$$\mathsf{Bilin}(V, (W \times U); Z) \cong \mathsf{Trilin}(V, W, U; Z) \cong \mathsf{Bilin}((V \times W), U; Z),$$

which is clear. □

Categories such as Vect_k, where there is an earmarked notion of a tensor product $\otimes_k$ that satisfies identities such as $V \otimes_k W \cong W \otimes_k V$, $k \otimes_k V \cong V \cong V \otimes_k k$ and $V \otimes_k (W \otimes_k U) \cong (V \otimes_k W) \otimes_k U$ play a special role in homological algebra. They are called *symmetric monoidal categories*. We will revisit them later in the course.

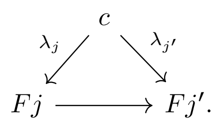
2. Limits and colimits

There is an algorithmic way in which geometry is built - a statement that will become more and more transparent through this course. Take for instance algebraic geometry - we can start with the ring $k[t]$ of polynomials in one variable over an algebraically closed field, take the n-fold tensor product $k[t_1] \otimes_k k[t_2] \otimes \cdots \otimes_k k[t_n] \cong k[t_1, \cdots, t_n]$, then take *quotients* by ideals $I = (f_1, \cdots, f_l)$. This defines an affine variety over the field k. Similarly, in *rigid analytic geometry*, we start with a complete, non-archimedean valued field k such as $\mathbb{Q}_p$ and repeat this algorithm by replacing the polynomial ring $k[t]$ with the Tate ring $k\langle t \rangle = \{\sum_{n=0}^{\infty} c_n t^n : |c_n| \to 0\}$. The resulting object is an affinoid variety over k. We can then *glue* affine or affinoid spaces together to form more complicated varieties (in algebraic geometry) and rigid analytic spaces (in non-archimedean geometry). One can do similar things in topology by starting with $\mathbb{R}$, taking *products* $\mathbb{R}^n$, *quotients* by equivalence relations and performing *gluing constructions* to build more complicated spaces. The point is that these constructions, performed internal to a category, are in a precise sense *universal* ways of constructing geometric objects from fundamental building blocks. We explore these constructions - called *limits* and *colimits* in this section.

Definition 2.1. Let $\mathcal{C}$ and J be categories, $c \in \mathcal{C}^0$ an object. The *constant functor* $F: J \to \mathcal{C}$ sends every object of J to c and every morphism of J to the identity 1_c at c.

Definition 2.2. A *cone over* $F: J \to \mathcal{C}$ with *summit* c is a natural transformation $\lambda: c \Rightarrow F$, where c denotes the constant functor at c indexed by J. Dually, a *cone under* F with *nadir* c is a natural transformation $\lambda: F \Rightarrow c$.

More explicitly, a cone over F is a family of morphisms $(\lambda_j : c \to Fj)_{j \in J}$, such that if $j \to j'$ is a morphism in J, then the following diagram commutes



In what follows, assume J and $\mathcal{C}$ are small and locally small, respectively. This ensures that the category of functors $\mathrm{Hom}(J, \mathcal{C})$ is locally small. For any diagram $F : J \to \mathcal{C}$, there is a functor

$$\mathsf{Cone}(-, F) : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$$

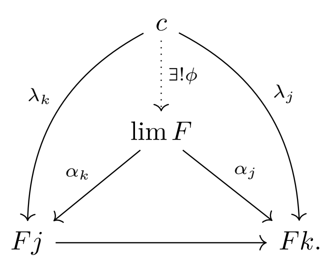
that assigns to each $c \in \mathcal{C}^0$, the set of cones over F with summit c. Dually, there is a functor $\mathsf{Cone}(F, -) : \mathcal{C} \to \mathsf{Set}$ that assigns to each c, the set of all cones under F with nadir c.

Definition 2.3. A *limit* of a diagram $F : J \to \mathcal{C}$, if it exists, is a representation for the functor $\mathsf{Cone}(-, F)$. A *colimit* of F, is a representation of $\mathsf{Cone}(F, -)$, if it exists.

More explicitly, the Yoneda Lemma implies that if a diagram $F : J \to \mathcal{C}$ has a limit, then it is given by an object $\lim F \in \mathcal{C}^0$ together with a universal cone $\alpha : \lim F \Rightarrow F$ implementing the natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(-, \lim F) \cong \mathsf{Cone}(-, F).$$

In other words, if $c \in \mathcal{C}^0$ is an object, and $\lambda : c \Rightarrow F$ is a cone over F with summit c, then there exists a unique morphism $\phi : c \to \lim F$ factorising $\lambda_j : c \to Fj$ for each j:



I leave it to you to work out a similar, concrete description for colimits of a diagram.

Exercise 2.4. Given two limit cones $c \Rightarrow F$ and $c' \Rightarrow F$ of a diagram $F : J \to \mathcal{C}$, show that there is a unique isomorphism $c \cong c'$ that commutes with the components of the limit cones.

2.1. Examples of limits and colimits. In this section, we discuss different types of limits and colimits that arise in the context of our course.

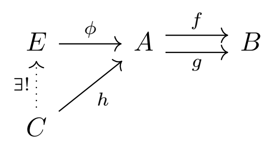
2.1.1. *Examples of limits.*

Example 2.5 (Products). A product is the limit of a diagram indexed by a discrete category, that is, a category with only identity morphisms. Let J be a discrete category and $F : J \to \mathcal{C}$ a diagram indexed by J. Then a cone over F with summit c is simply a collection of morphisms $(\lambda_j : c \to F_j)_{j \in J}$ with no further restrictions. If the limit exists, it is denoted by $\prod_{j \in J} F_j$ and the components $\prod_{j \in J} F_j \to F_i$ of the limit cone are called *projections*.

Example 2.6 (Products in sets and other concrete categories). In the category Set of sets, products exist and are given by the usual cartesian product of sets. Similarly, in Top, products of topological spaces are given by the cartersian product of their underlying sets, equipped the product topology. In the category Group of groups, products of groups are given by the cartesian products of the sets underlying the groups, equipped with pointwise operations. I leave it to you to check that these candidate products actually satisfy the appropriate universal properties.

Example 2.7 (Terminal objects). The terminal object is a trivial case of a product, where the indexing category is empty. In this case, the limit cone is simply an object t in a category $\mathcal{C}$ such that for any other object $x \in \mathcal{C}^0$, there is a unique morphism $x \to t$. In the category Set of sets, singletons $\{x\}$ are terminal objects. In Group, any trivial group is a terminal object.

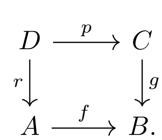
Example 2.8 (Equalisers). Let J_e be the category with two objects and two non-identity morphisms $\bullet \rightrightarrows \bullet$. An *equaliser* is the limit of a diagram $F\colon J_e \to \mathcal{C}$ indexed by the category J_e. Concretely, the cone over such a diagram F with summit C is a parallel pair of morphisms $f, g\colon A \rightrightarrows B$ with a morphism $h\colon C \to A$ such that $f \circ h = g \circ h$. The equaliser $\phi\colon E \to A$ is the universal arrow with this property, that is,



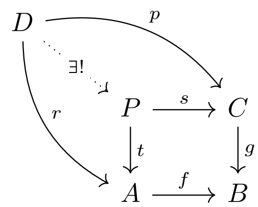
commutes.

Example 2.9 (Equalisers in sets, groups and other concrete categories). In Set, the equaliser of a pair of maps $f, g\colon X \rightrightarrows Y$ is given by the subset $E := \{x \in X : f(x) = g(x)\}$ together with the canonical inclusion into A. In Group, the equaliser of a pair of group homomorphisms $\phi, \psi\colon G \rightrightarrows H$ is given by the subgroup $\{g \in G : \phi(g) = \psi(g)\}$. In particular, if $\psi(x) = e_H$ is the trivial homomorphism mapping every element of G to the identity of H, then the equaliser is just the *kernel* of ϕ. In the category of abelian groups, the equaliser of a pair of maps is given by the kernel of the difference $(\phi - \psi)(x) := \phi(x) - \psi(x)$ homomorphism.

Example 2.10 (Pullbacks). Pullbacks are limits of diagrams indexed by the category $\bullet \to \bullet \leftarrow \bullet$ consisting of three objects and two non-identity morphisms with a common codomain. Let $A \xrightarrow{f} B \xleftarrow{g} C$ be the image of a diagram of this shape in a category $\mathcal{C}$. A cone over such a diagram with summit D consists of morphisms from D to each of the three objects in the image of the diagram, such that the following diagram commutes:



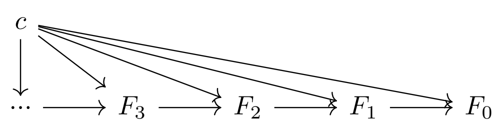
The universal property of limits then says given any diagram as above, there exists a unique factorisation of the legs of the cone summited by D. Diagrammatically, the following diagram



commutes in the category $\mathcal{C}$. The pullback P is often also called the *fibre product* and is denoted $A \times_B C$.

EXAMPLE 2.11 (Pullbacks in sets). Pullbacks in sets is the usual fibre product you might have encountered in differential geometry or algebraic geometry. If f and g are as above, then $A \times_B C := \{(a,c) : f(a) = g(c)\}$ and the morphisms s and t are the usual projections to C and A, respectively. It is instructive to check that the above candidate really satisfies the universal property of pullbacks.

EXAMPLE 2.12 (Inverse limits). Let $\mathcal{P} = (P, \leq)$ be a set with a preorder, viewed as a category as in Example 1.9. An *inverse limit* is the limit of a diagram indexed by $\mathcal{P}^{\mathrm{op}}$. Let us suppose P is a countable set, which is often the situation in practice. Then a cone over a diagram $F\colon \mathcal{P}^{\mathrm{op}} \to \mathcal{C}$ with summit $c \in \mathcal{C}^0$ is a commuting diagram of the form:



The limit of such a diagram is often denoted $\varprojlim F_n$.

EXERCISE 2.13. Express the p-adic integers $\mathbb{Z}_p := \{\sum_{n=0}^{\infty} c_n t^n : |c_n|_p \to 0\}$ as an inverse limit of the canonical projections $\mathbb{Z}/p^{k+1}\mathbb{Z} \to \mathbb{Z}/p^k\mathbb{Z}$.

2.1.2. *Examples of colimits.* Examples 2.5-2.12 can all be dualised to yield colimit constructions:

- a *coproduct* $\coprod_{j\in J} A_j$ is the colimit of a diagram $(A_j)_{j\in J}$ indexed by a discrete category J;
- an *initial object* is the colimit of an empty diagram;
- a *coequaliser* is the colimit of a diagram indexed by a category of the form $\bullet \rightrightarrows \bullet$
- a *pushout* is the colimit of a diagram indexed by $\bullet \leftarrow \bullet \to \bullet$;
- The notion dual to an inverse limit is the *direct limit* $\varinjlim_{p\in P} F_i$ of a $\mathcal{C}$-valued diagram $F\colon \mathcal{P} \to \mathcal{C}$ indexed by $\mathcal{P}$.

From the perspective of this course, we discuss colimits in the category of abelian groups Ab. There is a unique group homomorphism $\{0\} \to G$ from a trivial group into any group, so these are the initial objects. The coproduct of abelian groups $(G_j)_{j\in J}$ is given by their direct sum $\bigoplus_{j\in J} G_j$ defined as sequences $(g_j)_{j\in J}$ of elements with $g_j \in G_j$, such that at most finitely many of the g_j are non-zero. The coequaliser of a parallel pair $f, g\colon G \rightrightarrows H$ is the quotient of H by the subgroup generated by $\{f(x) - g(x) : x \in G\}$. In particular, if g is the trivial morphism mapping every element of G to identity 0_H in H, then the coequaliser of the result pair is the *cokernel* of f.

Exercise 2.14 (Direct limits of groups). Let $\mathcal{I} = (I, \leq)$ be a directed set, viewed as a category.

(1) Show that a functor $F: \mathcal{I} \to \mathsf{Group}$ corresponds to a collection of groups $(F_i)_{i \in I}$ together with group homomorphisms $\alpha_{ij}: F_i \to F_j$, for $i \geq j$, such that $\alpha_{ik} = \alpha_{jk} \circ \alpha_{ij}$ $(i \leq j \leq k)$ and α_{ii} is the identity;

(2) Show that the set $G := \coprod_{i \in I} F_i / \sim$, with $g_i \sim g_j$ if and only if there is a $k \geq i, j$ such that $\alpha_{ik}(g_i) = \alpha_{jk}(g_j)$, with the multiplication
$$[g_i] \cdot [g_j] := [\alpha_{ik}(g_i)\alpha_{jk}(g_j)]$$
defines a group.

(3) Show that the group defined above is the direct limit of the functor F above.

Remark 2.15. The same arguments can be bootstraped to categories such as Mod_R.

What we have done in the course of these examples is that we have shown that the categories of abelian groups, modules over a ring, sets, topological spaces actually possess all limits and colimits. Such categories are particularly important: they are examples of complete and cocomplete categories.

Definition 2.16. A category is said to be *(co)complete* if it contains all (co)limits. If a category has all limits and colimits, it is called *bicomplete*.

Lemma 2.17. *The categories* Set*,* Top*,* Group *and* Mod_R *are bicomplete categories.*

Exercise 2.18. Is the category of fields a bicomplete category?

We end this section with the remark that in order to check that a category is complete or cocomplete, it simply suffices to check that it has all (co)products and (co)equalisers. This is due to the following important result:

Theorem 2.19. *Any small (co)limit in* Set *may be expressed as a (co)equaliser of a pair of maps between (co)products.*

2.1.3. *Geometric realisation of a simplicial set.* Let $\triangle$ be the *simplex category* whose objects are ordinals $[n] := \{0, 1, \cdots, n\}$ and whose morphisms are order-preserving functions.

To acquaint ourselves with the category $\triangle$ better, let us first take a look at two examples of morphisms in it:

Example 2.20. For each $n \geq 0$, there are $n+1$ injections called *coface* maps $d^i: [n-1] \to [n]$ defined by
$$d^i(k) := \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } k \geq i, \end{cases}$$
for every $0 \leq i \leq n$. Each such map d^i misses i in its image.

Example 2.21. Similarly, there are $n+1$ surjections called *codegeneracy* maps $s^i: [n+1] \to [n]$ defined by
$$s^i(k) := \begin{cases} k & \text{if } k \leq i \\ k-1 & \text{if } k > i, \end{cases}$$

for $0 \le i \le n$. Each such map s^i hits i twice in its image.

Remark 2.22. It can be shown that every morphism in $\triangle$ is a composition of coface and codegeneracy maps.

Definition 2.23. A *simplicial set* is a functor $X\colon \triangle^{\mathrm{op}} \to \mathsf{Set}$. We denote by sSet the category of simplicial sets, and an object in it by $(X_n)_{n \in \mathbb{N}}$.

Example 2.24 (Standard simplex). The *standard n-simplex* is the simplicial set defined by $\triangle^n\colon \triangle^{\mathrm{op}} \to \mathsf{Set}$, $[m] \mapsto \mathrm{Hom}([m],[n])$. It is equivalently the image of the Yoneda embedding

$$\triangle \hookrightarrow \mathrm{Hom}(\triangle^{\mathrm{op}}, \mathsf{Set}), \quad [n] \mapsto ([m] \mapsto \mathrm{Hom}([m],[n])),$$

for each $n \in \mathbb{N}$.

Definition 2.25. The *simplex category of X* is defined as the category $\triangle \downarrow X$ whose objects are natural transformations $\triangle^n \to X$, and whose morphisms are defined as follows: a morphism in $\mathrm{Hom}(\triangle^n \overset{\eta}{\to} X, \triangle^m \overset{\zeta}{\to} X)$ is a natural transformation $\alpha\colon \triangle^n \to \triangle^m$ induced by a morphism $[n] \to [m]$ in $\triangle$, such that

$$\begin{array}{ccc} \triangle^n & \xrightarrow{\alpha} & \triangle^m \\ & {\scriptstyle\eta}\searrow \quad \swarrow{\scriptstyle\zeta} & \\ & X & \end{array}$$

commutes.

Let X be a simplicial set. We would like to functorially associate to it a topological space. And to do this, we first construct a functor

$$\delta\colon \triangle \to \mathsf{Top}.$$

For every ordinal $[n] \in \triangle$, we can define the *topological n-simplex* as the topological space $|\triangle^n| := \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} : 0 \le x_i \le 1, \sum_{i=0}^n x_i = 1\}$. This defines the action of δ on objects. To define its action on morphisms, we invoke Remark 2.22. Consequently, it suffices to say what δ does on coface and codegeneracy maps. Each codegeneracy map $s^i\colon [n] \to [n-1]$ maps to $|\triangle^{n-1}| \hookrightarrow |\triangle^n|$, $(x_0, \cdots, x_{n-1}) \mapsto (x_0, \cdots, 0, \cdots, x_{n-1})$, while each coface map $d^i\colon [n-1] \to [n]$ maps to $|\triangle^n| \to |\triangle^{n-1}|$, $(x_0, \cdots, x_n) \mapsto (x_0, \cdots, x_i + x_{i+1}, \cdots, x_n)$.

Now consider the diagram $F\colon \triangle \downarrow X \to \mathsf{Top}$, that assigns to a natural transformation $\triangle^n \to X$, the topological n-simplex $|\triangle^n| := \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} : 0 \le x_i \le 1, \sum_{i=0}^n x_i = 1\}$. On morphisms, it sends an $\alpha\colon \triangle^n \to \triangle^m$ induced by $[n] \to [m]$, to the continuous map $|\alpha|\colon |\triangle^n| \to |\triangle^m|$, obtained from the functoriality of δ. The colimit of this diagram is called the *geometric realisation* $|X|$ of X. By construction, we have obtained a functor

$$|-|\colon \mathsf{sSet} \to \mathsf{Top},$$

as desired.

The geometric realisation of a simplicial set actually lies in a particularly nice subcategory of Top - the category of compactly generated, Hausdorff topological spaces. A topological space X is said to be compactly generated if it satisfies the following: a subset $U \subseteq X$ is closed if and only if it $U \cap K$ is closed for all compact subsets $K \subseteq X$.

3. Adjunctions

DEFINITION 3.1. An opposing pair of functors $F: C \rightleftarrows D: G$ between two categories are said to be in *adjunction* if there are natural isomorphisms

$$\mathrm{Hom}_{\mathcal{D}}(Fx, y) \cong \mathrm{Hom}_{\mathcal{C}}(x, Gy)$$

in both variables $x \in \mathcal{C}^0$ and $y \in \mathcal{D}^0$. We say F is *left adjoint* to G and G is *right adjoint* to F.

Several forgetful functors, such as those discussed in Examples 1.23, admit left adjoints, namely the relevant 'free functors'. Let us look at some examples:

EXAMPLE 3.2. Consider the forgetful functor $U: \mathsf{Ab} \to \mathsf{Set}$, that forgets the group structure on an abelian group. Its left adjoint is given by the functor that assigns to a set S, the *free abelian group* $F(S) := \bigoplus_S \mathbb{Z}$. More concretely, this group is defined as the set of all formal linear combinations of elements of S with integer coefficients. The same construction can be done for the forgetful functor $U: \mathsf{Mod}_R \to \mathsf{Set}$, by taking formal linear combinations with coefficients in R.

A slightly more complicated example of the same nature is as follows:

EXAMPLE 3.3. The left adjoint of the forgetful functor $U: \mathsf{Group} \to \mathsf{Set}$ is the functor that assigns to a set S, the *free group F_S generated by S*. This is constructed as follows: we first define a set $S^{-1} = \{s^{-1} : s \in S\}$ of formal inverses. Let $T := S \bigcup S^{-1}$. We define a *word* as a concatenation of finitely many symbols in T. Now declare that if in a word $\cdots s \cdot s^{-1} \cdots$, a symbol $s \in S$ is adjacent to $s^{-1} \in S^{-1}$, then we simply omit the block $s \cdot s^{-1}$. This gives us a way to simplify words. A word that cannot be simplified by this rule is said to be *reduced*. The free group F_S is defined as the set of all reduced words, with concatenation as multiplication. The empty word is the identity of the group. Almost by construction, we have that a map of sets $f: S \to G$ induces a unique group homomorphism $F_S \to G$, mapping a generator $s_1 \cdots s_n \mapsto f(s_1) \cdots f(s_n)$.

EXAMPLE 3.4. Let $\phi: R \to S$ be a ring homomorphism. We can view any S-module $M \in \mathsf{Mod}_S$ as an R-module by defining an R-module action as $r \cdot m := \phi(r) \cdot m$. This is a functor $\mathsf{Mod}_S \to \mathsf{Mod}_R$, which we call *restriction of scalars*. Its left adjoint is given by *extension of scalars* $\mathsf{Mod}_R \to \mathsf{Mod}_S$, which assigns to an R-module M, the S-module defined by $M_S := S \otimes_R M$. Here we view S as an R-module via the map ϕ.

EXAMPLE 3.5 (Tensor-Hom). Let R and S be rings and let X be a fixed R-S-bimodule. Consider the functor $F: \mathsf{Mod}_R \to \mathsf{Mod}_S$ that assigns to $M \in \mathsf{Mod}_R$, the S-module $M \otimes_R X$. This has a right adjoint functor defined by

$$G: \mathsf{Mod}_S \to \mathsf{Mod}_R \quad M \mapsto \mathrm{Hom}_S(X, M).$$

The functor G is called the *Hom-functor*. If we specialise to the case where $R = S$ (and is commutative, for simplicity), we obtain an example of a category (namely, $(\mathsf{Mod}_R, \otimes_R)$) equipped with a tensor product, and in which the tensor-Hom adjunction holds. Such categories are called *closed*, categories. Furthermore, recall from Example 1.42, that Vect_k or more generally, Mod_R is also a symmetric, monoidal category. As the course goes on, we will see that closed, symmetric monoidal categories provide an ideal setting for constructions in homological algebra.

We now make a note of two facts that we will need later in the course:

THEOREM 3.6. *A right adjoint functor preserves limits. Dually, a left adjoint functor preserves colimits.*

PROOF. See [**3**, Theorem 4.5.2, Theorem 4.5.3]. □

COROLLARY 3.7. *In the category* Mod_R*, we have*

$$X \otimes_R (\bigoplus_{i\in I} X_i) \cong \bigoplus_{i\in I} X \otimes_R X_i, \qquad \textit{and } \mathsf{Hom}(X, \prod_{i\in I} X_i) \cong \prod_{i\in I} \mathrm{Hom}(X, X_i),$$

where I is an arbitrary indexing set, and X is a fixed R-module.

PROOF. Use the tensor-hom adjunction 3.5 and Theorem 3.6. □

CHAPTER 2

Abelian Categories and their Derivations

Often in geometry, one wants to understand a geometric object (say, a topological space) via an algebraic object (such as, a vector space). For instance in algebraic topology, one associates to a topological space X, the singular chain complex $\mathsf{Sing}(X)$, which is a collection of abelian groups

$$\cdots \to C^n(X) \overset{d^n}{\to} C^{n-1}(X) \overset{d^{n-1}}{\to} \cdots,$$

whose *homology* with coefficients in an abelian group A is defined as the groups $H_n(X, A) \coloneqq \ker(d_n)/\mathsf{im}(d_{n+1})$ for each $n \in \mathbb{Z}$. Similarly, in differential geometry, one derives information about a smooth manifold M via its *de Rham cohomology*, defined as the cohomology of the complex

$$0 \to C^\infty(M) \overset{d_0}{\to} \Omega^1(M) \overset{d_1}{\to} \Omega^2(M) \overset{d^2}{\to} \Omega^3(M) \to \cdots,$$

where $\Omega^n(M)$ denotes differential n-forms. What we are doing in the two examples above is that we are associating to a space, or a manifold, a diagram of abelian groups

$$\cdots \to C_{n+1} \overset{d_{n+1}}{\to} C_n \overset{d_n}{\to} \cdots$$

that satisfies $d_{n+1} \circ d_n = 0$ for each n. This is called a *chain complex*, defined over the category Ab of abelian groups, and is the starting point of homological algebra. More specifically, the category of abelian groups is an example of an *abelian category*, which is defined as a category enriched in abelian groups. In this chapter, we explore fundamental objects of homological algebra such as chain complexes and their homology groups, in the generality of abelian categories.

