Synchronization in Random Geometric Graphs

Mixed Local and Nonlocal Operators: Analytical, Numerical and Probabilistic Perspectives (MiLNE Congress)

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November 28, 2025 — Buenos Aires, Argentina

Introduction

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Kuramoto Model (mean field version)

$$\begin{cases} \frac{d}{dt}u^n(t,x_i) & = \omega_i + \frac{K}{n}\sum_{j=1}^{n}\sin\left(u^n(t,x_j) - u^n(t,x_i)\right) \\ u^n(0,x_i) & = u_0^n(x_i), \qquad i = 1,2,\dots,n. \end{cases}$$

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Initial question

Synchronization or patterns?

Known results

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- Complete graph → Synchronizes ✓
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- Random Geometric Graphs on $\mathbb{T}^d$ → the scaling limit of the Kuramoto model is the heat equation ✗
  (Cirelli, Groisman, Huang, Vivas - 2025)

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Let M be a Riemannian manifold, and let $\epsilon_n = \epsilon(n)$ satisfy certain conditions.

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Definition (RGG on M)

A Random Geometric Graph on M with parameter ϵ_n is $G_n = (\mathcal{V}, \mathcal{E})$, where:

- $\mathcal{V} = \{x_1, x_2, \ldots, x_n\} \subseteq M$ is a set of n i.i.d. random points uniformly distributed on M;
- $e = \{x_i, x_j\} \in \mathcal{E} \iff d(x_i, x_j) < \epsilon_n$.

Case M = $\mathbb{S}^1$

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Kuramoto Model

$$(\star)\begin{cases} \frac{d}{dt}u^n(t,x_i) & = \dfrac{\pi}{n^2\epsilon_n^3}\displaystyle\sum_{j\sim i}\sin\left(u^n(t,x_j)-u^n(t,x_i)\right), \\ u^n(0,x_i) & = \bar{u}(x_i), \qquad i=1,2,\ldots,n. \end{cases}$$

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Kuramoto Model (Gradient system)

If

$$E_n(u^n)=E_n(u_1^n,u_2^n,\dots,u_n^n)=\frac{\pi}{2n^2\epsilon_n^3}\sum_{i=1}^{n}\sum_{j\sim i}\left(1-\cos(u_j^n-u_i^n)\right),$$

then

$$\frac{d}{dt}u^n=-\nabla E_n(u^n).$$

Geometry of the space $\mathbb{T}^n$

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Definition (Index - continuous case)

For a continuous function $u : [0, 2\pi] \to \mathbb{R}$ with $u(2\pi) = u(0) + 2q\pi$, we define its index by $I(u) = q$.

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Definition (Index - discrete case)

For each $\mathbf{z} \in \mathcal{D} = \{\mathbf{z} \in \mathbb{T}^n : z_{j+1} \neq -z_j, 1 \leq j \leq n\}$, we define its index by

$$I(\mathbf{z}) = \frac{1}{2\pi} \sum_{j=1}^{n} \theta_{j+1} \ominus \theta_j.$$

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We have the decomposition

$$\mathbb{T}^n = \bigcup_{q \in \mathbb{Z}} K_q \cup \left(\bigcup_{q \in \mathbb{Z}} \partial K_q \right).$$

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Theorem

In the regime

$$n\epsilon_n^2 \to 0, \qquad \frac{n\epsilon_n}{\log n} \to \infty, \qquad \text{as } n \to \infty$$

it holds that

$$\lim_{n\to\infty} \mathbb{P}\left((\star) \text{ has an asymptotically stable equilibrium with index } q\right) = 1$$

for each $q \in \mathbb{Z}$.

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I) $\overline{K_q}$ compact + E_n continuous $\Rightarrow$ there is a minimum in $\overline{K_q}$.