1. Abelian categories

1.1. Additive categories.

1.1.1. *Zero objects and kernels.* The minimal structure one needs to talk about chain complexes $(C_n, d_n)_{n\in\mathbb{Z}}$ is a category $\mathcal{C}$ with a *zero object* and a *zero morphism*.

Definition 1.1. A *zero object* in a category $\mathcal{C}$ is an object that is both initial and terminal. If $\mathcal{C}$ has a zero object, then a *zero morphism* is a morphism in $\mathcal{C}$ that factors through a zero object.

Example 1.2. The category Mod_R of modules over a ring has a zero object, namely, the zero module $\{0\}$.

Example 1.3. Does the category of sets have a zero object?

Proposition 1.4. *Let $\mathcal{C}$ be a category with a zero object. Then there exists a unique zero morphism between any two objects of $\mathcal{C}$.*

Proof. Let A and B be two objects and denote by 0 a zero object of $\mathcal{C}$. There exists a unique morphism $A \to 0$, by the terminality of 0. Since 0 is also initial, there is a unique morphism $0 \to B$. The composition of these two morphisms gives the desired morphism. □

Proposition 1.5. *Let $\mathcal{C}$ be a category with a zero object. The composition of an arbitrary morphism with a zero morphism is a zero morphism.*

Proof. The composition factors through a zero object. □

Recall from Example 2.8, that the *kernel* of a group homomorphism $f\colon G \to H$ is the equaliser of the pair $f, 0\colon G \rightrightarrows H$ in the category Group. We can now rephrase the definition of kernels in the terminology we have just introduced:

Definition 1.6. Let $\mathcal{C}$ be a category with a zero object. The *kernel* of a morphism $f\colon A \to B$, if it exists, is the equaliser of f and the zero morphism. The cokernel of f is defined dually, that is, using the coequaliser of the pair $(f, 0)$.

Proposition 1.7. *Let $\mathcal{C}$ be a category with a zero object.*

(1) *If $f \circ g = 0$ for some arbitrary morphism g and a monomorphism f, then $g = 0$;*
(2) *The kernel of a monomorphism $f\colon A \to B$ is the zero morphism $0 \to A$;*
(3) *The kernel of the zero morphism $0\colon A \to B$ is the identity $1_A\colon A \to A$.*

Proof. Left as a trivial exercise. □

1.1.2. *Enriched categories.* So far we have been working with *locally small* categories to ensure that the collection of morphisms $\mathrm{Hom}_{\mathcal{C}}(x, y)$ between objects x, $y \in \mathcal{C}^0$ in a category is manageable in terms of size - that is, it is a *set*. Often however, the Hom-sets in the categories that typically arise in homological algebra carry some additional structure of their own. For instance, if X and Y are vector spaces over a field k, then $\mathrm{Hom}(X, Y)$ of linear maps between them is itself a vector space: the scalar multiplication is defined as $(a \cdot T)(x) := aT(x)$ and the addition of two linear maps T, $U \in \mathrm{Hom}(X, Y)$ is defined by $T + U(x) := T(x) + U(x)$. Furthermore, we can use the monoidal structure $\otimes$ on Vect_k to compose two such Hom-vector spaces:

$$\circ\colon \mathrm{Hom}(Y, Z) \otimes \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z), \quad (T, U) \mapsto U \circ T. \tag{1.8}$$

The composition is in turn associative and unital. In other words, the category Vect_k is *enriched* in the monoidal category $(\mathsf{Vect}_k, \otimes)$ in the following precise sense.

Definition 1.9. Let $\mathcal{M}$ be a monoidal category with monoidal unit $\mathbb{1}$ and tensor product $\otimes$. The data of a (locally small) category $\mathcal{C}$ *enriched over* $\mathcal{M}$ is:

- for any two objects x and y, an object $\mathrm{Hom}_{\mathcal{C}}(x, y) \in \mathcal{M}^0$;
- for any three objects x, y and z in $\mathcal{C}^0$, a *composition morphism* $c_{x,y,z}\colon \mathrm{Hom}_{\mathcal{C}}(y, z) \otimes \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{C}}(x, z)$;
- for any $x \in \mathcal{C}^0$, an *identity morphism* $1_x\colon \mathbb{1} \to \mathrm{Hom}_{\mathcal{C}}(x, x)$.

This data is subject to the condition that the following diagrams commute for all objects x, y, z, w:

$$\begin{array}{ccc}
(\mathrm{Hom}_{\mathcal{C}}(z,w)\otimes \mathrm{Hom}_{\mathcal{C}}(y,z))\otimes \mathrm{Hom}_{\mathcal{C}}(x,y) & \xrightarrow{\text{ass}} & \mathrm{Hom}_{\mathcal{C}}(z,w)\otimes (\mathrm{Hom}_{\mathcal{C}}(y,z)\otimes \mathrm{Hom}_{\mathcal{C}}(x,y)) \\
\downarrow{\scriptstyle c_{y,z,w}\otimes 1} & & \downarrow{\scriptstyle 1\otimes c_{x,y,z}} \\
\mathrm{Hom}_{\mathcal{C}}(y,w)\otimes \mathrm{Hom}_{\mathcal{C}}(x,y) & & \mathrm{Hom}_{\mathcal{C}}(z,w)\otimes \mathrm{Hom}_{\mathcal{C}}(x,z) \\
{\scriptstyle c_{x,y,w}}\searrow & & \swarrow{\scriptstyle c_{x,z,w}} \\
 & \mathrm{Hom}_{\mathcal{C}}(x,w) &
\end{array}$$

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{C}}(y,y)\otimes \mathrm{Hom}_{\mathcal{C}}(x,y) & \xrightarrow{c_{x,y,y}} & \mathrm{Hom}_{\mathcal{C}}(x,y) & \xleftarrow{c_{x,x,y}} & \mathrm{Hom}_{\mathcal{C}}(x,y)\otimes \mathrm{Hom}_{\mathcal{C}}(x,x) \\
{\scriptstyle 1_y\otimes 1}\uparrow & \nearrow{\scriptstyle \cong} & & \nwarrow{\scriptstyle \cong} & \uparrow{\scriptstyle 1\otimes 1_a} \\
1\otimes \mathrm{Hom}_{\mathcal{C}}(x,y) & & & & \mathrm{Hom}_{\mathcal{C}}(x,y)\otimes 1.
\end{array}$$

EXERCISE 1.10. What is an arbitrary category enriched in Set? Here the category of sets is viewed as a monoidal category with the cartesian product of sets and with a singleton set as the monoidal unit.

DEFINITION 1.11. A *pre-additive category* $\mathcal{C}$ is a category enriched in abelian groups. That is, $\mathrm{Hom}_{\mathcal{C}}(x,y)$ is an abelian group, and the composition map as in (1.8) is a group homomorphism in each variable.

EXAMPLE 1.12. The category of abelian groups is a pre-additive category. More generally, the category Mod_R of modules over a ring is a pre-additive category.

PROPOSITION 1.13. *In a preadditive category $\mathcal{C}$, the following are equivalent:*

(1) *$\mathcal{C}$ has an initial object;*
(2) *$\mathcal{C}$ has a terminal object;*
(3) *$\mathcal{C}$ has a zero object.*

PROOF. If $\mathcal{C}$ has a zero object, then it clearly has an initial and a terminal object. By duality, it suffices to prove that (1) implies (3). So let 0 be an initial object. The set $\mathrm{Hom}_{\mathcal{C}}(0,0)$ has a unique element 1_0, which is hence the identity of this set viewed as a group. Now let A be an object. Then $\mathrm{Hom}_{\mathcal{C}}(A,0)$ has at least one element, namely, the zero element of the group. But if $f\colon A\to 0$ is any morphism, then $f = 1_0\circ f$ must be the zero element of $\mathrm{Hom}_{\mathcal{C}}(A,0)$, since 1_0 is the zero element of $\mathrm{Hom}_{\mathcal{C}}(0,0)$. Therefore, 0 is terminal as well. □

PROPOSITION 1.14. *Given two objects A and B in a pre-additive category $\mathcal{C}$, the following are equivalent:*

(1) *The product (P, p_A, p_B) of A and B exists;*
(2) *The coproduct (P, s_A, s_B) of A and B exists;*
(3) *There is an object P and morphisms*

$$p_A\colon P\to A,\quad p_B\colon P\to B,\quad s_A\colon A\to P,\quad s_B\colon B\to P,$$

such that

$$p_A\circ s_A = 1_A,\quad p_B\circ s_B = 1_B,\quad p_A\circ s_B = 0,\quad p_B\circ s_A = 0, s_A\circ p_A + s_B\circ p_B = 1_P.$$

Moreover, under these conditions $s_A = \ker(p_B)$, $s_B = \ker(p_A)$, $p_A = \operatorname{coker}(s_B)$, $p_B = \operatorname{coker}(s_A)$

PROOF. By duality, it suffices to show that (1) is equivalent to (3). Suppose (1) holds, define $s_A\colon A\to P$ to be the unique morphism such that $p_A\circ s_A = 1$ and

$p_B \circ s_A = 0$. Similarly, define $s_B: B \to P$ to be the unique morphism satisfying $p_A \circ s_B = 0$ and $p_B \circ s_B = 1$. It is then easy to see that

$$p_A \circ (s_A \circ p_A + s_B \circ p_B) = p_A, \quad p_B \circ (s_A \circ p_A + s_B \circ p_B) = p_B,$$

so that $s_A \circ p_A + s_B \circ p_B = 1$.

Now suppose (3) holds, consider $C \in \mathcal{C}^0$ and two morphisms $f: C \to A$ and $g: C \to B$. Define $h: C \to P$ as $h = s_A \circ f + s_B \circ g$. One has $p_A \circ h = f$ and $p_B \circ h = g$. Now given $h': C \to P$ such that $p_A \circ h' = f$ and $p_B \circ h' = g$, then

$$h' = 1_P \circ h' = (s_A \circ p_A + s_B \circ p_B) \circ h' = s_A \circ f + s_B \circ g = h.$$

We now assume (3) and show that $s_A = \ker(p_B)$. We already know that $p_B \circ s_A = 0$. Choose $x: X \to P$ such that $p_B \circ x = 0$. Then the composite $p_A \circ x$ satisfies $s_A \circ p_A \circ x = x$, that is, it factorises x. Furthermore, the factorisation is unique because $p_A \circ s_A = 1_A$, and thus s_A is a monomorphism.

Finally, the relation $s_B = \ker(p_A)$ is true by analogy and the coker-relations hold by duality. □

Definition 1.15. An *additive category* $\mathcal{C}$ is a pre-additive category with a zero object and so that any pair of objects A, $B \in \mathcal{C}^0$ has a biproduct $A \oplus B$.

Example 1.16. The biproduct of any two R-modules is given by their R-module direct sum $A \oplus B$, which is also their product. Consequently, Mod_R is an additive category. In particular, the category of abelian groups is additive.

Example 1.17. The category $\mathsf{Ban}_{\mathbb{C}}$ of Banach spaces with bounded linear maps as morphisms is additive.

Exercise 1.18. Is the category of groups additive?

The following result says that the abelian group structure on the morphism spaces of an additive category are completely determined by the category $\mathcal{C}$.

Proposition 1.19. *Two additive structures on a category are isomorphic.*

Proof. For $C \in \mathcal{C}^0$, we define the diagonal functor $\triangle_C: C \to C \oplus C$ as the unique morphism such that $p_1 \circ \triangle_C = 1_C$ and $p_2 \circ \triangle_C = 1_C$. Further, define $\sigma_C = p_1 - p_2$. It can be easily checked that $\sigma_C = \operatorname{coker}(\triangle_C)$, so that $p_1 - p_2$ is characterised by the limit-colimit structure of $\mathcal{C}$.

Now if $f, g: A \to C$, then there exists a unique $c: A \to C \oplus C$ such that $p_1 \circ c = f$ and $p_2 \circ c = g$. So $f - g = p_1 \circ c - (p_2 \circ c) = (p_1 - p_2) \circ c$, and hence $f - g$ is also characterised by the limit-colimit structure of $\mathcal{C}$. Finally, $f + g = f - (0 - g)$, so that the addition law on $\operatorname{Hom}(A, C)$ is internal to $\mathcal{C}$. □

We now turn to additive functors.

Definition 1.20. A functor $F: \mathcal{C} \to \mathfrak{D}$ between pre-additive categories is called *additive* if for all x, $y \in \mathcal{C}$,

$$\operatorname{Hom}(x, y) \to \operatorname{Hom}(Fx, Fy)$$

are group homomorphisms.

Proposition 1.21. *Let $F: \mathcal{C} \to \mathfrak{D}$ be a functor between additive categories.*

- *F is additive;*
- *F preserves biproducts;*

- *F preserves finite products;*
- *F preserves finite coproducts.*

PROOF. By definition, finite products, coproducts and biproducts coincide in an additive category. Therefore, it suffices to show that F is additive if and only if $F(A \oplus B) \cong F(A) \oplus F(B)$. If F is additive, then it clearly preserves the identities defining a biproduct from Proposition 1.14. So (1) implies (2).

Conversely, suppose F preserves biproducts. Then from the proof of Proposition 1.19, we know that the addition of two morphisms can be expressed in terms of their difference, which in turn can be expressed as the difference between the projections $p_1, p_2 \colon A \oplus A \to A$. Consequently, we have that

$$F(p_1 - p_2) \circ F(s_1) = F((p_1 - p_2) \circ s_1) = F(1_A) = 1_{F(A)} = (F(p_1) - F(p_2)) \circ F(s_1),$$

and similarly for s_2. □

EXAMPLE 1.22. Let M be an R-bimodule. Then the functor $M \otimes_R - \colon \mathsf{Mod}_R \to \mathsf{Mod}_R$, that maps $N \mapsto M \otimes_R N$ is an additive functor. To see this, we can make use of the fact that $M \otimes_R -$ is a left adjoint functor to the internal Hom-functor. Since left adjoint functors preserve colimits (see Theorem 3.6), Proposition 1.21 implies the result. By the same type of reasoning, the internal Hom-functor $\mathrm{Hom}(M, -) \colon \mathsf{Mod}_R \to \mathsf{Mod}_S$ is additive.

1.2. Abelian categories. Since the definition of homology of a chain complex involves kernels and quotients of differentials, we need to ensure that such objects exist in our category. Moreover, for a chain complex to convey as much information as its homology, we need to ensure that the homology functor is an equivalence of categories. All this is achieved by the following special type of category:

DEFINITION 1.23. An additive category $\mathcal{C}$ is called an *abelian category* if:

- every morphism in $\mathcal{C}$ is has a kernel and a cokernel;
- every monomorphism is the kernel of its cokernel;
- every epimorphism is the cokernel of its kernel.

The second and the third condition seem rather innocuous. Let us explain this more categorically. Consider an additive category $\mathcal{C}$ in which every morphism has a kernel and a cokernel. Let $f \colon A \to B$ be an arbitrary morphism. We then have the following diagram:

$$\begin{array}{ccccccc} \ker(f) & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & \mathrm{coker}(f) \\ & & \downarrow & & \uparrow & & \\ & & \mathsf{coim}(f) & \xrightarrow{h} & \mathsf{im}(f), & & \end{array} \tag{1.24}$$

where $\mathsf{im}(f) \coloneqq \ker(B \to \mathrm{coker}(f))$ and $\mathsf{coim}(f) \coloneqq \mathrm{coker}(\ker(f) \to A)$. To see that there indeed exists a unique map $h \colon \mathsf{coim}(f) \to \mathsf{im}(f)$, we first note that there exists a unique map $f_1 \colon \mathsf{im}(f) \to B$, since f is an equaliser for the diagram $(\mathrm{coker}(f), 0) \colon B \rightrightarrows \mathrm{coker}(f)$. The map f_1 is then a coequaliser for the diagram $(\ker(f), 0) \colon \ker(f) \rightrightarrows A$ since $\mathsf{im}(f) \circ (f_1 \circ \ker(f)) = f \circ \ker(f) = 0 = \mathsf{im}(f) \circ 0$, and $\mathsf{im}(f)$ is a monomorphism (being a kernel). Consequently, an additive category with kernels and cokernels is abelian precisely when the map h is an isomorphism.

Example 1.25. The category Mod_R is the prototypical example of an abelian category. Let $f\colon A \to B$ be an R-module map. Its kernel is given by $\ker(f) = \{a \in A\colon f(a) = 0\}$ with the inclusion map into A. The cokernel of f is given by the quotient of B by the image $f(A)$, with the quotient map $B \twoheadrightarrow B/f(A)$. For the other claims, let $f\colon A \to B$ be a monomorphism, which is the same as an injective R-module map. Then $\mathrm{coker}(f) = B/A$, where $A \cong f(A)$ is identified as a B-submodule. So the kernel of the quotient map $B \twoheadrightarrow B/A$ is precisely $A \cong f(A) \subseteq B$. By a similar argument, every epimorphism is the cokernel of its kernel.

Example 1.26. Let X be a topological space and let $\mathsf{Op}(X)$ denote the category whose objects are open subsets of X and whose morphisms are inclusions between open subsets. Recall that a pre-sheaf valued in a category $\mathcal{C}$ is a functor $\mathcal{F}\colon \mathsf{Op}(X)^{\mathrm{op}} \to \mathcal{C}$. The usual choice of category $\mathcal{C}$ is the category of sets. But if we choose here an abelian category, then the resulting category of pre-sheaves $\mathsf{PSh}(X,\mathcal{C})$ is an abelian category.

Example 1.27 (A non-example). The category of all groups is not abelian: consider the inclusion $i\colon N \hookrightarrow G$ of a subgroup into a group. Then i is the kernel of its cokernel if and only if N equals its normal closure - that is, if it is a normal subgroup.

Exercise 1.28. Show that the category of Banach spaces with bounded linear maps is not an abelian category.

2. Chain complexes over abelian categories

We can now finally define and study the most important object of homological algebra:

Definition 2.1. A *$\mathbb{Z}$-graded chain complex* (denoted $C \coloneqq (C_n, d_n)$) over an abelian category $\mathcal{A}$ is a diagram of the form

$$\cdots \to C_{n+1} \overset{d_{n+1}}{\to} C_n \overset{d_n}{\to} C_{n-1} \to \cdots,$$

such that $d_{n+1} \circ d_n = 0$ for each $n \in \mathbb{Z}$. The maps $d_n\colon C_n \to C_{n-1}$ are called the *differentials* of the chain complex.

Remark 2.2. The above definition can actually be made in any category with a zero object. But it will soon become apparent why we need the generality of an abelian category.

Exercise 2.3. Let $C_n = \mathbb{Z}/8\mathbb{Z}$ for $n \geq 0$ and $C_n = 0$ for $n \leq 0$. For $n > 0$, define $d_n\colon C_n \to C_{n-1}$, $x \mapsto 4x \mod 8$. Show that (C,d) is a chain complex and compute its homology.

In what follows, we construct our first category of chain complexes $\mathsf{Kom}(\mathcal{C})$ over an abelian category $\mathcal{C}$. The objects of this category are chain complexes (C, d^C). A morphism between chain complexes $f\colon (C, d^C) \to (D, d^D)$ is a collection $(f_n\colon C_n \to D_n)_{n\in\mathbb{Z}}$ of morphisms, called *chain maps*, such that the following diagram commutes:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & C_n & \xrightarrow{d_n^C} & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & D_n & \xrightarrow{d_n^D} & D_{n-1} & \longrightarrow & \cdots
\end{array}$$

for all $n \in \mathbb{Z}$.

Now consider a chain complex $C = (C_n, d_n)$ in an abelian category $\mathcal{C}$. More specifically, consider the maps $C_{n+1} \overset{d_{n+1}}{\to} C_n \overset{d_n}{\to} C_{n-1}$, for each $n \in \mathbb{Z}$. Since $\mathcal{C}$ is abelian, the coimage and the image of f from diagram (1.24) are isomorphic. Consequently, there are morphisms $C_{n+1} \twoheadrightarrow \mathsf{im}(d_{n+1})$, $\ker(d_n) \rightarrowtail C_n$ and $\varphi\colon \mathsf{im}(d_{n+1}) \rightarrowtail \ker(d_n)$ such that

(2.4)
$$\begin{array}{ccccccc} & & \mathsf{im}(d_{n+1}) & \xrightarrow{\varphi} & \ker(d_n) & & \\ & \nearrow & & \searrow \quad \swarrow & & \searrow & \\ C_{n+1} & & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & & C_{n-1} \end{array}$$

commutes.

Existence of φ: To see that exists a unique φ, consider the equaliser diagram:

$$\ker(d_n) \xrightarrow{\ker(d_n)} C_n \underset{0}{\overset{d_n}{\rightrightarrows}} C_{n-1}.$$

Since C is a chain complex, we have $d_n \circ d_{n+1} = 0$. By the image-coimage factorisation, we have $d_{n+1} = \mathsf{im}(d_{n+1}) \circ \mathsf{coim}(d_{n+1})$. Therefore, we have

$$d_n \circ \mathsf{im}(d_{n+1}) \circ \mathsf{coim}(d_{n+1}) = 0 = 0 \circ \mathsf{coim}(d_{n+1}).$$

And since $\mathsf{coim}(d_{n+1})$ is an epimorphism (being a cokernel), we must have $d_n \circ \mathsf{im}(d_{n+1}) = 0$. So, $\mathsf{im}(d_{n+1})$ is an equaliser for the diagram above. Hence, by the universal property of equalisers, there exists a unique map $\mathsf{im}(d_{n+1}) \to \ker(d_n)$, as required.

Separately, the morphism φ is also a monomorphism. To see this, let $\varphi \circ g = \varphi \circ h$. Then $\ker(d_n) \circ \varphi \circ g = \ker(d_n) \circ \varphi \circ h$. Therefore, $\mathsf{im}(d_{n+1}) \circ g = \mathsf{im}(d_{n+1}) \circ h$. Since the image of a morphism is a monomorphism, we have that $g = h$. We call a chain complex C *exact* if the map $\varphi\colon \mathsf{im}(d_{n+1}) \to \ker(d_n)$ is an isomorphism in $\mathcal{C}$. The failure of exactness of the chain complex C means that the monomorphism φ has a non-trivial cokernel. This motivates the following definition:

DEFINITION 2.5. The *n-th homology* of a chain complex $C = (C_n, d_n)_{n\in\mathbb{Z}}$ in an abelian category $\mathcal{C}$ is defined as the object $H_n(C) \coloneqq \operatorname{coker}(\mathsf{im}(d_{n+1}) \to \ker(d_n))$ in $\mathcal{C}$.

REMARK 2.6. In the category Mod_R, the image of an R-module map $f\colon M \to N$ is given by the usual set-theoretic image $f(M) = \{f(m) : m \in M\}$, viewed as a submodule of N. In this case, the homology of a chain complex $C = (C_n, d_n)$ of R-modules is given by the familiar form

$$H_n(C) = \ker(d_n)/\mathsf{im}(d_{n+1}).$$

In fact, if you are generally overwhelmed by categorical arguments, you do not lose a lot by restricting your attention to the category Mod_R, where notions such as images, kernels, cokernels correspond closer to your prior intuition. There is also the following deep result, which we will not prove:

THEOREM 2.7 (Freyd-Mitchell Embedding Theorem). *Every small abelian category is a full subcategory of the category Mod_R, for some unital, not necessarily commutative ring R.*

We now promote the homology of a chain complex to a functor on the category of chain complexes over an abelian category $\mathcal{C}$. By definition, the homology of a chain complex is an object of $\mathcal{C}$. It remains to see what happens at the level of morphisms. Let $f\colon C \to D$ be a chain map. Then there exist unique morphisms such that the diagram below commutes:

$$\begin{array}{ccccccc}
C_{n+1} & \xrightarrow{d^C_{n+1}} & \mathsf{im}(d^C_{n+1}) & \xrightarrow{\varphi^C} & \ker(d^C_n) & \longrightarrow & C_n \\
\downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle \exists!} & & \downarrow{\scriptstyle \exists!} & & \downarrow{\scriptstyle f_n} \\
D_{n+1} & \xrightarrow{d^D_{n+1}} & \mathsf{im}(d^D_{n+1}) & \xrightarrow{\varphi^D} & \ker(d^D_n) & \longrightarrow & D_n,
\end{array}$$

proving the functoriality of H_n for each n.

Remark 2.8. The functoriality of H_n breaks down to the existence of the unique maps, claimed above. To see this, suppose there exists such maps $\alpha\colon \mathsf{im}(d^C_{n+1}) \to \mathsf{im}(d^D_{n+1})$ and $\beta\colon \ker(d^C_n) \to \ker(d^D_n)$ such that $\beta \circ \varphi^C = \varphi^D \circ \alpha$. Then consider the coequaliser diagrams

$$\begin{array}{ccccc}
\mathsf{im}(d^C_{n+1}) & \underset{0}{\overset{\varphi^C}{\rightrightarrows}} & \ker(d^C_n) & \longrightarrow & \operatorname{coker}(\varphi^C) = H_n(C) \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \vdots \\
\mathsf{im}(d^D_{n+1}) & \underset{0}{\overset{\varphi^D}{\rightrightarrows}} & \ker(d^D_n) & \xrightarrow{\operatorname{coker}(\varphi^D)} & \operatorname{coker}(\varphi^D) = H_n(D).
\end{array}$$

Then $\operatorname{coker}(\varphi^D) \circ \beta \circ \varphi^C = \operatorname{coker}(\varphi^D) \circ \varphi^D \circ \alpha = 0$. So by the universal property of coequalisers, there exists a unique morphism $H_n(C) = \operatorname{coker}(\varphi^C) \to \operatorname{coker}(\varphi^D) = H_n(D)$.

Exercise 2.9. Show that there really exist α and β as in the remark above. (Hint: justify and then use the statement that kernels and cokernels are functorial in an appropriate sense. The kernel part was already done in class.)

Remark 2.10. Note that the existence of φ and the functoriality assertions above are purely 'formal', that is, they follow completely from universal properties of the defining objects. This gives us the latitude of working in the general context of an abelian category (rather than specific examples such as Mod_R).

Definition 2.11. A chain map $f\colon C \to D$ between two chain complexes is called a *quasi-isomorphism* if $H_n(f)\colon H_n(C) \to H_n(D)$ is an isomorphism for each $n \in \mathbb{Z}$.

Lemma 2.12. *Let $\mathcal{C}$ be an abelian category and $C \in \mathsf{Kom}(\mathcal{C})$. The following statements are equivalent:*

- *C is exact;*
- *$H_n(C) \cong 0$ for each $n \in \mathbb{Z}$;*
- *the zero chain map $0 \to C$ is a quasi-isomorphism.*

Proof. If C is exact, then by definition $\operatorname{coker}(\mathsf{im}(d_{n+1}) \to \ker(d_n)) \cong 0$. Conversely, if the cokernel of a monomorphism in an abelian category is zero, then the morphism must be an isomorphism. Clearly if $H_n(C) = 0$, then the unique map $0 \to H_n(C) = 0$ must be an isomorphism. This $0 \to C$ is a quasi-isomorphism. Finally, if $0 \to C$ is a quasi-isomorphism, then $0 \to H_n(C)$ is an isomorphism. $\square$

2.1. Operations on chain complexes. Let $C = (C_n, d_n)$ be a chain complex. A chain complex B is called a *subcomplex* of C if each B_n is a subobject of C_n, and d_n^B is the restriction of d_n^C to B_n. Equivalently, the monomorphisms $i_n\colon B_n \to C_n$ assemble into a chain map $B \to C$. The cokernels of these map yield a chain complex

$$\cdots \to \operatorname{coker}(i_{n+1}) \to \operatorname{coker}(i_n) \to \cdots$$

called the *quotient complex* C/B. If $f\colon C \to D$ is a chain map, then $\ker(f) := (\ker(f_n))_{n\in\mathbb{Z}}$ is a subcomplex of C, called the *kernel* of a chain map. Dually, $\operatorname{coker}(f) := (\operatorname{coker}(f_n))_{n\in\mathbb{Z}}$ is a quotient complex called the *cokernel* of a chain map.

PROPOSITION 2.13. *Suppose $\mathcal{C}$ is an abelian category. Then the category of chain complexes $\mathsf{Kom}(\mathcal{C})$ with chain maps as morphisms is also an abelian category.*

PROOF. Kernels and cokernels are defined termwise as above. Note that these exist since $\mathcal{C}$ is an abelian category. Now let $f\colon A \to B$ be a chain map. We first claim that f is a monomorphism if and only if each $f_n\colon A_n \to B_n$ is a monic, that is, A is isomorphic to a subcomplex of B. This follows from the fact that the composite $\ker(f) \to B$ is zero, so if f is monic, then $\ker(f) = 0$. Therefore, if f is monic, it is isomorphic to the kernel of $B \to B/A$. Similarly, f is an epimorphism if and only if each $f_n\colon A_n \to B_n$ is an epimorphism. That is, B is isomorphic to the cokernel of the map $\ker(f) \to A$. □

DEFINITION 2.14. A diagram

$$A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C \tag{2.15}$$

in an abelian category $\mathcal{C}$ is called an *extension* or a *short exact sequence* if $f = \ker(g)$ and $g = \operatorname{coker}(f)$. An *extension of chain complexes* is a diagram as above in the category $\mathsf{Kom}(\mathcal{C})$, where the kernel and cokernel of a chain map are defined as above.

EXERCISE 2.16. A diagram of chain complexes $A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$ is an extension if and only if for each n, the diagram $A_n \overset{f_n}{\rightarrowtail} B_n \overset{g_n}{\twoheadrightarrow} C_n$ is an extension.

EXERCISE 2.17. A chain complex $C = (C_n, d_n)$ is exact if and only if for each n, the diagram $\ker(d_{n+1}) \to C_{n+1} \to \ker(d_n)$ is exact. Here $\ker(d_{n+1}) \subseteq C_{n+1}$ is the canonical inclusion, and $C_{n+1} \to \ker(d_n)$ is the map induced by $d_{n+1}\colon C_{n+1} \to C_n$.

DEFINITION 2.18. Let $C = (C_n, d_n)$ be a chain complex. The *translation* or *shift* of C by an integer m, is the chain complex defined by $C[m]_n := C_{m+n}$ with differential $d'_{n+1} := (-1)^m d_{n+m+1}\colon C[m]_{n+1} \to C[m]_n$.

EXERCISE 2.19. Show that for an integer m, the shifted chain complex has shift homology as

$$H_n(C[m]) = H_{m+n}(C)$$

for each $n \in \mathbb{Z}$.

2.2. The Snake Lemma and long exact sequences. In this subsection, we will prove some useful diagram lemmas in homological algebra. This will be used to prove that a short exact sequence of chain complexes induce a long exact sequence in homology. Throughout this section, $\mathcal{C}$ is an abelian category, which you can assume is a module category over a ring to fix intuition. Recall that we have come across the notion of an exact chain complex. These are diagrams

$$\cdots \to C_{n+1} \overset{d_{n+1}}{\to} C_n \overset{d_n}{\to} C_{n-1} \overset{d_{n-1}}{\to} \cdots$$

in $\mathcal{C}$ is called an *exact sequence* if the canonical map $\mathsf{im}(d_{n+1}) \to \ker(d_n)$ is an isomorphism. A *short exact sequence* or an *extension* is an exact sequence of the form

$$0 \to A \to B \to C \to 0.$$

Lemma 2.20. *Let $A \overset{p}{\to} B \overset{q}{\to} C \overset{r}{\to} D$ be an exact sequence in $\mathcal{C}$. Then for any morphism $f\colon E \to B$, the induced sequence*

$$A \to B/E \to C/E \to D$$

is also exact.

Here by B/E (respectively, C/E), we mean the cokernel of the map f and $q \circ f$.

Proof. We first note that exactness of the given sequence is equivalent to $\operatorname{coker}(p) \cong \ker(r)$. Now since $C \to C/E$ is an epimorphism, we have $\mathsf{im}(C/E \to D) \cong \mathsf{im}(C \to D) = \mathsf{im}(r)$. By exactness at C, $\mathsf{im}(r) \cong \operatorname{coker}(q) \cong \operatorname{coker}(B \to C/E)$. Finally, since $B \to B/E$ is also an epimorphism, we have $\operatorname{coker}(B \to C/E) \cong \operatorname{coker}(B/E \to C/E)$. Combining all these isomorphisms, we get exactness at C/E. To see exactness at B/E, we have a string of isomorphisms

$$\begin{aligned}\operatorname{coker}(A \to B/E) &\cong \operatorname{coker}(A \oplus E \to B) \cong \operatorname{coker}(E \to \operatorname{coker}(A \to B)) \\ &\cong \operatorname{coker}(E \to \mathsf{im}(B \to C)) \cong \mathsf{im}(B \to C/E) \cong \mathsf{im}(B/E \to C/E)\end{aligned}$$

which can be reasoned in a similar manner. □

Dually, we have the following:

Lemma 2.21. *Let $A \overset{p}{\to} B \overset{q}{\to} C \overset{r}{\to} D$ be an exact sequence in $\mathcal{C}$. Then for any morphism $C \to E$, the induced sequence*

$$A \to \ker(B \to E) \to \ker(C \to E) \to D$$

is exact.

Proof. $\mathcal{C}^{\mathrm{op}}$ is an abelian category, so the previous lemma applies. □

Theorem 2.22 (Snake Lemma). *Let $\mathcal{C}$ be an abelian category. Consider the following commuting diagram*

$$\begin{array}{ccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & & \end{array} ,$$

where the rows are exact sequences. Then there is an induced exact sequence relating the kernels and the cokernels of the maps a, b and c as follows:

$$\ker(a) \to \ker(b) \to ker(c) \overset{d}{\to} \operatorname{coker}(a) \to \operatorname{coker}(b) \to \operatorname{coker}(c).$$

Proof. See [**1**, Section 12.1]. □

We now apply the Snake Lemma to prove an important result about homology groups of exact sequences of chain complexes.

PROPOSITION 2.23. *Let $0 \to C \to D \to E \to 0$ be a short exact sequence of chain complexes. Then there is an induced long exact sequence of homology groups:*

$$\cdots \to H_n(C) \to H_n(D) \to H_n(E) \to H_{n-1}(C) \to H_{n-1}(D) \to H_{n-1}(E) \to \cdots.$$

PROOF. We first consider the diagram of exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_n & \xrightarrow{f_n} & D_n & \xrightarrow{g_n} & E_n & \longrightarrow & 0 \\ & & \downarrow d_n^C & & \downarrow d_n^D & & \downarrow d_n^E & & \\ 0 & \longrightarrow & C_{n-1} & \xrightarrow{f_{n-1}} & D_{n-1} & \xrightarrow{g_{n-1}} & E_{n-1} & \longrightarrow & 0. \end{array} \tag{2.24}$$

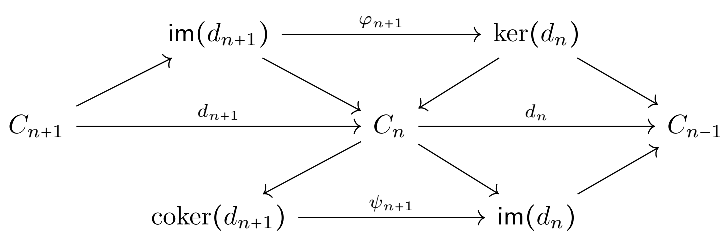
This induces the following commuting diagram with exact rows

$$\begin{array}{ccccccccc} & & \mathrm{coker}(d_{n+1}^C) & \longrightarrow & \mathrm{coker}(d_{n+1}^D) & \longrightarrow & \mathrm{coker}(d_{n+1}^E) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \ker(d_{n-1}^C) & \longrightarrow & \ker(d_{n-1}^D) & \longrightarrow & \ker(d_{n-1}^E). & & \end{array}$$

More concretely, the rows are exact by an application of the Snake Lemma to (2.24). The functoriality of cokernels and kernels induces the horizontal maps. The (unique) existence of the vertical maps follows from the fact that the map $d_n^C : C_n \to \ker(d_{n-1})$ coequalises the pair $(d_{n+1}^C, 0) : C_{n+1} \to C_n$ - and of course, likewise, for D and E. Denote the vertical maps above by ϕ_n^C, ϕ_n^D and ϕ_n^E. We first prove the following lemma:

LEMMA 2.25. *Let (C, d) be a chain complex over an abelian category $\mathcal{C}$. Consider the unique map $\phi_n : \mathrm{coker}(d_{n+1}) \to \ker(d_{n-1})$ constructed above. Then $H_n(C) = \ker(\phi_n)$ and $H_{n-1}(C) = \mathrm{coker}(\phi_n^C)$.*

PROOF. Consider again the diagram



Let u denote the composition $\ker(d_n) \to C_n \to \mathrm{coker}(d_{n+1})$. We then have unique monomorphisms $\mathsf{im}(d_{n+1}) \rightarrowtail \ker(u) \rightarrowtail \ker(C_n \to \mathrm{coker}(d_{n+1})) = \mathsf{im}(d_{n+1})$. Therefore, $\ker(u) \cong \mathsf{im}(d_{n+1})$. Similarly, $\mathrm{coker}(u) \cong \mathsf{im}(d_n)$. Since $\mathcal{C}$ is an abelian category, we have $\mathsf{coim}(u) \cong \mathsf{im}(u)$. Consequently,

$$\begin{aligned} H_n(C) :=\cong \mathrm{coker}(\mathsf{im}(d_{n+1}) \to \ker(d_n)) = \mathsf{coim}(u) \\ \cong \mathsf{im}(u) \cong \mathrm{coker}(C_{n+1} \to \ker(d_n)) \cong \ker(\mathrm{coker}(d_{n+1}) \to \mathsf{im}(d_n)) \cong \ker(\mathrm{coker}(d_{n+1}) \to C_{n-1}) \\ \cong \ker(\mathrm{coker}(d_{n+1}) \to \ker(d_{n-1})) = \ker(\phi_n). \end{aligned}$$

The part for $H_{n-1}(C) = \mathrm{coker}(\phi_n)$ is similar. $\square$

By the Snake Lemma, this implies the long exact sequence as desired, since for instance, $H_n(C)$ is $\ker(\operatorname{coker}(d_n^C) \to \ker(d_n^C))$ and $H_{n-1}(C)$ is $\operatorname{coker}(\operatorname{coker}(d_n^C) \to \ker(d_n^C))$. □

In the Lemma contained in the proof of the Proposition above, we tacitly used the following short observations, which I'll leave as an exercise:

Exercise 2.26. Let $f: A \to B$ be a morphism in a pointed category with kernels and cokernels. If $g: C \to A$ is an epimorphism, then $\operatorname{coker}(f) \cong \operatorname{coker}(f \circ g)$. Similarly, if $g': B \to E$ is a monomorphism, then $\ker(g' \circ f) = \ker(f)$.

2.3. Chain homotopy and bivariant homology. Let (C, d^C) and (D, d^D) be two chain complexes over an additive category $\mathcal{C}$. We define the *mapping complex* $\operatorname{Hom}(C, D) \in \mathsf{Kom}(\mathsf{Ab})$ as the chain complex of abelian groups defined by

$$\operatorname{Hom}(C, D)_n := \prod_{m \in \mathbb{Z}} \operatorname{Hom}(C_m, D_{m+n}),$$
$$\delta_n((f_m)) := d^D_{m+n} \circ f_m - (-1)^n f_{m-1} \circ d^C_m.$$

Exercise 2.27. Check that the mapping complex really is a chain complex. That is, check that the square of the differential is the zero map.

The *n-cycles*, that is, elements of $\ker(\delta_n)$ are precisely the chain maps $C \to D[n]$, where $C[n]$ is the shifted chain complex defined previously. The n-th homology

$$H_n(C, D) := H_n(\operatorname{Hom}(C, D))$$

is defined as the *n-th bivariant homology* of the chain complexes C and D.

We introduce some notation for what follows. If f, $g \in \operatorname{Hom}(C, C)$, denote by $[f, g]$ the *graded* commutator

$$[f, g] := fg - (-1)^{|f| \cdot |g|} gf.$$

The boundary map on $\operatorname{Hom}(C, C)$ maps $f \mapsto [f, \delta] = [\delta, f]$. By abuse of notation, we denote the boundary map on $\operatorname{Hom}(C, D)$ by $[\delta, f]$, although this involves two boundary maps d^C and d^D.

Definition 2.28. Two chain maps f, $g: C \to D$ are *chain homotopic* if there exists an $h \in \operatorname{Hom}(C, D)_1$ such that $[\delta, h] = f - g$. The map h is called a *chain homotopy* between f and g, and is often denoted $f \sim_h g$.

It is easy to see that chain homotopy is an equivalence relation on the set of chain maps $C \to D$. The set of equivalence classes is precisely $H_0(C, D)$. We shall use this set to define a more refined class of morphisms between chain complexes, leading to the following important category:

Definition 2.29. Let $\mathcal{C}$ be an additive category. The *homotopy category* $\mathsf{HoKom}(\mathcal{C})$ of chain complexes is the category with the same objects as $\mathsf{Kom}(\mathcal{C})$, and morphisms $H_0(C, D)$.

A chain complex C is called *contractible* if $C \cong 0$ in $\mathsf{HoKom}(\mathcal{C})$, or equivalently, $1_C \sim 0$. More explicitly, there exists a chain homotopy $h: C \to C[1]$ such that $[\delta, h] = 1_C$. A chain map $f: C \to D$ is called a *chain homotopy equivalence* if it is an isomorphism in $\mathsf{HoKom}(\mathcal{C})$. Explicitly, there exists a chain map $g: D \to C$, and chain homotopies $h^C: C \to C[1]$ and $h^D: D \to D[1]$, such that $[\delta, h^C] = 1_C - g \circ f$ and $[\delta, h^D] = 1_D - f \circ g$.

Remark 2.30. At the moment, we have two notions of homology. One is the (co)homology of a (co)chain complex, internal to an abelian category. The other is the bivariant homology of two chain complexes over arbitrary additive categories. The latter is defined as the homology of a certain chain complex in the category of abelian groups. In what generality can one specialise bivariant homology to the homology of a chain complex?

Definition 2.31. Suppose $\mathcal{C}$ is a symmetric, monoidal abelian category with unit object 1. View the unit object as a chain complex supported in degree 0. The *homology* of a chain complex C is defined as $H_n(C) := H_n(1, C)$ and the *cohomology* of C is defined as $H^n(C) := H_n(C, 1)$, for $n \in \mathbb{Z}$.

What does the homology of a chain complex in Definition 2.31 have to do with the definition of homology that we have grown accustomed to? Consider the case where $\mathcal{C}$ is the category of R-modules, where R is a ring. Let (C, d_n) be a chain complex of R-modules. Then $\mathrm{Hom}(R, C)$ is simply the chain complex C, viewed as a complex of abelian groups, and $\mathrm{Hom}(C, R)$ is the *dual complex* of R-valued linear functionals on C. This is a reasonable notion of homology when we work over a module category. However, as is probably evident, significant amount of information about the underlying objects of the category is lost if we work over, say, the category of Banach spaces. In this case, we will need the more fine grained notion of homology defined by the cokernel of the natural map $\mathsf{im}(d_{n+1}) \to \ker(d_n)$.

We now talk about an important construction, motivated from algebraic topology. Let (C, d^C) and (D, d^D) be chain complexes and let $f: C \to D$ be a chain map between them. The *mapping cone* of f is defined as the chain complex $\mathsf{cone}(f)_n := C_n \oplus D_{n+1}$ with differential

$$\delta_n^f := \begin{bmatrix} -d_n^C & 0 \\ f_n & d_{n+1}^D \end{bmatrix} : C_n \oplus D_{n+1} \to C_{n-1} \oplus D_n.$$

The coordinate maps $D \overset{\iota_f}{\to} \mathsf{cone}(f)$ and $\pi_f : \mathsf{cone}(f) \to C[1]$ are chain maps. The resulting diagram

$$C \overset{f}{\to} D \overset{\iota_f}{\to} \mathsf{cone}(f) \overset{\pi_f}{\to} C[-1]$$

is called the *mapping cone triangle*.

Theorem 2.32 (Puppe sequence). *Let $f: A \to B$ be a chain map between two chain complexes. Then for any chain complex D, the maps in the mapping cone sequence induces a natural long exact sequence of abelian groups*

$$\cdots \to H_1(D, A) \to H_1(D, B) \to H_1(D, \mathsf{cone}(f)) \to H_0(D, A) \to H_0(D, B) \to H_0(D, \mathsf{cone}(f)) \to \cdots,$$

and

$$\cdots \leftarrow H_1(A, D) \leftarrow H_1(B, D) \leftarrow H_1(\mathsf{cone}(f), D) \leftarrow H_0(A, D) \leftarrow H_0(B, D) \leftarrow H_0(\mathsf{cone}(f), D) \leftarrow \cdots.$$

Let $F: \mathcal{C} \to \mathcal{A}$ be an additive functor into an abelian category $\mathcal{A}$, and let $\bar{F}$ be its extension to a functor $\mathsf{Kom}(\mathcal{C}) \to \mathsf{Kom}(\mathcal{A})$. Then there is a natural long exact sequence

$$\cdots \to H_n(\bar{F}(A)) \to H_n(\bar{F}(B)) \to H_n(\bar{F}(\mathsf{cone}(f))) \to H_{n-1}(\bar{F}(A)) \to \cdots,$$

and

$$\cdots \leftarrow H^n(\bar{F}(A)) \leftarrow H^n(\bar{F}(B)) \leftarrow H^n(\bar{F}(\mathsf{cone}(f))) \leftarrow H^{n-1}(\bar{F}(A)) \to \cdots.$$

PROOF. We first see a simpler version of the result in the category $\mathsf{Kom}(\mathsf{Ab})$. Let $f: A \to B$ be a chain map. Then there is an exact sequence of chain complexes

$$0 \to B \xrightarrow{\iota_f} \mathsf{cone}(f) \to A[-1] \to 0,$$

where the first map is $b \mapsto (0, b)$, and the second map is $(a, b) \mapsto -a$. This induces a long exact sequence of homology groups by Proposition 2.23

$$\cdots \to H_n(B) \to H_n(\mathsf{cone}(f)) \to H_n(A[-1]) \cong H_{n-1}(A) \to H_{n-1}(B) \to \cdots.$$

To prove the general case, consider the functor $\mathrm{Hom}(D, -): \mathsf{HoKom}(\mathcal{C}) \to \mathsf{HoKom}(\mathsf{Ab})$. By the naturality of the cone construction, if $f: A \to B$ is a chain map, then the cone of the induced chain map $\mathrm{Hom}(D, f): \mathrm{Hom}(D, A) \to \mathrm{Hom}(D, B)$ is equal to the chain complex $\mathrm{Hom}(D, \mathsf{cone}(f))$. So the functor $\mathrm{Hom}(D, -)$ preserves mapping cone triangles. We have now reduced the problem to the case of a mapping cone triangle in the category $\mathsf{Kom}(\mathsf{Ab})$, which we have just proven.