II) Proposition: If $u \in C^2([0, 2\pi], \mathbb{R})$, then

$$\lim_{n \to \infty} E_n(u) = \frac{1}{12\pi} \int_0^{2\pi} |u'(x)|^2 dx, \qquad \text{in probability.}$$

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III) The following limit holds:

$$\lim_{n \to \infty} \inf_{\theta \in \partial K_q} E_n(\theta) = +\infty \qquad \text{almost surely.}$$

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Kuramoto Model

$$\begin{cases} \frac{d}{dt}u^n(t,x_i) & = \dfrac{4\pi}{c_2\epsilon_n^2 n}\displaystyle\sum_{j\sim i}\sin(u^n(t,x_j)-u^n(t,x_i)), \\ u^n(0,x_i) & = u_0^n(x_i), \qquad i=1,2,\dots,n. \end{cases}$$

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Heat Equation on the sphere

$$\begin{cases} \frac{d}{dt}u(t,x) &= \kappa\Delta_{\mathbb{S}^2}u(t,x)\\ u(0,x) &= u_0(x), \end{cases}$$

where $\Delta_{\mathbb{S}^2}$ denotes the Laplace-Beltrami operator on $\mathbb{S}^2$ and

$$\kappa := \frac{1}{4}\int_{\mathbb{R}^2} \|z\|^2 \, 1_{\{\|z\|^2<\epsilon_n\}}\, dz.$$

Remark: *Here we consider u as an $\mathbb{S}^1$-valued function.*

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Kuramoto Model

$$\text{(KM)} \begin{cases} \frac{d}{dt}u^n(t,x_i) & = \frac{4\pi}{c_2\epsilon_n^2 n}\sum_{i=1}^n \sum_{j\sim i} \sin\left(u^n(t,x_j) - u^n(t,x_i)\right) \\ u^n(0,x_i) & = u_0^n(x_i), \qquad i = 1,2,\dots,n. \end{cases}$$

Intermediate Equation

$$\text{(IE)} \begin{cases} \frac{d}{dt}u^{l,\epsilon_n}(t,x) & = \frac{1}{c_2\epsilon_n^2}\int_{\mathbb{S}^2} \sin\left(u^{l,\epsilon_n}(t,y) - u^{l,\epsilon_n}(t,x)\right) \mathbb{1}_{\{\|x-y\|^2<\epsilon_n\}}\, d\sigma(y), \\ u^{l,\epsilon_n}(0,x) & = u_0(x). \end{cases}$$

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Theorem (Scaling limit)

Assume that $\epsilon_n \to 0$ as $n \to \infty$ and $\liminf_{n\to\infty} \frac{\epsilon_n n^2}{\log n} = \infty$. Let $u_0 \in C^{2,\alpha}(\mathbb{S}^2)$ for some $\alpha \in (0,1)$. If

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\sup_{1\le i\le n} |u_0^n(x_i) - u_0(x_i)| > \delta \right) < \infty$$

for every $\delta > 0$, then

$$\lim_{n\to\infty} \sup_{1\le i\le n} \sup_{t\in[0,T]} |u^n(t,x_i) - u(t,x_i)| = 0, \qquad \text{almost surely.}$$

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Theorem (Synchronization)

Let A_n be the event that equation **(KM)** with initial condition u_0^n achieves phase synchronization. Then,

$$\mathbb{P}(A_n \text{ occurs for all } n \text{ large enough}) = 1.$$

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$$\|u(T, \cdot) - \iota\|_{L^\infty(\mathbb{S}^2)} < \pi/8.$$

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II) There exists a set $\tilde{\Omega}$ with $\mathbb{P}(\tilde{\Omega}) = 1$ such that

$$\max_{1 \leq i,j \leq n} |u^n(T, x_i) - u^n(T, x_j)| < \pi/2$$

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IV) It is known that $\tilde{\Omega} \cap B_n \subseteq A_n$, and by Borel-Cantelli lemma

$$\mathbb{P}\left((\tilde{\Omega} \cap B_n)^c \text{ i.o}\right) = 0.$$

THANKS! :)