Finally, for the last claim, it is easy to see that the exactness of a chain complex in an abelian category is equivalent to the exactness of the functor $\mathrm{Hom}(X, -)$ for all objects X in the category. Therefore, the exact sequences for the functors $H_*(\bar{F}(-))$ also reduces to the case of abelian groups. □

DEFINITION 2.33. An extension of chain complexes $K \rightarrowtail E \overset{p}{\twoheadrightarrow} Q$ in an abelian category $\mathcal{C}$ is called *semi-split* if there exists a sequence of morphisms $s_n: Q_n \to E_n$ such that $p_n \circ s_n = 1_{Q_n}$ for each n.

THEOREM 2.34. *Let $K \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} Q$ be a semi-split extension of chain complexes in $\mathcal{C}$. Then there are chain homotopy equivalences $\mathsf{cone}(p) \sim K[-1]$ and $\mathsf{cone}(i) \sim Q$.*

PROOF. We only construct the chain homotopy equivalence $K[1] \sim \mathsf{cone}(p)$, leaving the reader to dualise the other equivalence. The coordinate embedding $E[1] \to \mathsf{cone}(p)$ restricts to a chain map $K[1] \to \mathsf{cone}(p)$, since $p \circ i = 0$. Now a section $s: Q \to E$ induces a decomposition $E \cong K \oplus Q$, so that $\mathsf{cone}(p)_n \cong K_{n-1} \oplus Q_{n-1} \oplus Q_n$. The boundary map becomes

$$\begin{bmatrix} -d^K_{n-1} & -[\delta, s]_n & 0 \\ 0 & d^Q_{n-1} & 0 \\ 0 & 1_{Q_{n-1}} & d^Q_n, \end{bmatrix}$$

where $[\delta, s] = \delta \circ s - s \circ \delta$. Now we decompose

$$\mathsf{cone}(p)_n \cong K_{n-1} \oplus Q_{n-1} \oplus ([\delta, s], \delta, 1)^T \cdot Q_n \cong K[1] \oplus \mathsf{cone}(1_Q) \cong K[1],$$

since $\mathsf{cone}(1_Q)$ is contractible. □

Since homology preserves chain homotopy equivalences, the Puppe sequence implies that a semi-split extension of chain complexes induces a long exact sequence in homology. This is a rather important property for invariants to have, in order to facilitate computations.

LEMMA 2.35. *Let $f: A \to B$ be a chain map between chain complexes in an additive category $\mathcal{C}$. Then the following are equivalent:*

- *f is a chain homotopy equivalence;*

- *$f^*: H_*(D,A) \to H_*(D,B)$ is an isomorphism for each $*$, and for all $D \in \mathsf{HoKom}(\mathcal{C})$;*
- *the chain complex $\mathsf{cone}(f)$ is contractible.*

Proof. The first two are equivalent by the Yoneda Lemma. The last two are equivalent by the Puppe sequence. □

3. The triangulated category structure on the homotopy category of complexes

There are several formal similarities between the homotopy category of (nice) topological spaces, such as CW-complexes, and the homotopy category of chain complexes. For instance, they both have analogous notions of mapping cones and suspensions, which are used to obtain long exact sequences of homotopy groups and homology groups. The latter is a consequence of the Puppe sequence, that we saw previously. More concretely, the homotopy category of chain complexes $\mathsf{HoKom}(\mathcal{C})$ over an additive category, comes equipped with the distinguished mapping cone sequences

$$A \xrightarrow{f} B \to \mathsf{cone}(f) \to A[-1],$$

for any chain map $f: A \to B$. The same diagram in the stable homotopy category of spaces is the mapping cofibre sequence.

Definition 3.1. A *triangulated category* consists of the following data:

- an additive category $\mathcal{T}$;
- an auto-equivalence of categories $\Sigma: \mathcal{T} \to \mathcal{T}$ called a *translation* or *suspension* functor;
- a class of *distinguished* or *exact triangles*

subject to the following axioms:

1. every triangle isomorphic to an exact triangle is exact;
2. for every object $X \in \mathcal{T}^0$, the diagram $X \xrightarrow{1_X} X \xrightarrow{0} 0 \to \Sigma(X)$ is exact;
3. for every morphism $f: X \to Y$, there exists an exact triangle $X \xrightarrow{f} Y \to Z \to \Sigma(X)$. The object Z is called the *cofibre* of the morphism f;
4. **Rotation axiom:** if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$ is an exact triangle, then so are
$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X) \xrightarrow{-\Sigma f} \Sigma(Y), \quad \Sigma^{-1} Z \xrightarrow{-\Sigma^{-1} h} X \xrightarrow{f} Y \xrightarrow{g} Z$$
;
5. Suppose we have exact triangles as rows, and morphisms α and β as in the following diagram
$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\exists\gamma} & & \downarrow{\scriptstyle\Sigma\alpha} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X', \end{array}$$
then there exists a morphism $\gamma: Z \to Z'$ such that all the squares commute.
6. **Octahedral axiom:** See [**2**, Proposition 1.4.6, Remark 1.4.7].

Example 3.2. As already motivated in the introduction, the most interesting example of a triangulated category in homological algebra is the homotopy category of chain complexes $\mathsf{HoKom}(\mathcal{C})$ over an additive category $\mathcal{C}$. In this category, a triangle is exact if it is isomorphic to a mapping cone triangle. The auto-equivalence is given by the shift of a chain complex $\Sigma C \coloneqq C[-1]$.

Example 3.3 (Stable homotopy category). Another interesting example arises in algebraic topology. Let CTop^* be the category of pointed compact spaces. Recall, the objects of this category are compact spaces (X, x) with a distinguished base-point, and its morphisms $f\colon (X,x) \to (Y,y)$ are base-point preserving continuous maps.

Let $X \wedge Y$ denote the smash product of two pointed spaces (X,x) and (Y,y). The *suspension* of a space X is defined as

$$\Sigma X \coloneqq S^1 \wedge X, \quad \Sigma^n X = S^n \wedge X.$$

This is not an automorphism, so we need to formally invert it. To do this, let X be a pointed space and let $n \in \mathbb{Z}$. Consider the category Stable with objects (X,n) and define $\Sigma(X,n) \coloneqq (X, n-1)$. Its arrows are given by

$$\mathrm{Hom}((X,n),(Y,m)) \coloneqq \varinjlim_{k \geq m,n,0} [S^{k-n} \wedge X, S^{k-m} \wedge Y],$$

where $[\cdot,\cdot]$ denotes homotopy classes of based continuous maps. The category Stable is called the *stable homotopy category* of pointed spaces. On this category, suspension is indeed an automorphism. Furthermore, it is an additive category using the wedge sum $X \vee Y \coloneqq X \coprod Y / \sim$ as addition, where $\sim$ is the equivalence relation generated by (x,y). For some more details related to the reduced mapping cone, see Lecture Slides 10.

4. Tensor product and internal Hom

Let $\mathcal{C}$ be a closed, symmetric monoidal category, whose tensor product commutes with countable coproducts. Then the category $\mathsf{Kom}(\mathcal{C})$ inherits a symmetric, monoidal structure. The tensor product of two chain complexes is defined by $(C \otimes D)_n \coloneqq \oplus_{k \in \mathbb{Z}} C_k \otimes D_{n-k}$, with differential given by

$$\delta_{|C_n \otimes D_m} \coloneqq d_n^C \otimes 1_{D_m} + (-1)^n 1_{C_n} \otimes d_m^D.$$

It is easy to see that this is a chain complex, and that $\otimes$ defines a symmetric monoidal structure. Finally, since $\mathcal{C}$ is closed, it has an internal Hom-functor that is adjoint to the tensor product functor. This internal Hom in the definition of the mapping complex of two chain complexes yields the internal-Hom in the category $\mathsf{Kom}(\mathcal{C})$.

5. Homological algebra in abelian categories

So far we have studied two ways of identifying two chain complexes: up to chain homotopy equivalences and quasi-isomorphisms. It is often practical, however, to approximate a given chain complex (or a module, viewed as a chain complex) by a chain complex with simpler entries. These simpler objects are called *projective* objects, and these approximations are called *projective resolutions*. Furthermore, a projective resolution is quasi-isomorphic to the original module or chain complex. The process of passing from the world of modules or the homotopy category of chain complexes to this richer world which identifies chain complexes with projective resolutions is the core of homological algebra.

Example 5.1 (Motivating example). Let us first see an example of why mere chain homotopies do not always suffice. Consider the following chain map in the category of chain complexes of abelian groups:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}_2 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \xrightarrow{\text{mod } 2} & \mathbb{Z}_2.
\end{array}$$

The domain of this chain map is an exact sequence, and is hence quasi-isomorphic to the 0-complex. Consequently, any chain map into it must be null-homotopic. But the chain map in question is only chain homotopic to itself, since the degree-0 component of any chain homotopy must be a group homomorphism $\mathbb{Z}_2 \to \mathbb{Z}$, which must be the zero morphism. On the other hand, consider the chain map

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \text{mod } 2} \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}_2,
\end{array}$$

between a chain complex whose entries consist only of free abelian groups, into the domain of the original complex. Composing with the original chain map yields a chain map

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \text{mod } 2} \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \xrightarrow{\text{mod } 2} & \mathbb{Z}_2
\end{array}$$

that *is* null-homotopic: the maps in degrees 0 and 1 are the identity maps $\mathbb{Z} \to \mathbb{Z}$. What we have done is that we have *resolved* a chain map between two chain complexes of abelian groups with a chain complex of *free* abelian groups. This will be made precise slightly later in the section, after we have established some basic vocabulary.

In the example above, we resolved a chain map by a chain complex of free modules. But in practical situations, it suffices to only insist that there is a resolution by direct summands of free modules.

Definition 5.2. An object P of an abelian category $\mathcal{A}$ is called *projective* if the following condition holds: for any morphism $f\colon P \to B$, and any epimorphism $A \overset{p}{\twoheadrightarrow} B$, there exists a morphism $\hat{f}\colon P \to A$ such that

$$\begin{array}{ccc}
 & & P \\
 & {\scriptstyle \hat{f}} \swarrow & \downarrow{\scriptstyle f} \\
A & \xrightarrow{p} & B
\end{array}$$

commutes.

Lemma 5.3. *Any free R-module is projective.*

Proof. This result explicitly requires the axiom of choice, which we tacitly assume as part of our set-theoretic setup. Now recall that the free-forgetful adjuction says that set-theoretic maps $S \to U(N)$ is in unique bijection with an R-module map $F(S) \cong \oplus_S R \to N$, where N is an R-module, and U is the forgetful functor from R-modules to the category of sets.

Let $M \twoheadrightarrow N$ be an epimorphism of R-modules. Then the underlying map $U(M) \to U(N)$ is a set-theoretic surjection. Now choose a splitting of this map $U(N) \to U(M)$ - and this is where the axiom of choice comes in - to obtain a set-theoretic map $S \to U(M)$. By adjunction, there exists a unique R-module map $F(S) \to M$, as required. □

Remark 5.4. In the category of sets, every object is projective if and only if the axiom of choice holds. This is a homological algebraic formulation of this axiom, which I find quite enlightening.

Lemma 5.5. *An R-module is projective if and only if it is a direct summand of a free R-module.*

Proof. If a module M is the direct summand of a free module, then there exists an M' and an N (free) such that $N \cong M \oplus M'$. To show that M is projective, let $A \twoheadrightarrow B$ be an R-module epimorphism, and let $M \to B$ be an R-module map. Since N is free by the previous lemma, for any map $N \to B$, there exists a lifting $N \to A$. Furthermore, there exists a map $M \to N$ that splits the sequence $M' \rightarrowtail M \oplus M' \twoheadrightarrow M$. Consequently, there exists a lifting $M \to A$ of the original surjection $A \twoheadrightarrow B$. The converse is left as an exercise. □

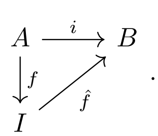
Definition 5.6. A category $\mathcal{A}$ has enough projectives if for any object X, there is a projective object $P \in \mathcal{A}^0$ and an epimorphism $P \twoheadrightarrow X$.

Proposition 5.7. *The category Mod_R has enough projectives.*

Proof. Let M be an R-module. By Lemma 5.3, $F(U(M))$ is a projective R-module, where $U(M)$ is the set underlying M. Furthermore, $F(U(M)) \cong \oplus_{m \in U(M)} R \to M$, mapping 1 in each summand (indexed by m) to m is a surjective R-module map. □

There is a notion that is dual to projective objects in a category. These are called *injective objects.*

Definition 5.8. An object I in a category is called *injective* if for any monomorphism $i\colon A \rightarrowtail B$ and an morphism $f\colon A \to I$, there exists a morphism $\hat{f}\colon I \to B$ such that the following diagram commutes



Similarly, a category $\mathcal{C}$ has *enough injectives* if for every object $X \in \mathcal{C}^0$, there is a monomorphism $X \to I$, where I is an injective object. Finally, dual to the projective case, we have the following result in the category of R-modules, whose proof we omit:

Proposition 5.9. *The category Mod_R has enough injectives.*

We now come to the notion of 'simpler approximations' referred to in the motivation of this section.

DEFINITION 5.10 (Projective resolutions). Let $\mathcal{C}$ be an abelian category, and let X be an object in $\mathcal{C}$. A *projective resolution* of X is a chain complex $P_\bullet$ together with a morphism $P_0 \to X$, such that P_i are projective, and the resulting complex

$$\cdots P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to X \to 0$$

is exact.

PROPOSITION 5.11. *Let $\mathcal{C}$ be an abelian category with enough projectives. Then every object $X \in \mathcal{C}$ has a projective resolution.*

PROOF. We proceed by induction on n. Since $\mathcal{C}$ has enough projectives, there is an epimorphism $P_0 \twoheadrightarrow X$. This is part of an exact sequence $P_0 \to X \to 0$. Now suppose the result holds for $n = k$, then there is an exact sequence

$$P_k \xrightarrow{d_k} P_{k-1} \to \cdots \to P_0 \to X \to 0,$$

where P_i are projective objects in $\mathcal{C}$. By the hypothesis of enough projectives, there is an epimorphism $P_{k+1} \overset{p}{\twoheadrightarrow} \ker(d_k)$. Now form the diagram

$$P_{k+1} \xrightarrow{\ker(d_k)\circ p} P_k \to \cdots \to P_0 \to X.$$

This is exact at P_k, which is what we wanted to show. □

We end this section linking the theory developed so far with the problem motivated at the start.

LEMMA 5.12. *Let $f\colon P_\bullet \to X_\bullet$ be a chain map between chain complexes (supported in non-negative degree). Suppose $P_\bullet$ is degree-wise a projective chain complex, and $X_\bullet$ is an exact chain complex. Then f is null-homotopic.*

PROOF. This result is formally dual to the result for injective resolutions that will be proven below. □

6. The derived category of an abelian category

6.1. Localisation of a category. Let R be a commutative, unital ring, and let $f \in R$ be a non-unit in R. A typical situation in algebraic geometry requires us to enlarge the ring R in order that f becomes invertible in this larger ring. This is called the *localisation* R_f of the ring R at f. The localisation of a ring satisfies the following universal property: suppose S is a commutative unital ring then there is a natural bijection between ring homomorphisms $R_f \to S$ and ring homomorphisms $R \to S$ that map f to a unit in S.

We would like to do something similar in our context of chain complexes $\mathsf{Kom}(\mathcal{C})$ over an abelian category $\mathcal{C}$, where we would like to treat quasi-isomorphisms between complexes as *isomorphisms* in a larger category. This is done by localising the category of complexes at the quasi-isomorphisms. But before we get there, let us first talk about the localisation of a category. But since the definition of localisation of an arbitrary category is rather inexplicit, we restrict ourselves to the localisation of a triangulated category.

DEFINITION 6.1. Let $\mathcal{C}$ and $\mathcal{D}$ be two triangulated categories. A *triangle functor* is an additive functor $F{:}\,\mathcal{C} \to \mathcal{D}$, together with a natural isomorphism $F \circ \Sigma \cong \Sigma \circ F$, such that F maps exact triangles in $\mathcal{C}$ to exact triangles in $\mathcal{D}$.

DEFINITION 6.2. Let $\mathcal{D}$ be a triangulated category. A full subcategory $\mathcal{C}$ of $\mathcal{D}$ is called a *triangulated subcategory* if

- it is closed under isomorphisms;
- the suspension Σ on $\mathcal{D}$ restricts to a suspension on $\mathcal{C}$, that is, $\Sigma(\mathcal{C}) \cong \mathcal{C}$;
- for any triangle $X \to Y \to Z \to \Sigma X$ with X, $Y \in \mathcal{C}^0$, we have $Z \in \mathcal{C}^0$.

DEFINITION 6.3. Given a triangulated functor $F{:}\,\mathcal{D} \to \mathcal{E}$ between two triangulated categories, its *kernel* is the full subcategory with objects $\ker(F) := \{X \in \mathcal{D} : FX \cong 0\}$.

DEFINITION 6.4. A triangulated subcategory $\mathcal{C} \subseteq \mathcal{D}$ is called *thick* if for all X, $Y \in \mathcal{D}$ such that $X \oplus Y \in \mathcal{C}$, we have X and $Y \in \mathcal{C}^0$.

LEMMA 6.5. *Let $F{:}\,\mathcal{C} \to \mathcal{D}$ be a triangle functor between two triangulated categories. Then* $\ker(F)$ *is a thick triangulated subcategory.*

Let $\mathcal{C} \subseteq \mathcal{D}$ be a triangulated subcategory. We define a collection

$$\mathsf{Mor}_{\mathcal{C}} := \{f{:}\,X \to Y \in \mathcal{D}^1 : \mathsf{cone}(f) \in \mathcal{C}\},$$

consisting precisely of morphisms of $\mathcal{D}$, which if completed to an exact triangle

$$X \to Y \to Z \to \Sigma X,$$

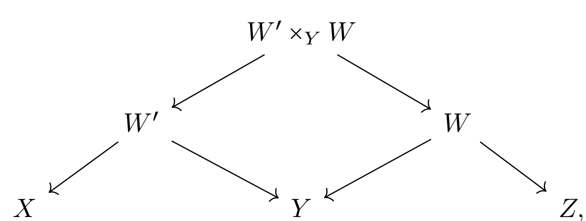
we have $Z \in \mathcal{C}$. We shall use this to define the morphisms of the "localisation of $\mathcal{D}$ at $\mathcal{C}$".

DEFINITION 6.6. For objects X and $Y \in \mathcal{D}$, we define a collection of diagrams

$$\hat{\mathrm{Hom}}_{\mathcal{D}}(X,Y) := \{X \xleftarrow{f} W \to Y : f \in \mathsf{Mor}_{\mathcal{C}}\},$$

called *roofs* in $\mathcal{D}$.

The composition of two roofs $Y \leftarrow W \to Z$ and $X \leftarrow W' \to Y$ in $\mathrm{Hom}_{\mathcal{D}}(X,Y)$ is given as follows:



where the object at the top is the (homotopy) pullback of the morphisms $W' \to Y$ and $W \to Y$. This always exists, but is only defined up to a non-canonical isomorphism. Consequently, compositions of roofs are only defined up to isomorphism. This composition is associative, but again, only up to isomorphism.

The final thing we need is an equivalence relation that encodes the notion of fractions - similar to how the localisation of a ring is constructed. This is given as follows: two roofs $X \leftarrow Z \to Y$ and $X \leftarrow Z' \to Y$ are equivalent if and only if there exists a roof $Z \leftarrow W \to Z'$ such that the following diagram

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow & \uparrow & \searrow & \\ X & \longleftarrow & W & \longrightarrow & Y \\ & \nwarrow & \downarrow & \nearrow & \\ & & Z' & & \end{array}$$

commutes. The equivalence class of $X \overset{g}{\leftarrow} W \overset{f}{\rightarrow} Y$ is denoted $g \circ f^{-1}$. We can now define the notion of localisation that we have been after:

Definition 6.7. The *Verdier localisation* $\mathcal{D}/\mathcal{C}$ (or *Verdier quotient*) of a triangulated category $\mathcal{D}$ at a triangulated subcategory $\mathcal{C}$ is defined as the category with the same objects as $\mathcal{D}$, and with morphisms $\mathrm{Hom}_{\mathcal{D}/\mathcal{C}}(X, Y) := \widehat{\mathrm{Hom}}_{\mathcal{D}}(X, Y)/\sim$ for all X, $Y \in \mathcal{D}$. There is a functor, called the *localising functor*, $F\colon \mathcal{D} \to \mathcal{D}/\mathcal{C}$ that is the identity on objects, and maps $f\colon X \to Y$ to the roof $X \overset{1}{\leftarrow} X \overset{f}{\rightarrow} Y$.

We first show that the Verdier quotient $\mathcal{D}/\mathcal{C}$ of a triangulated category by a triangulated subcategory is itself a triangulated category. This ensures that we continue to have long exact or Puppe sequences in our larger category, where quasi-isomorphisms have been inverted. The triangulated category structure is defined as follows: the *suspension* functor is defined as

$$\Sigma\colon \mathcal{D}/\mathcal{C} \to \mathcal{D}/\mathcal{C}, \quad X \mapsto \Sigma_{\mathcal{D}}(X), \quad [X \overset{f}{\leftarrow} W \overset{g}{\rightarrow} Y] \mapsto [\Sigma X \overset{\Sigma f}{\leftarrow} \Sigma W \overset{\Sigma g}{\rightarrow} \Sigma Y],$$

and a triangle is exact if it is isomorphic to the image of a triangle in $\mathcal{D}$ under F.

Exercise 6.8. Check that the above really defines a triangulated category structure on $\mathcal{D}/\mathcal{C}$.

The following result shows that the Verdier quotient category that we have constructed indeed has the expected universal property:

Proposition 6.9. *The functor $F\colon \mathcal{D} \to \mathcal{D}/\mathcal{C}$ is triangulated, mapping diagrams in $\mathsf{Mor}(\mathcal{C})$ to isomorphisms in $\mathcal{D}/\mathcal{C}$. Moreover, if $T\colon \mathcal{D} \to \mathcal{E}$ is any other triangulated functor mapping digrams in $\mathsf{Mor}(\mathcal{C})$ to isomorphisms in $\mathcal{E}$, then there exists a unique triangle functor $\mathcal{D}/\mathcal{C} \to \mathcal{E}$ factorising T.*

Proof. The functor F is triangulated by contruction as the exact triangles in $\mathcal{D}/\mathcal{C}$ are precisely those mapped under the image of F on exact triangles in $\mathcal{D}$. Now let $f\colon X \to Y$ lie inside $\mathsf{Mor}(\mathcal{C})$. Then the roof $Y \overset{f}{\leftarrow} X \to X$ is inverse to the roof $X \leftarrow X \overset{f}{\rightarrow} Y$. That is, $F(f)$ is invertible in the category $\mathcal{D}/\mathcal{C}$. Finally, to check that F is universal among triangle functors mapping morphisms in $\mathsf{Mor}(\mathcal{C})$ to isomorphisms in a triangulated category, let $T\colon \mathcal{D} \to \mathcal{E}$ be such a functor. Then clearly T extends to any roof in $\widehat{\mathrm{Hom}}_{\mathcal{D}}(X, Y)$, and if two such roofs are equivalent, then their images under T are isomorphic. $\square$

We can now finally talk about the derived category of an abelian category. Recall that given any additive category $\mathcal{C}$, its homotopy category $\mathsf{HoKom}(\mathcal{C})$ of chain complexes is a triangulated category. The suspension of a chain complex

$\Sigma X = X[1]$, and its distinguished triangles are diagrams of chain complexes that are isomorphic to the mapping cone diagram

$$X \to Y \to \mathsf{cone}(f) \to X[1],$$

where $f\colon X \to Y$ is a chain map (up to chain homotopy).

LEMMA 6.10. *The full subcategory* $\mathsf{Exact}(\mathcal{C})$ *of exact chain complexes is a triangulated subcategory. In the notation above,* $\mathsf{Mor}(\mathsf{Exact}(\mathcal{C}))$ *consists of quasi-isomorphisms.*

PROOF. By a previous result, a chain map is a quasi-isomorphism precisely when it is exact. So the latter claim is immediate. To prove the first claim, it is easy to check that the category of exact complexes is thick. Now given a triangle

$$X \to Y \to Z \to X[1],$$

where X and Y are exact, the Puppe long exact sequence yields that $H_i(Z) = 0$ for all i, and consequently, Z is also exact. □

DEFINITION 6.11. The *derived category* $\mathsf{Der}(\mathcal{C})$ of an abelian category is defined as the Verdier localisation of the homotopy category $\mathsf{HoKom}(\mathcal{C})$ at the triangulated subcategory $\mathsf{Exact}(\mathcal{C})$ of exact chain complexes.

7. Derived functors

Let $\mathcal{C}$ be an additive category. We can view any object $X \in \mathcal{C}^0$ as a chain complex $\cdots 0 \to X \to 0 \to \cdots$ supported at degree 0 by X. This yields canonical functors

$$\mathcal{C} \to \mathsf{Kom}(\mathcal{C}) \to \mathsf{HoKom}(\mathcal{C}) \xrightarrow{q} \mathsf{Der}(\mathcal{C}).$$

Here q is the localisation functor constructed in the previous section. Now suppose we have an additive functor $F\colon \mathcal{C} \to \mathcal{D}$. This extends to a triangle functor $\tilde{F}\colon \mathsf{HoKom}(\mathcal{C}) \to \mathsf{HoKom}(\mathcal{D})$, by entry-wise application. Composing with the localisation functor to $\mathsf{Der}(\mathcal{D})$ yields a triangle functor $\mathcal{C} \to \mathsf{Der}(\mathcal{D})$. We would like to extend this to a functor $\mathsf{Der}(\mathcal{C}) \to \mathsf{Der}(\mathcal{D})$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow q & & \downarrow q \\ \mathsf{Der}(\mathcal{C}) & \xrightarrow{\tilde{F}} & \mathsf{Der}(\mathcal{D}). \end{array}$$

In general, such a functor does not exist. In this section, we explore important examples when a functor can be 'derived' in the sense above. But first, we look at some general techniques for investigating the derivation of a functor. Let $F\colon \mathsf{HoKom}(\mathcal{C}) \to \mathsf{E}$ be a triangle functor into a triangulated category $\mathcal{A}$. The category E is for instance the derived category of an abelian category. Likewise, the functor F could be assumed to be coming from an additive functor between abelian categories.

DEFINITION 7.1. The *total right derived functor* of F, if it exists, is a triangle functor $RF\colon \mathsf{Der}(\mathcal{C}) \to \mathsf{E}$ together with a natural transformation $\eta\colon F \Rightarrow RF$. This is universal in the sense that if there is a another triangle functor $G\colon \mathsf{Der}(\mathcal{C}) \to \mathsf{E}$ with a natural transformation $\eta'\colon F \Rightarrow G$, then there is a unique natural transformation $\theta\colon RF \Rightarrow G$ such that $\theta \circ \eta = \eta'$.

Dually, the *total left derived functor* of F, if it exists, is a functor $LF: \mathsf{Der}(\mathcal{C}) \to \mathsf{E}$ with a natural transformation $\eta: LF \Rightarrow F$ satisfying a similar universal property.

REMARK 7.2. The pair (RF, η) is also called the *left Kan extension* of the functor F along the functor $\mathsf{HoKom}(\mathcal{C}) \to \mathsf{Der}(\mathcal{C})$. Dually, the total left derived functor pair (LF, η) is called the *right Kan extension* of F along $\mathsf{HoKom}(\mathcal{C}) \to \mathsf{Der}(\mathcal{C})$. By the universality of the natural transformations $F \Rightarrow RF$ and $LF \Rightarrow F$, the total derived functors are unique, if they exist.

In what follows, we only work out arguments and definitions for right derived functors, leaving it for the reader to figure out dual statements and definitions.

DEFINITION 7.3.
- A *right deformation* on $\mathsf{HoKom}(\mathcal{C})$ is a triangle endofunctor $I: \mathsf{HoKom}(\mathcal{C}) \to \mathsf{HoKom}(\mathcal{C})$ with a natural isomorphism $i: 1_{\mathsf{HoKom}(\mathcal{C})} \overset{\sim}{\Rightarrow} I$;
- A *right deformation of a triangle functor* $F: \mathsf{HoKom}(\mathcal{C}) \to \mathcal{E}$ is a right deformation (i, I) such that F preserves quasi-isomorphisms between objects in a full triangulated subcategory J containing the image of I.

THEOREM 7.4. *If F has a right deformation, it has a total right derived functor.*

PROOF. The proof specific to our situation was done in class. For a more general statement and proof, see [**3**, Propositions 6.4.11, 6.4.12]. □

We have therefore reduced the derived functor problem to the problem of finding a deformation for the functor $F: \mathsf{HoKom}(\mathcal{C}) \to \mathsf{E}$. Let us suppose that $\mathcal{C}$ is a category with enough injective objects (for example, $\mathcal{C} = \mathsf{Mod}_R$).

LEMMA 7.5. *Let $\mathcal{C}$ be an abelian category with enough injectives. Then every object $X \in \mathcal{C}^0$ has an injective resolution $0 \to I^0 \to I^1 \to \cdots$.*

PROOF. Simply dualise the proof as in the projective case, which is dealt with in Proposition 5.11. Observe that an object in $\mathcal{C}$ is injective if it is projective in $\mathcal{C}^{\mathrm{op}}$, and that a chain complex in fC is a cochain complex in $\mathcal{C}^{\mathrm{op}}$. □

The most natural candidate for a right deformation, is the 'functor' $I: \mathsf{HoKom}(\mathcal{C}) \to \mathsf{HoKom}(\mathcal{C})$ that 'resolves' a chain complex by injective objects in $\mathcal{C}$. But before we define such a functor at the level of chain complexes, we first define a functor $I: M \mapsto I^\bullet$, where $I^\bullet = (I^n)_{n \geq 0}$ is a resolution with injective I^n. However, in order that such a functor is well-defined, we need to show that the association of an injective resolution to an object is unique up to homotopy. This turns out to be a rather important result in homological algebra.

THEOREM 7.6. *Let $f: X \to Y$ be a morphism in $\mathcal{C}$. Let $i_Y: Y \to Y^\bullet$ be an injective resolution of Y and let $i_X: X \to X^\bullet$ be a monomorphism that is a quasi-isomorphism. Then there is a chain map $f^\bullet: X^\bullet \to Y^\bullet$ extending f. Moreover, any two such liftings are unique up to a chain homotopy.*

PROOF. Done in class. □

The next question that arises is about uniqueness of the lifting of a morphism $f: X \to Y$ to a chain map into an injective resolution.

THEOREM 7.7. *Let $f^\bullet: X^\bullet \to Y^\bullet$ be a chain map between cochain complexes in non-negative degree, where $X^\bullet$ is an exact chain complex and $Y^\bullet$ is termwise injective. Then $f^\bullet$ is null-homotopic.*

THEOREM 7.8. *The lifting $f^\bullet$ in the Fundamental Theorem is unique up to a chain homotopy.*

PROOF. Done in class. □

As a consequence of Theorem 7.6, we see that the assignment $\mathcal{C} \ni M \mapsto I(M)^\bullet \in \mathsf{HoKom}(\mathcal{C})$ of an object to a choice of injective resolution is well defined as a functor $\mathcal{C} \to \mathsf{Der}(\mathcal{C})$. This is because any two choices of injective resolutions are chain homotopy equivalent, and hence isomorphic in the derived category. At this point of time, one can ask whether an analogous result holds at the level of chain complexes. That is, given a chain complex $X \in \mathsf{HoKom}(\mathcal{C})$, can one resolve it via 'injective chain complexes'? The cleanest way to answer this is by generalising the notion of an injective object in the homotopy category of chain complexes.

We only sketch the contents of what follows, leaving the interested reader to look up more details in [**1**, Chapter 14]. The main idea, however, remains similar in spirit to the approximation of an object by an injective resolution, and showing that the choice of such a resolution does not matter.

DEFINITION 7.9. Let $\mathcal{C}$ be an abelian category.

- An object $I \in \mathsf{HoKom}(\mathcal{C})$ is called *K-injective* if for every exact $X \in \mathsf{HoKom}(\mathcal{C})$, the mapping complex $\mathrm{Hom}(X, I)$ is exact;
- Given $X \in \mathsf{HoKom}(\mathcal{C})$, a *$K$-injective resolution* of X is a quasi-isomorphism $X \to I$, where I is K-injective;
- We say $\mathsf{HoKom}(\mathcal{C})$ has *enough K-injectives* if every object has a K-injective resolution.

EXAMPLE 7.10. The homotopy category of complexes over $\mathcal{C} = \mathsf{Mod}_R$ has enough K-injectives. More generally, if $\mathcal{C}$ is any *Grothendieck abelian category*: (ie, $\mathcal{C}$ is closed under arbitrary direct sums, the direct limit functor preserves short exact sequences, and for every object $X \in \mathcal{C}^0$, there is an epimorphism $\bigoplus_I R \to X$), then $\mathsf{HoKom}(\mathcal{C})$ has enough K-injectives. This is a rather lengthy result, so I will skip its proof in the interest of time. See details in [**1**, Corollary 14.1.8] in case you are interested.

Let $\mathsf{HoKom}(\mathcal{C})_{\mathrm{inj}}$ be the full subcategory of $\mathsf{HoKom}(\mathcal{C})$ consisting of K-injective complexes.

THEOREM 7.11. *Let $\mathcal{C}$ be a Grothendieck abelian category.*

- *for any $X \in \mathsf{HoKom}(\mathcal{C})$, there is a quasi-isomorphism $X \to I$, where $I \in \mathsf{HoKom}(\mathcal{C})$ is K-injective;*
- *The localisation functor $Q\colon \mathsf{HoKom}(\mathcal{C}) \to \mathsf{Der}(\mathcal{C})$ induces an equivalence of categories $\mathsf{HoKom}(\mathcal{C})_{\mathrm{inj}} \cong \mathsf{Der}(\mathcal{C})$;*
- *any triangulated functor $F\colon \mathsf{HoKom}(\mathcal{C}) \to E$ has a total right derived functor.*

PROOF. The proof of the first two claims are detailed in [**1**, Theorem 14.3.1]. We only show the construction of the right derived functor. Let $R\colon \mathsf{Der}(\mathcal{C}) \to \mathsf{HoKom}(\mathcal{C})_{\mathrm{inj}}$ be the inverse of Q above. Then the right derived functor is given by the composition

$$\mathsf{Der}(\mathcal{C}) \xrightarrow{R} \mathsf{HoKom}_{\mathrm{inj}}(\mathcal{C}) \hookrightarrow \mathsf{HoKom}(\mathcal{C}) \xrightarrow{F} E.$$

□

We now move to a more concrete situation, which will ultimately lead us to some of the most important geometric invariants.

DEFINITION 7.12. Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories, and let $F: \mathcal{C} \to \mathcal{D}$ be an additive functor between them. We call the functor F

- *left exact* if whenever $f: X \to Y$ is a monomorphism in $\mathcal{C}$, the induced map $F(f): F(X) \to F(Y)$ is a monomorphism in $\mathcal{D}$;
- *right exact* if whenever $f: X \to Y$ is an epimorphism, the induced map $F(f): F(X) \to F(Y)$ is an epimorphism;
- *exact* if it is both left and right exact.

We first start with left exact functors.

PROPOSITION 7.13. *Let $F: \mathcal{C} \to \mathcal{D}$ be a left exact functor between abelian categories. Suppose $0 \to X \to Y \to Z \to 0$ is a short exact sequence in $\mathcal{C}$, then*

$$0 \to F(X) \to F(Y) \to F(Z)$$

is an exact sequence in $\mathcal{D}$. Dually, if F is a right exact functor, then $F(X) \to F(Y) \to F(Z) \to 0$ is an exact sequence in $\mathcal{D}$.

PROOF. Trivial. □

We can now define (and implicitly, find a way to compute) derived functors. Let $\mathsf{HoKom}(\mathcal{C}_{\mathrm{inj}})^{\geq 0}$ denote the homotopy category of chain complexes, whose entries are injective objects of the category $\mathcal{C}$. Suppose further that $\mathcal{C}$ has enough injectives. Then there is an equivalence of categories

$$\mathsf{HoKom}(\mathcal{C}_{\mathrm{inj}})^{\geq 0} \cong \mathsf{Der}(\mathcal{C})^{\geq 0} := \mathsf{HoKom}(\mathcal{C})^{\geq 0} / \mathsf{Exact}(\mathcal{C})^{\geq 0},$$

which uses that every bounded below chain complex has an injective resolution **- I will upload a proof of this**.

DEFINITION 7.14. Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories, and let $\mathcal{C}$ have enough injectives. The composite functor

$$\mathcal{C} \xrightarrow{I} \mathsf{HoKom}(\mathcal{C}_{\mathrm{inj}})^{\geq 0} \xrightarrow{F} \mathsf{HoKom}(\mathcal{D}) \xrightarrow{q} \mathsf{Der}(\mathcal{D}) \xrightarrow{H^n(-)} \mathcal{D}$$

is called the n-th *right derived functor* $R^n F: \mathcal{C} \to \mathcal{D}$. Here I is the injective resolution functor, and H^n is the n-th cohomology functor.

REMARK 7.15. Using the identification of the bounded derived category $\mathsf{Der}(\mathcal{C})^{\geq 0}$ with the bounded homotopy category of injective chain complexes above, we can define the n-th total right derived functor of a triangle functor $F: \mathsf{HoKom}(\mathcal{C})^{\geq 0} \to \mathsf{Der}(\mathcal{D})^{\geq 0}$ as the composition

$$\mathsf{Der}(\mathcal{C})^{\geq 0} \cong \mathsf{HoKom}(\mathcal{C}_{\mathrm{inj}})^{\geq 0} \xrightarrow{F} \mathsf{Der}(\mathcal{D})^{\geq 0},$$

thereby justifying its name as a derived functor.

Likewise, we can define the n-th *left derived functor* $L_n F: \mathcal{C} \to \mathcal{D}$, where we use the projective resolution functor $P: \mathcal{C} \to \mathsf{HoKom}(\mathcal{C})$ in place of the injective resolution functor. Here we assume that the abelian category $\mathcal{C}$ has enough projectives.

LEMMA 7.16. *Let $F: \mathcal{C} \to \mathcal{D}$ be a left exact functor between abelian categories, where $\mathcal{C}$ has enough injectives. Then for all $X \in \mathcal{C}^0$, there is a natural isomorphism $R^0 F(X) \cong F(X)$. Dually, if F is right exact and $\mathcal{C}$ has enough projectives, then $L_0 F(X) \cong F(X)$.*

PROOF. The hypothesis of enough injectives implies that X has an injective resolution $X \to I^\bullet$, so that

$$0 \to X \to I^0 \to I^1 \to \cdots$$

is an exact sequence. Since F is left exact, we have an exact sequence

$$0 \to F(X) \to F(I^0) \to F(I^1) \to \cdots,$$

and hence $R^0F(X) \cong \ker(F(I^0) \to F(I^1)) \cong F(X)$. □

PROPOSITION 7.17. *Let $F{:}\,\mathcal{C} \to \mathcal{D}$ be an additive functor.*

- *If F is right exact and N is projective, then $L_iF(N) \cong 0$ for all $i \geq 1$;*
- *If F is left exact and M is injective, then $R^iF(M) \cong 0$ for all $i \geq 1$.*

PROOF. If N is projective, choose the projective resolution $\cdots \to 0 \to N$. So for $i \geq 1$, $L_nF(N) \cong H_n(0) = 0$. □

PROPOSITION 7.18. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence in an abelian category with enough projectives. Then there exists a commuting diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_\bullet & \longrightarrow & B_\bullet & \longrightarrow & C_\bullet & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

where each vertical morphism is a projective resolution, and each horizontal row is a short exact sequence. A similar claim holds for morphisms out of a short exact sequence in a category with enough injectives.

PROPOSITION 7.19. *Let $\mathcal{C}$ be an abelian category with enough injectives, $F{:}\,\mathcal{C} \to \mathcal{D}$ a left exact functor, and $0 \to A \to B \to C \to 0$ a short exact sequence in $\mathcal{C}$. Then there is an induced long exact sequence in $\mathcal{D}$ as follows:*

$$0 \to F(A) \to F(B) \to F(C) \xrightarrow{\delta_0} R^1F(A) \to R^1F(B) \to R^1F(C) \to \cdots.$$

A dual claim holds for a right exact functor on an abelian category with enough projectives.

PROOF. Given a short exact sequence $0 \to A \to B \to C \to 0$, we can resolve them to obtain extensions of injective complexes $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$. Furthermore, by the way we constructed the complex $B^n = A^n \oplus C^n$, we actually have split exact sequences

$$0 \to A^n \to B^n \to C^n \to 0$$

for each n. Since F is additive, it preserves split exact sequences, and we have exact sequences $0 \to F(A^n) \to F(B^n) \to F(C^n) \to 0$ for each n. Now apply the Puppe sequence to the exact sequence of cochain complexes

$$0 \to F(A^\bullet) \to F(B^\bullet) \to F(C^\bullet) \to 0.$$

Combining this with Lemma 7.16, we are done. □

PROPOSITION 7.20. *Let $F{:}\,\mathcal{C} \to \mathcal{D}$ be an exact functor. Then $R^iF \cong 0$ and $L_iF \cong 0$ for all i.*

PROOF. Obvious. □

8. Ext, Tor, Hochschild homology and cohomology

In this section, we consider the derived functors of the Hom-functor, and its left adjoint, the tensor product functor. Let $\mathcal{C}$ be an abelian category. The Hom-functor

$$\mathrm{Hom}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Ab}$$

is a bifunctor. Now suppose $X \in \mathcal{C}^0$, then the functors

$$\mathrm{Hom}(X,-)\colon \mathcal{C} \to \mathsf{Ab}, \text{ and } \mathrm{Hom}(-,X)\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Ab}$$

are left exact. Indeed, if $0 \to A \to B \to C \to 0$ is an exact sequence, then

$$0 \to \mathrm{Hom}(X,A) \to \mathrm{Hom}(X,B) \to \mathrm{Hom}(X,C)$$

is an exact sequence.

Definition 8.1. Let $\mathcal{C}$ be an abelian category and $X \in \mathcal{C}^0$. We call the right derived functors (if they exist)

$$\mathsf{Ext}^n(-,X) \coloneqq R^n \mathrm{Hom}(-,X)\colon \mathcal{C} \to \mathsf{Ab}$$

the *Ext-functor* in the first variable. Similarly, we can define

$$\mathsf{Ext}^n(X,-) \coloneqq R^n \mathrm{Hom}(X,-)\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Ab},$$

and call it the Ext-functor in the second variable.

Remark 8.2. The way we compute the right derived functors $R^n \mathrm{Hom}(-,X)$ depends on whether we would like to view the left exact functor $\mathrm{Hom}(-,X)$ as a covariant functor on $\mathcal{C}^{\mathrm{op}}$ or as a contravariant functor on $\mathcal{C}$ (and likewise with $\mathrm{Hom}(X,-)$). This does not change anything as if $\mathcal{C}$ admits injective resolutions of objects, then $\mathcal{C}^{\mathrm{op}}$ admits projective resolutions. Therefore, if $M \in \mathcal{C}^0$ and $\mathcal{C}$ has enough injectives, then we can pick an injective resolution $0 \to M \to I^\bullet$, so that $\mathsf{Ext}^n(X,M) \cong H^n(\mathrm{Hom}(X,I^\bullet))$. On the other hand, if $M \in \mathcal{C}$, then as $\mathcal{C}^{\mathrm{op}}$ has enough projectives, we can build a projective resolution $P_\bullet \to M \to 0$. Then the right derived functors are given by $\mathsf{Ext}^n(M,X) \cong H_n(\mathrm{Hom}(P_\bullet,X))$. We will see below that it is often easier to construct projective resolutions more canonically, so it is worthwhile to have this dual perspective at hand already.

We now talk about the 'dual' construction, namely, the left derived functor of the tensor product functor. Here tensor product is meant to understand the left adjoint of the Hom-functor. To stay reasonably concrete, let us work in the category Mod_R of modules over a commutative ring R. Given an $N \in \mathsf{Mod}_R$, it is well known that

$$- \otimes_R N\colon \mathsf{Mod}_R \to \mathsf{Mod}_R$$

is an additive, left exact functor.

Definition 8.3. Given $N \in \mathsf{Mod}_R^0$, $n \in \mathbb{N}$, the left derived functors

$$\mathsf{Tor}_n^R(-,N) \coloneqq L_n(- \otimes_R N)\colon \mathsf{Mod}_R \to \mathsf{Mod}_R$$

are called the n-th *Tor-functors*.

8.1. Ext and group cohomology. Let $\mathcal{C}$ be an abelian category and let $A \in \mathcal{C}^0$ be an object. In this section, we interpret $\operatorname{Ext}(-, A)$ in terms of "extensions of a group by A". In order to make this precise, we shall take a brief detour and study group extensions and group cohomology.

Definition 8.4. A short exact sequence of groups of the form

$$0 \to A \to \hat{G} \to G \to 0$$

is called a *group extension of* G *by* A. If the monomorphism $A \to \hat{G}$ factors through the center of $\hat{G}$, we call this a *central extension*.

We say that a given a group extension $0 \to A \to \hat{G}_1 \to G \to 0$ is *equivalent* to another extension $0 \to A \to \hat{G}_2 \to G \to 0$ of G by A if there is an isomorphism $f \colon \hat{G}_1 \to \hat{G}_2$, inducing an isomorphism of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & \hat{G}_1 & \longrightarrow & G & \longrightarrow & 0 \\ & & \big\downarrow{=} & & \big\downarrow{f} & & \big\downarrow{=} & & \\ 0 & \longrightarrow & A & \longrightarrow & \hat{G}_2 & \longrightarrow & G & \longrightarrow & 0. \end{array}$$

Denote by $\operatorname{Ext}(G, A)$ and $Z\operatorname{Ext}(G, A)$ the set of equivalence classes of extensions (respectively, central extensions) of G by A, using the notion of equivalence above.

Let G be a group and A an abelian group (regarded as being equipped with the trivial G-action). A *group* 2*-cocycle on* G *with coefficients in* A is a function

$$c \colon G \times G \to A$$

satisfying the 2*-cocycle condition*: $c(x, y) - c(x, y \cdot z) + c(x \cdot y, z) - c(y, z) = 0$ for x, y, $z \in G$.

For two such cocycles c and $\tilde{c}$, a *coboundary* between them is a function $h \colon G \to A$ satisfying $\tilde{c}(x, y) = (c + dh)(x, y)$, where $dh(x, y) = h(x \cdot y) - h(x) - h(y)$. The *second group cohomology* $H^2(G, A) := \frac{\text{2-cocycles}}{\text{coboundaries}}$ is the set of equivalence classes of group 2-cocycles modulo co-boundaries. Actually, this is an abelian group under the operation $[c_1] + [c_2] := [c_1 + c_2]$, where $(c_1 + c_2)(x, y) = c_1(x, y) + c_2(x, y)$.

Theorem 8.5. *There is a natural bijection* $Z\operatorname{Ext}(G, A) \cong H^2(G, A)$.

Proof. The proof simplifies if we use *normalised 2-cocycles*, that is, a 2-cocycle $c \colon G \times G \to A$ satisfying $c(x, y) = 0$ whenever $x = 1$ or $y = 1$. As proven in the lecture, any 2-cocycle is cohomologous to a normalised 2-cocycle. So let $[c] \in H^2(G, A)$, where c is a normalised 2-cocycle. We can define a group $G \times_c A$ whose underlying set is the cartesian product $G \times A$, and whose group operation is $(g_1, a_1) \cdot (g_2, a_2) := (g_1 g_2, a_1 + a_2 + c(g_1, g_2))$. Then the assignment $[c] \mapsto (A \overset{i}{\rightarrowtail} G \times_c A \overset{p}{\twoheadrightarrow} G)$ is a well-defined group homomorphism.

Now let $A \rightarrowtail \hat{G} \overset{p}{\twoheadrightarrow} G$ be a central group extension. Let $s \colon G \to \hat{G}$ be a (set-theoretic) section of p. Then $c \colon G \times G \to A$, $c(g_1, g_2) := s(g_1)^{-1} s(g_2)^{-1} s(g_1 g_2)$ yields a normalised 2-cocycle. Furthermore, different choices of sections yield cohomologous cocycles. So we get a well-defined group homomorphism $[A \rightarrowtail \hat{G} \overset{p}{\twoheadrightarrow} G] \mapsto [c]$. It is easy to see now that the two constructions are inverse to each other. □

We now relate central extensions with right derived functors. Let $\mathbb{Z}[G]$ denote the group ring of G. This is the free abelian group with G as its basis, and whose ring multiplication is given by the group multiplication in G.

EXERCISE 8.6. Find an explicit relationship between a left $\mathbb{Z}[G]$-module M and a group homomorphism $G \to \mathsf{Aut}(M)$.

Let $\epsilon: Z[G] \to \mathbb{Z}$ be the surjection defined by $\sum_{g\in G} r_g g \mapsto \sum_{g\in G} r_g$. Here we view $\mathbb{Z}$ as a trivial $\mathbb{Z}[G]$-module (that is, we equip it with the group action $G \times \mathbb{Z} \to \mathbb{Z}$, $g \cdot n := n$). We would like the map ϵ to be part of a projective resolution $(P_n, \delta_n)_{n\in\mathbb{Z}}$ of $\mathbb{Z}$ by $\mathbb{Z}[G]$-modules. We define $P_n := F(U(G)^n)$, that is, the free $\mathbb{Z}[G]$-module over the set $U(G)^n$. Then with the differentials

$$\delta_n: P_n \to P_{n-1}, \quad \delta_n(g_1, \cdots, g_n) = \sum_{i=0}^{n} (-1)^i d_i,$$

$$d_i(g_1, \cdots, g_n) := \begin{cases} g_1(g_2, \cdots, g_n), & i = 0 \\ (g_1, \cdots, g_i g_{i+1}, \cdots, g_n), & \text{for } 1 \le i \le n-1 \\ (g_1, \cdots, g_{n-1}), & i = n \end{cases}$$

we get a chain complex of free $\mathbb{Z}[G]$-modules.

EXERCISE 8.7. Check that (P_n, δ_n) is indeed a chain complex.

In order to show that the complex defined above is a resolution, we need to show that the augmented complex

$$\cdots \to P_1 \to \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z}$$

is exact. This follows from the following:

LEMMA 8.8. *The maps* $s_i: \mathbb{Z}[G^i] \to \mathbb{Z}[G^{i+1}]$ *defined on basis elements by* $s(g_1, \cdots, g_i) := (1, g_1, \cdots, g_i)$ *define a contracting homotopy for the chain complex above.*

PROOF. Straightforward computation. □

Suppose A is an abelian group with a linear G-action. The group of *n-cochains* is defined as the set of functions $C^n(G, A) := \{f: G^n \to A\}$ with pointwise operations. Define $d_n: C^n(G, A) \to C^{n+1}(G, A)$,

$$d_n f(g_0, \cdots, g_n) := g_0 f(g_1, \cdots, g_n) + \sum_{i=1}^{n} (-1)^i f(g_1, \cdots, g_{i-1} g_i, \cdots, g_n) + (-1)^{n+1} f(g_0, \cdots, g_{n-1}),$$

The *n-th group cohomology* $H^n(G, A)$ *of* G *with coefficients in* A is defined as the cohomology of the above cochain complex. We can now finally relate group cohomology with the right derived functors of the Hom-functor $\mathrm{Hom}_{\mathbb{Z}[G]}(-, A): \mathsf{Mod}_{\mathbb{Z}[G]} \to \mathsf{Ab}$.

THEOREM 8.9. *We have* $\mathrm{Hom}(P_{n+1}, A) \cong C^n(G, A)$ - *implying that*

$$H^n(G, A) \cong \mathsf{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, A)$$

for all n.

PROOF. Define $\psi: \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^{i+1}], A) \to C^i(G, A)$,

$$\psi_i(\varphi)(g_1, \cdots, g_i) = \varphi(1, g_1, \cdots, g_1 \ldots g_i).$$

To see that this is injective, let $\psi_i(\varphi) = 0$. Then $\varphi(1, g_1, \cdots, g_1 \ldots g_i) = 0$ for all $g_i \in G$. Let $g_j = h_{j-1}^{-1} h_j$ for $1 \le j \le i$ and $h_0, \cdots, h_i \in G$. Then $\varphi(h_0, \cdots, h_i) = 0$, implying injectivity.

For surjectivity, if $f \in C^i(G,A)$, then defining $\varphi(h_0,\cdots,h_i) = h_0 f(h_0^{-1}h_1,\cdots,h_{i-1}^{-1}h_i)$, we can check that $\psi_i(\varphi) = f$. Finally, it is easy (but tedious) to check that the maps ψ are compatible with the differentials of the two chain complexes above - that is, they are chain maps. □

Remark 8.10. We have the following consequences:

- if $n = 0$, $H^0(G,A) \cong A^G := \{a \in A : ga = a, \forall g \in G\} = \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z},A)$;
- if $n = 1$, A a trivial G-module, $H^1(G,A) = \mathrm{Hom}(G,A)$;
- $n = 2$, $H^2(G,A)$ is extensions of G by A;
- finally, higher Ext groups measure G-invariants of injective approximations: if $A \to I_\bullet$ is an injective resolution, then we get a long exact sequence
$$0 \to I_0^G \to I_1^G \to I_2 \to \cdots,$$
using the computation for $n = 0$ and the long exact sequence result about right derived functors.

If we work over an arbitrary abelian category $\mathcal{C}$, we can similarly define equivalence classes of extensions
$$0 \to A \to \hat{G} \to G \to 0$$
of G by A. Denote these by $\mathrm{Ext}(G,A)$. We then have a similar result as in the case where $G = \mathbb{Z}$:

Theorem 8.11. *Let $\mathcal{C}$ be an abelian category with enough projectives, and let G and $A \in \mathcal{C}^0$. There is a natural bijection* $\mathrm{Ext}(G,A) \cong \mathrm{Ext}^1_{\mathbb{Z}[G]}(G,A)$.

8.2. Ext and Hochschild cohomology. In this section, we show that the right derived functors of the Hom-functor are related to another important geometric invariant, namely *Hochschild cohomology.* We provide the definition in the category Mod_R, but the definition makes sense in any symmetric monoidal abelian category.

Let R be commutative unital ring, A a unital R-algebra, and let M be a fixed unital A-bimodule. Using this as the basic setup, we would like to study derived functors of the functor 'Hom$(-,M)$'. The reason for the functor "Hom$(-,M)$" being in quotation marks is because it is the Hom-functor on the category of A-bimodules. If we want to recover the previous version of $\mathrm{Ext}^n(-,M)$, we need a left/right module structure on M over some R-algebra. This is done as follows: define the opposite algebra A^{op} as the algebra with the same underlying set as A, but with reversed multiplication $m_{A^{\mathrm{op}}}(a \otimes b) := m(b \otimes a)$, where $m: A \otimes A \to A$ is the original multiplication on A. Using this, we can define the *enveloping algebra* $A^e := A \otimes_R A^{\mathrm{op}}$.

Lemma 8.12. *An A-bimodule is equivalently an A^e-module. In particular, A viewed as an A-bimodule is an A^e-module.*

Proof. Given an A-bimodule M, we can give it the structure of a left A^e-module by defining $(a \otimes b) \cdot m := a \cdot m \cdot b$. Conversely, given an A^e-module structure on M, define $a \cdot m \cdot b := (a \otimes b) \cdot m$. □

Definition 8.13. The n-th *Hochschild cohomology of A with coefficients in M* is defined as $\mathrm{HH}^n(A,M) := \mathrm{Ext}^n_{A^e}(A,M)$. When $A = M$, we denote $\mathrm{HH}^n(A,A)$ simply by $\mathrm{HH}^n(A)$.

Unravelling what this means, the Hochschild cohomology of A can be computed by taking a projective resolution $P_\bullet \to A$ by (unital) A^e-modules, and then taking the cohomology of the cochain complex $\mathrm{Hom}_{A^e}(P_\bullet, M)$. Furthermore, since any two projective resolutions are quasi-isomorphic, the definition makes sense. So far, however, we have not found any explicit projective resolution.

PROPOSITION 8.14. *Let A be as above, and assume additionally that A is projective as an R-module. Define $P_n := A \otimes_R A^{\otimes_R n+1}$ and $b_n : P_n \to P_{n-1}$, $b_n := \sum_{i=0}^n (-1)^i d_i$, where*

$$d_i(a \otimes a_1 \otimes \cdots \otimes a_{n+1}) := \begin{cases} (aa_1 \otimes a_2 \otimes \cdots \otimes a_{n+1}) & i = 0 \\ (aa_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) & \text{for } 1 \le i \le n \end{cases}$$

Then (P_n, b_n) is a resolution of A by projective A^e-modules.

PROOF. Since A is projective as an R-module, so is $A \otimes_R A$. Similarly, $P_n = A \otimes_R A^{\otimes_R n} \otimes_R A$ is projective as an A^e-module for $n \ge 1$. To see that the complex $P_\bullet \to A \to 0$ is exact, we use the contracting homotopy

$$s_n : A^{\otimes_R n} \to A^{\otimes_R n+1}, \quad s_n(a_1 \otimes \cdots \otimes a_n) := (1 \otimes a_1 \otimes \cdots \otimes a_n)$$

as in Lemma 8.8. ☐

One can actually take the analogy with group cohomology further and ask whether Hochschild cohomology can be interpreted in terms of cochains, coboundaries and cocycles. This is indeed the case:

DEFINITION 8.15. Let A be a unital R-algebra, and let M be an A-bimodule. We define *Hochschild n-cochains* as the R-module $C^n(A, M)$ of R-linear maps $A^n \to M$. For an n-cochain $f : A^n \to M$, we define its *Hochschild coboundary* by

$$\delta_n(f)(a_0, \cdots, a_n) := a_0 f(a_1, \cdots, a_n) + \sum_{i=1}^n (-1)^i f(a_1, \cdots, a_{i-1} a_i, \cdots, a_n) + (-1)^{n+1} f(a_0, \cdots, a_{n-1}) a_n.$$

The map $\delta_n : C^n(A, M) \to C^{n+1}(A, M)$ is called the *Hochschild differential.*

By convention, we define $C^0(A, M) := M$ and $\delta_0(m) : A \to M$, $\delta_0(m)(a) := a \cdot m - m \cdot a$.

LEMMA 8.16. *Let A be a unital R-algebra and let M be an A-bimodule. Then $(C^n(A, M), \delta_n)$ is a chain complex of R-modules.*

PROPOSITION 8.17. *There is an isomorphism of R-modules* $\mathrm{Hom}_{A^e}(P_{n+1}, M) \cong C^n(A, M) = \mathrm{Hom}_R(A^{\otimes_R n}, M)$. *This induces an isomorphism in homology*

$$HH^n(A, M) \cong H^n((C^n(A, M), \delta_n))$$

for all n.

PROOF. In one direction, we define $\psi_n : \mathrm{Hom}_{A^e}(P_{n+1}, M) \to \mathrm{Hom}_R(A^{\otimes_R n}, M)$, $f \mapsto \psi(f)(a_1 \otimes \cdots \otimes a_n) := f(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1)$. In the other direction, given a Hochschild n-cochain $f : A^{\otimes_R n} \to M$, we can define $\tilde{f} : A \otimes_R A^{\otimes_R n} \otimes_R A \to M$ by $\tilde{f}(a_0 \otimes \cdots \otimes a_{n+1}) := a_0 \cdot f(a_1 \otimes \cdots \otimes a_n) \cdot a_{n+1}$. These two maps are inverse to each other and commute with the boundary maps of the Bar resolution and the Hochschild differentials. ☐

We now compute Hochschild cohomology in some lower dimensional cases.

EXAMPLE 8.18. For $n = 0$, $\mathrm{HH}^n(A, M) \cong M^A := \{m \in M : a \cdot m = m \cdot a, \forall a \in A\}$.

EXAMPLE 8.19. For $n = 1$, a 1-cocycle (i.e., $\ker(\delta_1)$) is an R-linear map $f\colon A \to M$ satisfying $f(a \cdot b) = af(b) + f(a)b$. Such an R-module map is called a *R-derivation.* Denote the module of R-derivations by $\mathsf{Der}(A, M)$. A 1-cochain $A \to M$ is a Hochschild coboundary if it is of the form $\mathsf{Ad}_m(a) = am - ma$. This is a derivation, and we call such special derivations *inner derivations* and denote them by $\mathsf{Inn}(A, M)$. Therefore,

$$\mathrm{HH}^1(A, M) = \frac{\mathsf{Der}(A, M)}{\mathsf{Inn}(A, M)}.$$

EXAMPLE 8.20. For $n = 2$, there is a similar relation with extensions as we previously had. An *abelian extension of A by M* is an R-split extension of the form

$$0 \to M \to E \to A \to 0$$

such that $M^2 = 0$. Denoting by $s\colon A \to E$ the R-linear splitting of the projection $E \to A$, we can define an A-bimodule structure on M as follows: $a{\cdot}m{\cdot}a' := s(a)ms(a')$ with product inside E. Two such extensions with M and A fixed are *equivalent* if there is a commuting diagram of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E_1 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow{f} & & \downarrow{=} & & \\ 0 & \longrightarrow & M & \longrightarrow & E_2 & \longrightarrow & A & \longrightarrow & 0, \end{array}$$

where $f\colon E_1 \to E_2$ is an R-algebra homomorphism. Denote by $\mathsf{Ext}(A, M)$ the set of equivalence classes of extensions of A by the bimodule M using the equivalence relation above.

We first see how such extensions arise from Hochschild 2-cocycles. Let $f\colon A \otimes_R A \to M$ be a 2-cocycle. As an R-module, define $E = A \oplus M$. The product rule is given by $(a_1, m_1) \cdot (a_2, m_2) = (a_1 a_2, a_1 m_2 + m_1 a_2 + f(a_1 \otimes a_2))$. As in the group cocycle case, the cocycle condition for f yields associativity of the product.

THEOREM 8.21. *Let A be a unital R-algebra and M an A-bimodule. There is a canonical bijection $\mathrm{HH}^2(A, M) \cong \mathsf{Ext}(A, M)$.*

PROOF. In class. □

Beyond lower dimensional Hochschild cohomology groups, the other computations of Hochschild cohomology are rather specialised. We will look at one of these examples - namely, $\mathrm{HH}^k(\mathrm{C}^\infty(M))$ - at a slightly later point.

8.3. Tor and torsion subgroups. Just as Ext studies extensions of a group, Tor studies the torsion subgroups of a group. Recall that given an $N \in \mathsf{Mod}_R$, we defined the n-th left derived functors $\mathsf{Tor}_n^R(-, N)$ of the right exact functor $- \otimes_R N$ as the composition

$$\mathsf{Mod}_R \to \mathsf{Der}(\mathsf{Mod}_R)^{\geq 0} \overset{-\otimes_R N}{\to} \mathsf{Der}(\mathsf{Mod}_R) \overset{H_n}{\to} \mathsf{Mod}_R.$$

REMARK 8.22. Often the functor above is interesting in its own right even before taking homology. It defines the *derived tensor product*: if M and N are two R-modules, we define

$$M \otimes_R^{\mathbb{L}} N := P(M) \otimes_R N \cong M \otimes_R Q(N) \cong P(M) \otimes_R Q(N) \in \mathsf{Der}(\mathsf{Mod}_R),$$

where P and Q are the projective resolution functors.

Let A be an abelian group and let $\mathbb{Z}/n$ denote the cyclic group for $n \in \mathbb{N}$. The subgroup of A defined by $A_n := \{a \in A : na = 0\}$ is called the *torsion subgroup* of A.

PROPOSITION 8.23. *We have* $\mathsf{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, A) \cong A_n$ *when* $n \geq 1$ *and* $\mathsf{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, A) \cong 0$.

PROOF. Consider the projective resolution

$$0 \to \mathbb{Z} \xrightarrow{n\cdot} \mathbb{Z} \xrightarrow{\bmod n} \mathbb{Z}/n \to 0$$

of $\mathbb{Z}/n$. Tensoring with A, the complex $0 \to \mathbb{Z} \otimes_{\mathbb{Z}} A \xrightarrow{n \otimes 1_A} \mathbb{Z} \otimes_{\mathbb{Z}} A$ computes the i-th left derived functors. In particular, $\mathsf{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, A) \cong \ker(n \otimes 1_A) \cong A_n$. For the second claim, we use that $\mathbb{Z}$ is already free. □

PROPOSITION 8.24. *Let* $N \in \mathsf{Mod}_R$. *Then* $\mathsf{Tor}_n^R(-, N)$ *preserves direct sums.*

PROOF. Before taking homology, we have the functor $- \otimes_R^{\mathbb{L}} N$, which preserves direct sums. Finally, homology preserves direct sums as it is a cokernel. □

PROPOSITION 8.25. *Let A be a finite abelian group, and let B be any abelian group. Then* $\mathsf{Tor}_1^{\mathbb{Z}}(A, B)$ *is a direct sum of torsion-subgroups of A.*

PROOF. Any finite abelian group is a direct sum of cyclic groups by the Fundamental Theorem of finite abelian groups. So we have $A \cong \bigoplus_k \mathbb{Z}/n_k$ for some k's. Now use Propositions 8.23 and 8.24. □

REMARK 8.26. Actually, with some more work one can show that if A is any abelian group, then $\mathsf{Tor}_1^R(A, B)$ is a filtered colimit of direct sums of torsion-subgroups of either A or B. That is, the above result holds even without the finiteness of A.

We end this subsection by showing that just as $\mathsf{Ext}^n(-, N)$ measures the failure of projectivity of N, $\mathsf{Tor}_n(-, N)$ measures the obstruction to flatness:

DEFINITION 8.27. An R-module N is called *flat* if the tensor product functor

$$- \otimes_R N : \mathsf{Mod}_R \to \mathsf{Mod}_R$$

(or $N \otimes_R -$) is a left exact (and hence exact) functor.

LEMMA 8.28. *An R-module N is flat if and only if* $\mathsf{Tor}_i^R(-, N)(M) \cong 0$ *for all* $M \in \mathsf{Mod}_R$ *and* $i \geq 1$.

PROOF. Flatness of N is equivalent to $- \otimes_R N$ being exact. Consequently, if $P_\bullet \to M \to 0$ is any projective resolution of M by R-modules, then $P_\bullet \otimes_R N$ is exact. □

8.4. Tor and Hochschild homology. In this section, we define the Hochschild homology of a unital associative algebra A over a unital commutative ring R, with coefficients in an A-bimodule M. Again, it is possible to define Hochschild homology using the same approach as we point out below in the generality of any symmetric monoidal category with some mild assumptions that provide the existence of left derived functors.

The chain complex of R-modules that computes Hochschild homology is given in degree n by $C_n(A, M) := M \otimes_R A^{\otimes_R n}$, and with differentials $b_n : C_n(A, M) \to$

$C_{n-1}(A,M)$, $b_n := \sum_{i=0}^n (-1)^i d_i$, where

$$d_i(m\otimes a_1\otimes\cdots\otimes a_n) := \begin{cases} (ma_1\otimes a_2\otimes\cdots\otimes a_n), & i=0 \\ (m\otimes a_1\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_n) & \text{for } 1\le i\le n-1 \\ (a_nm\otimes a_1\otimes\cdots\otimes a_{n-1}), & i=n. \end{cases}$$

Definition 8.29. The *Hochschild homology* of A with coefficients in M is the homology of the chain complex

$$\cdots \to C_n(A,M) \xrightarrow{b_n} C_{n-1}(A,M) \to \cdots \to C_1(A,M) \to M,$$

denoted by $\mathrm{HH}_n(A,M)$. In the specific case where $M=A$, we denote the resulting complex by $C_n(A,A)$, and call its homology $\mathrm{HH}_n(A)$ the *Hochschild homology of* A.

To relate Hochschild homology with left derived functors, we fix an A-bimodule (or equivalently, an A^e-module) M. Assume A is a projective R-module. We can use the bar resolution $P_\bullet \to A \to 0$ of A by projective A^e-modules to compute the left derived functors of $M\otimes_{A^e} - : \mathsf{Mod}_{A^e} \to \mathsf{Mod}_R$.

Proposition 8.30. *With A, R and M as above, we have* $\mathrm{HH}_n(A,M) \cong \mathsf{Tor}_n^{A^e}(M,A)$ *for all n.*

Proof. It is easy to see that $M\otimes_{A^e} A\otimes_R A^{\otimes_R n}\otimes_R A \cong M\otimes_{A^e} A\otimes_R A^{\mathrm{op}}\otimes_R A^{\otimes_R n} \cong M\otimes_R A^{\otimes_R n}$. □

As already mentioned, the complex computing Hochschild homology is model for the derived tensor product: that is, we have

$$C_\bullet(A,M) \cong M\otimes^{\mathbb{L}}_{A^e} A \cong M\otimes_{A^e} P_\bullet(A),$$

where $P_\bullet$ is any projective bimodule (or A^e-module) resolution of A. In particular, if $M=A$, we have $C_\bullet(A) \cong A\otimes^{\mathbb{L}}_{A^e} A$. This has an interesting meaning if A is commutative:

Corollary 8.31. *Let $X=\mathrm{Spec}(A)$ be an affine scheme over a field of characteristic zero. Then the global sections of*

$$X\times_{X\times X} X \cong \mathrm{Spec}(A\otimes^{\mathbb{L}}_{A\otimes_R A} A)$$

computes the Hochschild homology of A.

In the corollary above, Spec of the simplicial, commutative R-algebra $A\otimes^{\mathbb{L}}_{A\otimes_R A} A$ is simply an object in the dual category $\mathsf{sCAlg}_R^{\mathrm{op}}$ of the category of simplicial commutative R-algebras. It is in the sense above that the Hochschild 'algebra' models the space of (derived) self-intersections of a space with itself. This is treated in the subject of *derived algebraic geometry.*

Let us first see a few basic computations of Hochschild homology:

Example 8.32. $\mathrm{HH}_0(A,M) \cong M_A := M/\{am-ma : m\in M, a\in A\}$.

Example 8.33. For any associative R-algebra A, $HH_0(A,A) = A/[A,A]$. If A is commutative, this means that $HH_0(A) = A$.

Example 8.34. When $A=R$, then $HH_0(A)=R$ and $HH_n(A)=0$ for $n\ge 1$.

Hochschild homology and differential forms. For a commutative R-algebra A, we have already seen that $HH_0(A) = A$. When $n = 1$, b_1 vanishes, so that $\mathrm{HH}_1(A) = A \otimes_R A / \langle \{ab \otimes c - a \otimes bc + ca \otimes b : a, b, c \in A\} \rangle$. The relations that we divide out from $A \otimes_R A$ to get the first Hochschild homology resembles the Lebnitz rule from differential calculus. This is made precise by the following:

Definition 8.35. The A-module of *Kähler differentials* $\Omega^1_{A/R}$ is defined as the A-module generated by the symbols $a\mathrm{d}(b)$ subject to the relations: $\mathrm{d}(r) = 0$, $\mathrm{d}(a \cdot b) = a\mathrm{d}(b) + \mathrm{d}(a)b$, $\mathrm{d}(a + b) = \mathrm{d}(a) + \mathrm{d}(b)$, where $a, b \in A$ and $r \in R$

It is now easy to see that $\mathrm{HH}_1(A) \cong \Omega^1_{A/R}$, via $a \otimes b \mapsto a\mathrm{d}(b)$. We can now ask whether the higher Hochschild homology groups of a commutative R-algebra are in some sense related to *higher Kähler differentials*, defined by taking exterior powers of $\Omega^1_{A/R}$

$$\Omega^n_{A/R} := \bigwedge_A^n \Omega^1_{A/R}.$$

As an A-module, $\Omega^n_{A/R}$ is spanned by elements of the form $a_0\mathrm{d}(a_1) \wedge \cdots \wedge \mathrm{d}(a_n)$. Now there is an explicit, canonical map $\epsilon_n : \Omega^n_{A/R} \to \mathrm{HH}_n(A)$ given by

$$a_0\mathrm{d}(a_1) \wedge \cdots \wedge \mathrm{d}(a_n) \mapsto \sum_{\sigma \in S_n} (-1)^{\mathrm{sgn}(\sigma)} a_0 \otimes a_{\sigma(1)} \otimes \cdots a_{\sigma(n)},$$

called the *anti-symmetrisation map.* We would like to show that this is an isomorphism for a suitable class of algebras, called *smooth algebras.*

Definition 8.36. A commutative R-algebra A is called *smooth* if for any extension

$$N \rightarrowtail E \twoheadrightarrow B,$$

of commutative R-algebras, where $N^2 = 0$, and for any R-algebra homomorphism $A \to B$, there is a lifting $A \to E$.

Note that the lifting $A \to E$ above need not be unique. When such a lifting is unique, we call A an *étale* over R.

Example 8.37. For any set S, the free R-algebra $R[S]$ over S is smooth. In particular, polynomial algebras $R[x_1, \cdots, x_n]$ are smooth.

The main source of examples comes from the following:

Example 8.38. For any finitely generated R-algebra A, $\mathrm{Spec}(A)$ is a smooth scheme over R if and only if A is a smooth R-algebra.

Finally, we state the relationship between Hochschild homology and de Rham cohomology for smooth commutative algebras.

Theorem 8.39 (Hochschild-Kostant-Rosenberg, 1962). *Let A be a smooth R-algebra. Then $\mathrm{HH}_*(A/R) \cong \Omega^*_{A/R}$ as A-modules for each $* \geq 0$.*

Proof. We will only sketch the proof, leaving it only for those with some background in algebraic geometry to fill in details. The case $* = 1$ has been checked by hand. By functoriality of the exterior product, we need to show that $\bigwedge_A^* \mathrm{HH}_1(A/R) \cong \mathrm{HH}_*(A/R)$. Equivalently, we need to show that $\bigwedge_A^* \mathsf{Tor}_1^{A \otimes_R A}(A, A) \cong \mathsf{Tor}_*^{A \otimes_R A}(A, A)$. Now $\mathsf{Tor}_1^{A \otimes_R A}(A, A) = I/I^2$, where $I = \ker(A \otimes A \to A)$. If A is a polynomial

R-algebra, the Tor-claim above can be checked by hand. We then use that $\mathrm{HH}_*(A/R) \otimes_A A[f^{-1}] = \mathrm{HH}_*(A[f^{-1}]/R)$, $f \in A$, to reduce to the case where A' étale over A - this follows from the claim that $\mathrm{HH}(A/R) \otimes_A A' \cong \mathrm{HH}(A'/R)$ if A' is étale over a flat commutative R-algebra A. The proof is complete by reducing from the case where A is smooth to the case where A is étale over a polynomial algebra - this can always be done - and then using the result for polynomial algebras. □

CHAPTER 3

Cyclic and periodic cyclic homology

In Chapter 2, Section 8, we defined Hochschild homology and cohomology as the derivation of the tensor product and internal Hom-functors. The most important computation that we saw, namely, the Hochschild-Kostant-Rosenberg Theorem (see 8.39) identified the Hochschild homology of a smooth affine scheme with its modules of Kähler differentials. That is, Hochschild homology generalises Kähler differentials to the setting of associative algebras. In this chapter, we shall define invariants of associative algebras that specialise to de Rham cohomology when we restrict to smooth commutative algebras over a field of positive characteristic. Along the way, we will redefine Hochschild homology as the homology groups of "noncommutative differential forms", thereby making precise what it means for Hochschild homology to generalise differential forms to the noncommutative setting.

1. Noncommutative differential geometry

In this section we study some analogies between differential forms on manifolds and formal derivations on associative algebras. To simplify the approach, we work over the abelian category of $\mathbb{C}$-vector spaces, and algebras refer to algebra objects in this category. The core definitions and arguments, however, remain the same over any $\mathbb{Q}$-linear symmetric monoidal category and monoids in such categories.

Recall that in Example 8.19, we defined linear maps $A \to M$ from a unital algebra into an A-bimodule M satisfying $f(ab) = af(b) + f(a)b$. These maps are called derivations, and the set of all such derivations is denoted $\mathsf{Der}(A, M)$. Consider the bimodule $\Omega^1(A) := \ker(A \otimes A \overset{\text{mult}}{\to} A)$ obtained by taking the kernel of the multiplication of A. Define the map

$$\mathrm{d}\colon A \to \Omega^1(A), a \mapsto 1 \otimes a - a \otimes 1;$$

this is an A-bimodule map.

Exercise 1.1. Show that $\Omega^1(A)$ is spanned by elements of the form $a\mathrm{d}(b) := a\mathrm{d}(b)$ for $a \in A$ and $b \in A/\mathbb{C}\cdot 1$. Deduce that $\Omega^1(A) \cong A \otimes A/\mathbb{C}\cdot 1$ and as a consequence, $\Omega^1(A)$ is a free left A-module.

We will identify the elements of $\Omega^1(A)$ with terms of the form $a\mathrm{d}b$ as in Exercise 1.1. The right module structure on $\Omega^1(A)$ is given explicitly by the Leibniz rule $(a\mathrm{d}b) \cdot c := a\mathrm{d}(bc) - ab\mathrm{d}(c)$.

Lemma 1.2. *Let A be a unital algebra, M a unital A-bimodule, and $D\colon A \to M$ a derivation. Then there exists a unique A-bimodule map $\Omega^1(A) \to M$ factorising D. That is,* $\mathsf{Der}(A, M) \cong \mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(\Omega^1(A), M)$.

Proof. The map is given explicitly by $f(a\mathrm{d}b) := aD(b)$. Fill in the details. $\square$

In the language of category theory, what we have shown is that the functor $\mathsf{Der}(A,-)\colon \mathsf{Mod}_{A\otimes A^{\mathrm{op}}} \to \mathsf{Set}$ is representable; it is represented by the A-bimodule $\Omega^1(A)$. We call $\Omega^1(A)$ the A-bimodule of *noncommutative differential forms*. The name comes from the fact that it plays a role similar to Kähler differentials in algebraic geometry, where if A is a unital commutative algebra, and M is an A-module, then derivations $A \to M$ are in bijection with A-linear maps $\Omega^1_A \to M$.

Exercise 1.3. Let A be a commutative unital algebra. Show that we have an A-module isomorphism $\Omega^1_A \cong \Omega^1(A)/\Omega^1(A)^2$.

Now suppose A is non-unital, we define $A^+ := A \oplus \mathbb{C}$ with the multiplication $(a,\lambda)\cdot(b,\mu) := (ab + \lambda\cdot b + a\cdot\mu, \lambda\mu)$. This is a unital algebra with unit given by $(0,1)$. In fact, it is the universal way to turn a non-unital algebra into a unital algebra: any algebra homomorphism $A \to B$ into a unital algebra extends to a unital homomorphism $A^+ \to B$.

Exercise 1.4. Show that any A-bimodule derivation $A \to M$ on a unital algebra extends to an A^+-bimodule derivation into a unital A-bimodule M.

The universal derivation on the unitalisation is defined by the obvious composition $A \to A^+ \to \Omega^1(A^+)$. So for a nonunital algebra A, we could redefine $\Omega^1(A) := \Omega^1(A^+)$. This definition is somewhat confusing notationally if A is a unital algebra, as in that case $\Omega^1(A)$ is *not* isomorphic to $\Omega^1(A^+)$. The only way to remedy this is by choosing a different notation for the unital and the nonunital versions of noncommutative differential forms. However, we shall sweep this under the rug as the algebras we shall concern ourselves with in this course will generally be unital. Also, most results that are true about $\Omega^1(A)$ continue to remain true about $\Omega^1(A^+)$, so we shall stick to the former.

Just as higher Kähler differentials Ω^n_A are defined by taking exterior powers $\wedge_{i=1}^n \Omega^1_A$, we define *noncommutative differential n-forms* by taking n-fold tensor products

$$\Omega^n(A) := \Omega^1(A) \otimes_A \cdots \otimes_A \Omega^1(A)$$

of $\Omega^1(A)$. Now since $\Omega^1(A) \cong A \otimes \bar{A}$ as a left A-module, where $\bar{A} = A/\mathbb{C}$, we can explicitly describe the higher n-forms as follows:

Lemma 1.5. *We have an isomorphism of left A-modules $\Omega^n(A) \cong A \otimes \bar{A}^{\otimes n}$ for all $n \geq 1$, such that the right A-module structure becomes*

$$(a_0 \otimes a_2 \otimes \cdots \otimes a_n)\cdot b := a_0 \otimes \cdots \otimes (a_n \cdot b) - a_0 \otimes \cdots \otimes (a_{n-1}a_n) \otimes b + \cdots \pm (-1)^n a_0 a_1 \otimes \cdots \otimes b.$$

Proof. Use induction and Exercise 1.1. □

We shall use the isomorphism of Lemma 1.5 to identify pure tensors $a_0 \otimes \cdots \otimes a_n \leftrightarrow a_0 \mathrm{d}a_1 \mathrm{d}a_2 \cdots \mathrm{d}a_n$. Now just as we classified maps $\Omega^1(A) \to M$ into an A-bimodule M as bimodule derivations $A \to M$, the following result is a universal property for noncommutative differential n-forms:

Proposition 1.6. *The map $d^{\cup n}\colon A^n \to \Omega^n(A)$ defined by $(a_1,\ldots,a_n) \mapsto \mathrm{d}a_1 \ldots \mathrm{d}a_n$ is the universal normalised Hochschild n-cocycle with values in an A-bimodule. In other words, any other normalised Hochschild n-cocycle $f\colon A^n \to M$ into an A-bimodule M factorises uniquely through an A-bimodule map $f^*\colon \Omega^n(A) \to M$.*

Proof. Since $\Omega^n(A) \cong A \otimes \bar{A}^n$ by Lemma 1.5, it is a free left A-module. Consequently, there is a unique left A-module map $\Omega^n(A) \to M$ satisfying $f^* \circ d^{\cup n} = f$. It remains to check that this map is a right module map if f is a cocycle. This is left as an exercise. □

Continuing our analogy with differential geometry, we know that the complex computing de Rham cohomology is a prototypical example of a *commutative differential graded algebra* in the following sense:

Definition 1.7. A *graded algebra* A is an algebra together with a decomposition $A = \bigoplus_{n=0}^{\infty} A_n$ such that $A_n A_m \subseteq A_{n+m}$. A *graded derivation* is a graded algebra A together with a linear map $d\colon A \to A$ such that $d(A_n) \subseteq A_{n+1}$, and satisfying the *graded Leibniz rule*

$$d(a \cdot b) = d(a) \cdot b + (-1)^n a d(b)$$

for $a \in A_n$ and $b \in A_m$. A *differential graded algebra* is a graded algebra with a graded derivation that satisfies $d \circ d = 0$. A *commutative differential graded algebra* is a differential graded algebra satisfying $a \cdot b = (-1)^{nm} b \cdot a$, where $a \in A_n$ and $b \in A_m$.

Exercise 1.8. Show that commutative differential graded algebras are precisely the commutative algebra objects in the category of chain complexes with entries in the category of complex vector spaces. See Section 4 for the definition of the tensor product of two complexes.

Exercise 1.9. Show that the de Rham complex on a manifold with exterior product is a commutative differential graded algebra.

In light of Exercise 1.9, it is reasonable to expect that our noncommutative differential forms also assemble into a (noncommutative) differential graded algebra. This is indeed the case: define $\Omega(A) := \bigoplus_{n=0}^{\infty} \Omega^n(A)$ with the differential

$$d\colon \Omega(A) \to \Omega(A), \quad a_0 \mathrm{d}a_1 \cdots \mathrm{d}a_n \mapsto \mathrm{d}a_0 \mathrm{d}a_1 \cdots \mathrm{d}a_n$$

for each $n \in \mathbb{N}$, where it is understood that $\mathrm{d}(1) = 0$.

Exercise 1.10. Verify that $(\Omega(A), d)$ is a differential graded algebra.

Actually, way more is true for the de Rham complex $(\Omega^\bullet, d)$. Viewed as a commutative differential graded algebra, it is initial in the sense that for any other commutative differential algebra $(A^\bullet, \eta)$, any algebra homomorphism $\Omega^0 \to A^0$ into the degree zero part of A extends uniquely to a differential graded algebra homomorphism $\Omega^\bullet \to A$. The same holds for noncommutative differential forms:

Proposition 1.11. *The differential graded algebra* $(\Omega(A), d)$ *is initial in the sense that any for any unital differential graded algebra* (B, η), *differential graded algebra homomorphisms* $(\Omega(A), d) \to (B, \eta)$ *are in bijection with unital algebra homomorphisms* $A \to B_0$.

Proof. If $f\colon A \to B$ is an algebra homomorphism, then we define maps $f_n\colon \Omega^n(A) \to B_n$ by mapping $a_0 \mathrm{d}a_1 \cdots \mathrm{d}a_n \mapsto f(a_n)\eta(f(a_1)) \cdots \eta(f(a_n))$. This clearly satisfies $f_{n+1} \circ \mathrm{d} = \eta f_n$ for all $n \in \mathbb{N}$ - that is, it is a chain map. It is multiplicative because η satisfies the graded Leibniz rule, which dictates the multiplication of differential forms. And the maps f_n are the only maps that are compatible with the multiplication and the differentials. □

We have now seemingly built a differential graded algebra that is playing a role identical to the chain complex computing de Rham cohomology. It is therefore tempting to define "noncommutative de Rham cohomology" as the homology of this chain complex. Let us see what happens when we try to do that: consider first the splitting $A \cong \mathbb{C} \oplus \bar{A}$. Define

$$s_n\colon \Omega^{n+1}(A) \to \Omega^n(A), \quad \mathrm{d}a_0 \dots \mathrm{d}a_n := a_0\mathrm{d}a_1 \dots \mathrm{d}a_n, \text{ and } s_n(a_0\mathrm{d}a_1 \dots \mathrm{d}a_n) = 0$$

for $a_i \in \bar{A}$. This satisfies $\mathrm{d} \circ s + s \circ \mathrm{d} = \mathrm{id}$ on $\Omega^n(A)$ for $n \geq 1$, while $s \circ \mathrm{d} = \mathrm{d} \circ s + s \circ \mathrm{d}$ on $\Omega^0(A) \cong A$ is the projection onto $\bar{A}$. What we have therefore built is a contracting homotopy for the complex $(\Omega(A), \mathrm{d})$, that is,

Lemma 1.12. *For a unital algebra A, we have $H^0(\Omega(A), \mathrm{d}) = \mathbb{C}$ and for each $n \geq 1$, $H^n(\Omega(A), \mathrm{d}) = 0$.*

As a consequence, the most obvious way to generalise differential forms and de Rham cohomology to the setting of associative algebras does not work. The issue is that noncommutative differential forms do not specialise to ordinary differential forms in the commutative case, that is, when the algebra is $A = \mathrm{C}^\infty(M)$. They are, however, related in the following sense:

Lemma 1.13. *Let M be a smooth manifold and let $A = \mathrm{C}^\infty(M)$. Then the differential 1-forms $\Omega^1(M)$ are isomorphic as an A-module to the commutator quotient of $\Omega^1(A)/[A, \Omega^1(A)]$.*

Proof. The latter A-module has the same universal property as $\Omega^1(M)$. □

Lemma 1.13 suggests that taking commutator quotients of the complex $(\Omega(A), \mathrm{d})$ might be a better way to link noncommutative forms with the de Rham complex. This is quite close to being the case, as we shall now see. Consider the bar resolution $(P_\bullet, b)$ from Proposition 8.14. This is an explicit resolution of A by projective A-bimodules that is used to compute Hochschild homology. Remarkably, we have the following relationship with noncommutative differential forms:

Proposition 1.14. *The commutator quotient complex $(P_\bullet/[P_\bullet, A], \tilde{b'})$ of the bar resolution is isomorphic to $(\Omega^n(A), b)$, where*

$$b(a_0\mathrm{d}a_1 \dots \mathrm{d}a_{n+1}) = (-1)^n[a_0\mathrm{d}a_1 \dots \mathrm{d}a_n, a_{n+1}].$$

In general, if $Q_\bullet \to A$ is any resolution of A by projective A-bimodules, then the homology of the commutator quotient $Q_\bullet/[Q_\bullet, A]$ with the induced differential is isomorphic to the homology of $(\Omega^\bullet(A), b)$.

Proof. **Prove it.** □

Using Proposition 1.14, we can redefine the Hochschild homology of a unital algebra.

Definition 1.15. The *Hochschild homology of a unital algebra A* is defined as the homology of the chain complex $(\Omega^n(A), b)$. We shall call the maps $b_n\colon \Omega^{n+1}(A) \to \Omega^n(A)$ for $n \in \mathbb{N}$, the *Hochschild boundary maps.*

Notice that although we have found a chain complex of noncommutative differential forms computing Hochschild homology, which in turn yields usual differential forms in the case of a smooth manifold or an smooth affine scheme, the maps b_n are *degree-decreasing.* On the other hand, the maps in de Rham cohomology

$d\colon \Omega^n_A \to \Omega^{n+1}_A$ are *degree increasing.* This creates the need for another important character in our story - a degree increasing differential on noncommutative differential forms, which "specialises" to the de Rham differential.

2. Operators in cyclic homology

So far we have introduced two operators on the bimodule of noncommutative differential forms on an associative, unital algebra A - the universal derivation $\mathrm{d}\colon \Omega^n(A) \to \Omega^{n+1}(A)$ and the Hochschild boundary map $b\colon \Omega^n(A) \to \Omega^{n-1}(A)$. These are both differentials, that is, they satisfy $\mathrm{d}^2 = 0 = b^2$. To measure the extent to which they anti-commute, we define the *Karoubi operator*

$$\kappa := 1 - [\mathrm{d}, b] = 1 - (\mathrm{d}b + b\mathrm{d})\colon \Omega(A) \to \Omega(A),$$

which in terms of elements is given by

$$\kappa(\omega \mathrm{d}x) = \omega \mathrm{d}x - (-1)^n d([x,\omega]) - (-1)^{n+1}[x, \mathrm{d}\omega]$$
$$= \omega \mathrm{d}x - (-1)^n [\mathrm{d}x, \omega] = (-1)^{n-1}\mathrm{d}x\omega,$$

for $\omega \in \Omega^{n-1}(A)$ and $x \in A$. Note that since b is a degree -1 operator and d is a degree +1 operator, the Karoubi operator is degree 0.

We now introduce a degree +1-operator $B\colon \Omega(A) \to \Omega(A)$, which will generalise the de Rham differential. It is defined as

$$B = \sum_{j=0}^{n} \kappa^j \circ d, \quad x_0 \mathrm{d}x_1 \dots \mathrm{d}x_n \mapsto \sum_{j=0}^{n} (-1)^{jn} \mathrm{d}x_j \dots \mathrm{d}x_n \mathrm{d}x_0 \dots \mathrm{d}x_{j-1},$$

for $x_j \in A$. This is called the *Connes operator*.

THEOREM 2.1. *Let A be a unital, associative algebra. The operators* d*, b, κ and B satisfy the following relations on $\Omega^n(A)$:*

(1) $\kappa \circ \mathrm{d} = \mathrm{d} \circ \kappa$;
(2) $\kappa \circ b = b \circ \kappa$;
(3) $\mathrm{d} \circ \kappa^{n+1} = \kappa^{n+1} \circ \mathrm{d} = \mathrm{d}$;
(4) $b \circ \kappa^n = \kappa^n \circ b = b$;
(5) $\kappa \circ B = B \circ \kappa = B$;
(6) $\kappa^n = 1 + b \circ \kappa^n \circ \mathrm{d}$;
(7) $b \circ \mathrm{d} = \kappa^{n+1} - \kappa$;
(8) $\mathrm{d} \circ b = 1 - \kappa^{n+1}$;
(9) $(\kappa^n - 1)(\kappa^{n+1} - 1) = 0$;
(10) $b \circ B = -B \circ b = \frac{(\kappa^n - 1)\cdot(\kappa^{n+1}-1)}{1-\kappa}$, *where the right hand side is κ plugged into the polynomial* $\frac{(1-x^n)\cdot(1-x^{n+1})}{1-x}$.

PROOF. **Do it.** □

We now provide some conceptual explanation for the plethora of identities introduced above. Let D be the subring of operators on $\Omega(A)$ generated by b and d. The homogeneous operators in D are generated by linear combinations of $(\mathrm{d}b)^n$ and $(b\mathrm{d})^n$ for $n \in \mathbb{N}$ because $\mathrm{d}^2 = 0 = b^2$. Identities (7) and (8) in Theorem 2.1 show that this ring is generated by κ. Identities (1) and (2) show that this subring is central in D. Restricting to $\Omega^n(A)$, this subring is isomorphic to $\mathbb{Z}[\kappa]/(\kappa^n - 1)(\kappa^{n+1} - 1)$ by identity (9). In other words, (9) describes the minimal polynomial of $\kappa_{|\Omega^n(A)}$.

There is more happening here. Identity (10) could be used to deduce that $b \circ B + B \circ b = 0$. That is, b and B anti-commute on $\Omega^n(A)$. This implies that the subspaces $\ker(b)$ and $\mathrm{im}(b)$ are B-invariant, so that the Connes operator B descends to a map

$$B_n^*: \mathrm{HH}_n(A) \to \mathrm{HH}_{n+1}(A),$$

for each $n \in \mathbb{N}$. Its square is zero as $B^2 = 0$, so that $(\mathrm{HH}_n(A), B_n^*)$ is a cochain complex.

Exercise 2.2. Show that if A is a smooth, commutative algebra, then the homology of the cochain complex $(\mathrm{HH}_n(A), B_n^*)_{n \in \mathbb{N}}$ is the algebraic de Rham cohomology of A.

2.1. Computations for the polynomial and the Weyl algebra. Before moving on further, let us compute the Hochschild homology of the polynomial algebra $\mathbb{C}[x, y]$ and the Weyl algebra $W[p, q]$, which is the universal algebra generated by the *canonical commutation relation*

$$[p, q] = pq - qp = i\hbar,$$

where $\hbar$ of course denotes the Planck's constant. The Weyl algebra is an example of a *deformation quantisation*, which is a topic of interest to mathematical physicists.

Exercise 2.3. Show that

$$H_*(\mathrm{HH}_n(\mathbb{C}[x, y]), B_n^*) = \begin{cases} \mathbb{C} & \text{if } * = 0 \\ 0 & * > 0. \end{cases}$$

We now compute the homology of the complex $(\mathrm{HH}_n(W[p, q]), B_n^*)$ for the Weyl algebra. Before this, we need to compute the Hochschild homology of the Weyl algebra. Let $\mathrm{ad}_p(x) := [p, x] = px - xp$ (and similarly, $\mathrm{ad}_q(x) = [q, x]$) for $x \in W[p, q]$.

Exercise 2.4. Show that

$$0 \to W[p, q] \overset{(\mathrm{ad}_p, -\mathrm{ad}_q)}{\to} W[p, q] \oplus W[p, q] \overset{(\mathrm{ad}_p, \mathrm{ad}_q)}{\to} W[p, q] \to 0$$

is a projective bimodule resolution of $W[p, q]$. Use it to compute the Hochschild homology of $W[p, q]$. Deduce that

$$H_*(\mathrm{HH}_n(W[p, q]), B_n^*) = \begin{cases} \mathbb{C} & \text{if } * = 2 \\ 0 & \text{else.} \end{cases}$$

Exercises 2.3 and 3.8 show that $\mathbb{C}[x, y]$ and $W[p, q]$ have different homologies, despite being sufficiently close to each other (by way of a deformation quantisation). This forces us to conclude that the Hochschild complex with the Connes operator still lacks several properties that one might hope for. This takes us to cyclic and periodic cyclic homology.

3. Cyclic and periodic cyclic homology

Consider again the example that we ended the previous section with, namely, the Hochschild-Connes homology of the Weyl algebra and the polynomial algebra. The homology groups were not isomorphic, but were pretty close to being so. More concretely, we have

$$\prod_{k\in\mathbb{Z}} H_{n+2k}(\mathrm{HH}_*(\mathbb{C}[x,y]), B_*) \cong \prod_{k\in\mathbb{Z}} H_{n+2k}(\mathrm{HH}_*(W[p,q]), B_*),$$

which is just $\mathbb{C}$ when $n = 0$, and 0 for $n = 1$. This is probably an overkill, but it suggests what we could do with the complex $(\mathrm{HH}_*(A), B_*)$ to make it satisfy desirable properties - namely, *product periodification.* In what follows, we make this precise.

Recall the relations $b^2 = 0$, $B^2 = 0$ and $bB + Bb = 0$ between the Hochschild operator $b\colon \Omega^n(A) \to \Omega^{n-1}(A)$ and the Connes operator $B\colon \Omega^n(A) \to \Omega^{n+1}$ on the differential graded algebra $\Omega(A)$. In other words, we have two complexes $(\Omega(A), b)$ and $(\Omega(A), B)$, where the two boundary maps anti-commute. Such objects are called *bicomplexes*:

Definition 3.1. A *bicomplex* over the category of complex vector spaces is a $\mathbb{Z}\times\mathbb{Z}$-graded vector space, or equivalently, a family $(C_{n,m})_{(n,m)\in\mathbb{Z}^2}$ of vector spaces with endomorphisms δ_h and δ_v of degree $(-1,0)$ and $(0,-1)$, respectively, such that $\delta_h^2 = 0$ and $\delta_v^2 = 0$ and $\delta_h \circ \delta_v + \delta_v \circ \delta_h = 0$. The maps δ_h and δ_v are called the horizontal and vertical maps of the bicomplex $(C_{n,m})$.

The relations imply that for each fixed $n = N$ and $m = M$, the vector spaces $((C_{N,m})_{m\in\mathbb{Z}}, \delta_v)$ and $((C_{n,M})_{m\in\mathbb{Z}}, \delta_h)$ define chain complexes of vector spaces. Their homologies are called the *vertical homology* and the *horizontal homology* of the bicomplex, respectively. We however want a chain complex that incorporates both the vertical as well as the horizontal maps of a bicomplex. There are two ways of doing this:

Definition 3.2. The *direct product totalisation* of a bicomplex $C = ((C_{n,m}), \delta_h, \delta_v)$ is defined as the chain complex whose k-th term is given by $D_k = \prod_{n+m=k} C_{n,m}$ with the differential $\delta = \delta_v + \delta_h$. Similarly, the *direct sum totalisation* is given by $D_k = bigoplus_{n+m=k} C_{n,m}$, with differential again given by $\delta = \delta_h + \delta_v$.

Remark 3.3. Note that since $(n,m) \in \mathbb{Z}^2$, we have infinite direct sums and products in the entries of the total complexes.

As already mentioned, we have already proved that $(\Omega(A), b, B)$ is a bicomplex. This is called the *cyclic bicomplex* and can be viewed diagramatically as:

$$\begin{array}{ccccccc}
\cdots & \longleftarrow & \cdots & \longleftarrow & \cdots & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega^3(A) & \xleftarrow[B]{} & \Omega^2(A) & \xleftarrow[B]{} & \Omega^1(A) & \xleftarrow[B]{} & A \\
\downarrow b & & \downarrow b & & \downarrow b & & \\
\Omega^2(A) & \xleftarrow[B]{} & \Omega^1(A) & \xleftarrow[B]{} & A & & \\
\downarrow b & & \downarrow b & & & & \\
\Omega^1(A) & \xleftarrow[B]{} & A & & & & \\
\downarrow b & & & & & & \\
A, & & & & & &
\end{array}$$

where the diagram extends indefinitely wherever it can in the horizontal and vertical directions.

Definition 3.4. The total complex $\mathsf{HC}(A)$ of the cyclic bicomplex $(\Omega(A), b, B)$ above is called the *cyclic homology complex* of A. Its homology $\mathrm{HC}_*(A)$ is called the *cyclic homology* of A. Explicitly, the degree n vector space of the cyclic homology complex is

$$\Omega^n(A) \times \Omega^{n-1}(A) \times \cdots \times \Omega^{n \mod 2}(A),$$

where the last term is $\Omega^1(A)$ or A, depending on whether or not n is even or odd. The boundary map is $b + B$, except on the first summand, where it is just b.

By Definition 1.15, the first column $(\Omega^n(A), b_n)_{n\in\mathbb{N}}$ of the cyclic bicomplex computes the Hochschild homology of A. Let us now remove this first column; this new bicomplex is just the cyclic bicomplex, but shifted to the left and the top by 1. This produces a shift of degree two on the total complex. More precisely, we have a projection operator

$$S\colon \mathsf{HC}(A) \twoheadrightarrow \mathsf{HC}(A)[-2], \quad \Omega^n(A) \times \Omega^{n-1}(A) \times \cdots \to \Omega^{n-2}(A) \times \Omega^{n-3}(A) \times \cdots,$$

which we call the *periodicity operator*.

Exercise 3.5. Show that the periodicity operator is a chain map, whose kernel is the inclusion of the Hochschild complex $\mathsf{HH}(A) \coloneqq (\Omega^n(A), b_n)$ into the cyclic homology complex $\mathsf{HC}(A)$.

Exercise 3.5 leads to an exact sequence of chain complexes

$$0 \to \mathsf{HH}(A) \overset{\subseteq}{\to} \mathsf{HC}(A) \overset{S}{\to} \mathsf{HC}(A) \to 0,$$

which by the Proposition 2.23 induces a long exact sequence of homology groups

$$\cdots \to \mathrm{HC}_n(A) \overset{S}{\to} \mathrm{HC}_{n-2}(A) \overset{B}{\to} \mathrm{HH}_{n-1}(A) \overset{I}{\to} \mathrm{HC}_{n-1}(A) \to \cdots,$$

where I, S and B are the maps on homology induced by the inclusion $\mathsf{HH}(A) \subseteq \mathsf{HC}(A)$, the periodicity operator $S\colon \mathsf{HC}(A) \to \mathsf{HC}(A)[-2]$ and B is the boundary map from the Snake Lemma. This is called the *Connes SBI-sequence*.

Exercise 3.6. Show that the boundary map in the long exact sequence induced by the short exact sequence of chain complexes above is Connes' B-operator.

We can iterate the periodicity construction to produce an inverse system

$$\cdots \to \mathsf{HC}(A)[n] \overset{S}{\to} \mathsf{HC}(A)[n-2] \overset{S}{\to} \mathsf{HC}(A)[n-4] \to \cdots$$

of chain complexes of complex vector spaces. The inverse limit of this inverse system of chain complexes is a $\mathbb{Z}_2$-graded chain complex defined by

$$\cdots \to \Omega^{\mathrm{ev}}(A) \overset{b+B}{\to} \Omega^{\mathrm{odd}}(A) \overset{b+B}{\to} \Omega^{\mathrm{ev}}(A) \to \cdots,$$

where $\Omega^{\mathrm{ev}}(A) \coloneqq \prod_{n\in\mathbb{N}} \Omega^{2n}(A)$ and $\Omega^{\mathrm{odd}}(A) \coloneqq \prod_{n\in\mathbb{N}} \Omega^{2n+1}(A)$.

Definition 3.7. The chain complex $\mathsf{HP}(A) \coloneqq \varinjlim_n (\mathsf{HC}[-2n], S)$ is called the *periodic cyclic homology complex* of A. Its homology is called the *periodic cyclic homology* of A.

Before moving on, let us use the SBI-sequence to compute cyclic and periodic cyclic homology for some algebras.

EXAMPLE 3.8. Let $A = W[p,q]$ be the Weyl algebra. We have already seen in Exercise 3.8 that its Hochschild homology $\mathrm{HH}_n(A) = \mathbb{C}$ for $n = 2$ and is zero for all other n. It is trivial to observe that $\mathrm{HC}_n(A) = 0$ for $n < 0$. Now the SBI-exact sequence allows us to compute $\mathrm{HC}_n(A)$ recursively for all $n \in \mathbb{N}$. Namely, the exactness of the sequence forces $\mathrm{HC}_n(A) = 0$ for $n \leq 1$ and $\mathrm{HC}_n(A) = \mathbb{C}$ for $n \geq 2$, with an invertible map $S\colon \mathrm{HC}_{n+2}(A) \to \mathrm{HC}_n(A)$ for $n \geq 2$. A similar computation works whenever Hochschild homology is concentrated in one degree.

Recall that the Hochschild homology of a smooth commutative algebra is given by the Hochschild-Kostant-Rosenberg isomorphism

$$\mathrm{HH}_n(A) \cong \Omega^n(A)$$

for all n. The same result also holds for $A = \mathrm{C}^\infty(M)$. As mentioned previously, this means that Hochschild homology generalises differential forms. However, there seems to be something peculiar about this statement - the right hand side of the isomorphism above has homology groups, while the left hand side has the terms of a chain complex (before taking homology). The operator B motivated at the start of Section 2 is in some sense meant to correct precisely this:

THEOREM 3.9. *Let A be a smooth commutative algebra over a field of characteristic zero. We then have*

(1) $\mathrm{HP}_*(A) \cong \bigoplus_{n\in\mathbb{Z}} \mathsf{hdR}^{*+2n}(A)$ *for* $* = 0,1$*;*
(2) $\mathrm{HC}_*(A) \cong \Omega^*(A)/\mathrm{d}(\Omega^{*-1}(A)) \oplus \mathsf{hdR}^{*-2}(A) \oplus \cdots$ *for* $* \in \mathbb{N}$.

PROOF. By the Hochschild-Kostant-Rosenberg Theorem, the cyclic bicomplex becomes

$$\begin{array}{ccccccc}
\cdots & \longleftarrow & \cdots & \longleftarrow & \cdots & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega^3_A & \xleftarrow[\mathrm{d}]{} & \Omega^2_A & \xleftarrow[\mathrm{d}]{} & \Omega^1(A) & \xleftarrow[\mathrm{d}]{} & A \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \\
\Omega^2_A & \xleftarrow[\mathrm{d}]{} & \Omega^1_A & \xleftarrow[\mathrm{d}]{} & A & & \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & & & \\
\Omega^1_A & \xleftarrow[d]{} & A & & & & \\
\downarrow{\scriptstyle 0} & & & & & & \\
A. & & & & & &
\end{array}$$

The rest is left as an exercise. □

3.1. The Hodge filtration on periodic cyclic homology. We end this section with another tool to compute periodic cyclic homology of an algebra when some information about its Hochschild homology is known.

Let A be an algebra. For each n, define subspaces of $\mathsf{HP}(A)$

$$\mathcal{F}_n(\mathsf{HP}(A)) \coloneqq b(\Omega^n(A)) \times \prod_{k=n}^{\infty} \Omega^k(A).$$

Each such subspace is invariant under the maps b and B, so that the differential $b+B$ on $\mathsf{HP}(A)$ induces one on $\mathcal{F}_n(\mathsf{HP}(A))$. That is, each $\mathcal{F}_n(\mathsf{HP}(A))$ is a subcomplex of $\mathsf{HP}(A)$. Furthermore, for each n, $\mathcal{F}_{n+1}(\mathsf{HP}(A)) \subseteq \mathcal{F}_n(\mathsf{HP}(A))$. In other words, what we have defined is a *decreasing filtration* on $\mathsf{HP}(A)$ by the subcomplexes $(\mathcal{F}_n(\mathsf{HP}(A)))_{n\in\mathbb{N}}$ - this is called the *Hodge filtration.*

THEOREM 3.10. *Suppose there is an $n \in \mathbb{N}$ such that $\mathrm{HH}_N(A) = 0$ for all $N \geq n$. Then the chain complex $\mathcal{F}_n(\mathsf{HP}(A))$ is contractible. Consequently, $\mathrm{HP}_*(A)$ is isomorphic to the homology of the smaller complex defined by*

$$(\prod_{k=0}^{n-1} \Omega^k(A) \times \frac{\Omega^n(A)}{b(\Omega^{n+1}(A))}, b + B).$$

We first need a lemma that we state without proof:

LEMMA 3.11. *Let C be a chain complex and let $(\mathcal{F}_n(C))$ be a decreasing filtration by subcomplexes. Assume $\bigcap \mathcal{F}_n(C) = \{0\}$. If the quotient complexes $\mathcal{F}_n(C)/\mathcal{F}_{n+1}(C)$ are exact for all $n \in \mathbb{N}$, then C is exact.*

Returning to the proof of Theorem 3.10:

PROOF. Fix $N \geq n$. The subquotient $\mathcal{F}_N/\mathcal{F}_{N+1} = b(\Omega^N(A)) \oplus \frac{\Omega^N(A)}{b(\Omega^{N+1}(A))}$ with the boundary b; B vanishes on this subquotient because it maps $b(\Omega^N(A))$ to $b(\Omega^{N+1}(A))$. Consequently, the homology of $\mathcal{F}_N/\mathcal{F}_{N+1}$ is $\mathrm{HH}_N(A)$, which vanishes by hypothesis. So the subquotient $\mathcal{F}_N/\mathcal{F}_{N+1}$ is exact. By Lemma 3.11, the quotient $\mathcal{F}_n/\mathcal{F}_N$ vanishes for all $N \geq n$. Consequently, the product $\prod_{N\geq n} \mathcal{F}_n/\mathcal{F}_N$ is exact. We then have a short exact sequence of chain complexes

$$0 \to \varinjlim \mathcal{F}_n/\mathcal{F}_N \to \prod_{N\geq n} \mathcal{F}_n/\mathcal{F}_N \to \prod_{N\geq n} \mathcal{F}_n/\mathcal{F}_N \to 0,$$

where the second map is given by $(x_N)_{N\geq n} \mapsto (q_N(x_{N+1} - x_N))$, where $q_N: \mathcal{F}_n/\mathcal{F}_{N+1} \twoheadrightarrow \mathcal{F}_n/\mathcal{F}_N$ is the canonical projection. Finally, the kernel of a map between exact complexes is exact, so $\varinjlim_{N\geq n} \mathcal{F}_n/\mathcal{F}_N$ is exact. And this inverse limit is precisely $\mathcal{F}_n$ with the boundary $b + B$. Finally, dividing out by an exact chain complex does not change homology, so the conclusion about $\mathrm{HP}_*(A)$ follows. □

The reader might now wonder how restrictive the condition that $\mathrm{HH}_N(A) = 0$ for sufficiently large N is. By definition, Hochschild homology is computed by projective bimodule resolutions. For several classes of algebras, it happens that these resolutions can be chosen to be of finite length - as we have seen for $\mathbb{C}[x, y]$ and the Weyl algebra. When we have such a finite length resolution at our disposal, Hochschild homology vanishes after a finite level. A conceptual study of algebras for which this happens is in order:

DEFINITION 3.12. Let A be a unital algebra and M a unital module. We say that M has *projective dimension k* if it has a projective A-module resolution of length k, but none of shorter length. The *bidimension* of a unital algebra A is the projective dimension of A as an A-bimodule. For a non-unital algebra A, we define the bidimension of A as the dimension of its unitalisation A^+.

PROPOSITION 3.13. *Let A be a unital algebra. The following are equivalent:*

(1) *A has bidimension at most n;*
(2) *A has a projective resolution of length n;*
(3) *The bimodule $\Omega^n(A)$ is projective as an A-bimodule;*
(4) *There is a linear map $\nabla: \Omega^n(A) \to \Omega^{n+1}(A)$ that satisfies $\nabla(a \cdot \omega) = a\nabla(\omega)$ and $\nabla(\omega \cdot a) = \nabla(\omega)a + \omega \mathrm{d}(a)$ for $a \in A$ and $\omega \in \Omega^n(A)$.*

PROOF. **Do it.** □

The map $\nabla\colon \Omega^n(A) \to \Omega^{n+1}(A)$ is called an *n-connection* on A. It should be remniscient of connections in differential geometry, albeit only in a formal sense. The following result shows that for all smooth algebras, there exists such a connection. Equivalently, smooth algebras satisfy the conditions of Theorem 3.10.

PROPOSITION 3.14. *Let A be a smooth commutative algebra over a commutative unital ring R. Then A has finite bidimension.*

PROOF. By Proposition 3.13, we need to show that $\Omega^n(A)$ is projective for some n. This is equivalent to the showing that A has finite projective dimension as a module over $S = A \otimes_R A$. Now if A is smooth it is in particular flat and finitely generated over R, which implies that A is free as an R-module. Thus $\mathrm{HH}^*(A, -) = \mathrm{Ext}^*_S(A, -)$, and thus the previous statements are equivalent to

$$\mathrm{Ext}^n_S(A, -) = 0 \text{ for sufficiently large } n.$$

Now localization at a multiplicative set commutes with $\mathrm{Ext}_S(M, -)$ when M is finitely presented (since our S is noetherian, this is the same as finitely generated, so this follows from [**4**, Prop. 3.3.10]). Because A is smooth, $\Omega^1(A) = \ker(S \to A)$ is locally generated by a regular sequence of length d equal to rank of the projective A-module Ω^1_A (the commutative one) (see e.g. the proof of the Hochschild-Kostant-Rosenberg theorem). The Koszul complex associated to such a sequence gives a free resolution of length d, so the equation above is true for $n \geq d + 1$ locally and therefore globally by the fact explained before that $\mathrm{Ext}^n_S(A, -)$ localizes. □

The definition of periodic cyclic homology provided in 3.7 is one of several ways of defining periodic cyclic homology. It is also the one original definition provided by Connes. From what we have seen so far, it is computationally useful if the Hochschild homology, and consequently the cyclic homology of an algebra is accessible. This is because, by construction, periodic cyclic homology is essentially built by a sequence of categorical manipulations to the cyclic bicomplex that computes Hochschild and cyclic homology. On the other hand, when the Hochschild homology is not computable (which is the case for several noncommutative algebras), one typically needs to use properties intrinsic only to periodic cyclic homology. These properties do not follow from those of Hochschild or cyclic homology, so Definition 3.7 alone is rather restrictive. This will be addressed in the next chapter, where we shall see an alternative way to think about periodic cyclic homology.

CHAPTER 4

Periodic cyclic homology via nilpotent extensions

As explained in the last part of the previous chapter, the computations of periodic cyclic homology we have so far are only possible because they follow from the computation of Hochschild homology of the algebra in question. Such computations, however, are typically not accessible when the algebra in question is not sufficiently smooth; that is, when they lack derivations. For more complicated algebras, we need further properties that allow the reduction of such algebras to simpler algebras. These properties only hold for periodic cyclic homology, but are sadly not apparent from the definition of periodic cyclic homology presented so far. This will be corrected in this chapter.

To motivate the story, we start with Grothendieck's infinitesimal cohomology. Let A be the coordinate ring of a (not necessarily smooth) affine variety X over a field k of characteristic zero. In this situation, de Rham cohomology is replaced by *infinitesimal cohomology*. This is done by embedding the variety X into a smooth variety such as an affine hyperplane, and completing the de Rham complex along the subvariety. More precisely, and algebraically, we can form the quotient

$$I \rightarrowtail k[x_1, \dots, x_n] \twoheadrightarrow A,$$

where $\mathbb{A}_k^n \coloneqq \mathrm{Spec}(k[x_1, \cdots, x_n])$ denotes the embedding space of $X = \mathcal{O}(A)$. Then infinitesimal cohomology of X is defined as

$$H_{\mathrm{inf}}^*(X) \coloneqq \mathsf{hdR}^i(k[x_1, \dots, x_n]_{\widehat{I}}),$$

where $k[x_1, \dots, x_n]_{\widehat{I}} = \varprojlim k[x_1, \dots, x_n]/I^n$.

In the noncommutative situation, we replace a finite-type commutative algebra A by an associative algebra. Since the polynomial algebra is a free commutative algebra, a suitable replacement of it in the associative setting is the free noncommutative algebra, or the *tensor algebra* $\mathsf{T}(A) = \bigoplus_{n=0}^{\infty} A^{\otimes n}$ of an algebra. We then have a similar extension

$$\mathsf{J}(A) \rightarrowtail \mathsf{T}(A) \twoheadrightarrow A,$$

playing the same role as the polynomial extension above. The tensor algebra is isomorphic to the differential graded algebra $\bigoplus_{n=0}^{\infty} \Omega^{2n}(A)$ of even, noncommutative differential forms. However, as we have seen in the previous chapter, this is the wrong candidate for noncommutative de Rham cohomology. Instead, we take the *completed tensor algebra*

$$\mathcal{T}(A) \coloneqq \varprojlim \mathsf{T}(A)/\mathsf{J}(A)^n \cong \prod_{n=0}^{\infty} \Omega^{2n}(A),$$

which *does* give the right de Rham theory. Following the same path as Grothendieck's infinitesimal cohomology, we can define a complex

$$H_{\mathrm{inf,nc}}(A) \coloneqq X(\mathcal{T}(A))$$

called the *X-complex* of A. The suggestive notation is due to the fact that we would like to define a noncommutative variant of infinitesimal cohomology, which just gives de Rham cohomology for smooth, commutative algebras. This variant will turn out to be precisely periodic cyclic homology.

Continuing the analogy further, just as the polynomial algebra was used to smoothen a not-necessarily smooth algebra, the tensor algebra makes an associative algebra *quasi-free*, which is the right replacement of smoothness in noncommutative geometry. Furthermore, an important result about infinitesimal cohomology is that it does not depend on the choice of embedding space. The same is true for the functor $H_{\mathrm{inf,nc}}$.

CHAPTER 5

To be decided

Bibliography

[1] Masaki Kashiwara and Pierre Schapira, *Categories and sheaves*, Grundlehren der Mathematischen Wissenschaften, vol. 332, Springer, Berlin, 2006. doi: 10.1007/3-540-27950-4 MR **2182076**

[2] Amnon Neeman, *Triangulated categories*, Annals of Mathematics Studies, vol. 148, Princeton University Press, Princeton, NJ, 2001. MR **1812507**

[3] Emily Riehl, *Categories in context*, Dover, 2017.

[4] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. doi: 10.1017/CBO9781139644136 MR **1269324**