Compactness and dichotomy in nonlocal shape optimization

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We prove a general result about the behaviour of minimizing sequences for nonlocal shape functionals satisfying suitable structural assumptions. Typical examples include functions of the eigenvalues of the fractional Laplacian under homogeneous Dirichlet boundary conditions. Exploiting a nonlocal version of Lions' concentration-compactness principle, we prove that either an optimal shape exists, or there exists a minimizing sequence consisting of two "pieces" whose mutual distance tends to infinity. Our work is inspired by similar results obtained by Bucur in the local case.

1 Introduction

A significant task in Shape Optimization consists in proving existence of minimizing sets, in a suitable class, for shape functionals of the kind

$$\Omega \mapsto J(\Omega) = F(\lambda_1(\Omega), ..., \lambda_m(\Omega)),$$

where $m \in \mathbb{N}^*$, $\Omega \subset \mathbb{R}^N$, and $\lambda_1(\Omega), ..., \lambda_m(\Omega)$ are eigenvalues of some differential operator. In the case of the Laplacian under Dirichlet boundary conditions, and $J(\Omega) = \lambda_k(\Omega)$, existence of optimal shapes among all measurable sets with prescribed Lebesgue measure has been a challenging open problem for a long time. Apart from the simpler cases $k = 1$ and $k = 2$, where the Faber-Krahn inequality implies that the optimal shape is a ball (for $k = 1$) or the disjoint union of two equal balls (for $k = 2$), for the general case existence in the class of quasi-open sets has been proven only recently by Bucur in [8] and by Mazzoleni and Pratelli in [22] independently. It is still an open problem to identify the optimal shapes for $k \geq 3$, although numerical simulations support some conjectures.

When the differential operator under consideration is the fractional Laplacian, defined as

$$(-\Delta)^s u(x) := C_{s,N} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{N+2s}}\, dy,$$

where $s \in (0, 1)$ and $C_{s,N}$ is a normalization constant, the situation is quite different. While the ball minimizes again the first eigenvalue under a volume constraint, the problem

$$\min\{\lambda_2(\Omega) \,|\, \Omega \subset \mathbb{R}^N,\ |\Omega| = c\}, \tag{1}$$

where $c > 0$, and $|\Omega|$ is the Lebesgue measure of Ω, does not have a solution. Indeed, it was proven by Brasco and the first author [5] that, for every admissible set Ω,

$$\lambda_2(\Omega) > \lambda_1(\widetilde{B}),$$

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where $\widetilde{B}$ is a ball of volume $\frac{c}{2}$, and that a minimizing sequence $\{\Omega_n\}_{n\in\mathbb{N}}$ such that $\lambda_2(\Omega_n) \to \lambda_1(\widetilde{B})$ is given by the union of two disjoint balls of volume $\frac{c}{2}$, such that their mutual distance tends to infinity. This means that, in the nonlocal case, a general existence result as in [8] or [22] can not hold true. On the other hand, if one restricts the minimization to quasi-open sets which are contained in a fixed open set $D \subset \mathbb{R}^N$, a generalization of the existence result by Buttazzo and Dal Maso [10] holds true, as shown by Fernández Bonder, Ritorto and the second author in [16].

Inspired by the results obtained in [7] by Bucur, in this paper we prove that, in the case of the fractional Laplacian, for a minimizing sequence only two situations can occur: *compactness*, which implies, under some assumptions, existence of an optimal shape; or *dichotomy*, which means that the sequence essentially behaves as the union of two disconnected sets, whose mutual distance tends to infinity, as in Problem (1). To prove the result, we make use of a nonlocal version of the celebrated concentration-compactness principle of Lions [21], which we apply to the sequence of *torsion functions* w_{Ω_n}, where $\{\Omega_n\}_{n\in\mathbb{N}}$ is a minimizing sequence for the shape functional under consideration. We recall that w_{Ω_n} is defined as the weak solution of the problem

$$\begin{cases} (-\Delta)^s w_{\Omega_n} = 1 & \text{in } \Omega_n, \\ w_{\Omega_n} = 0 & \text{in } \mathbb{R}^N \setminus \Omega_n. \end{cases} \tag{2}$$

In order to introduce our main results, we recall that a sequence $\{\Omega_n\}_{n\in\mathbb{N}}$ of s-quasi-open sets of uniformly bounded Lebesgue measure is said to *γ-converge* to the s-quasi-open set Ω if the solutions w_{Ω_n} of (2) strongly converge in $L^2(\mathbb{R}^N)$ to the solution $w_\Omega \in H^s_0(\Omega)$ of the problem

$$\begin{cases} (-\Delta)^s w_{\Omega} = 1 & \text{in } \Omega, \\ w_{\Omega} = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

(see Section 2 for precise definitions of s-quasi-open sets). Moreover, we say that a sequence $\{\Omega_n\}_{n\in\mathbb{N}}$ of s-quasi-open sets of uniformly bounded Lebesgue measure *weakly γ-converges* to the s-quasi-open set Ω if the solutions w_{Ω_n} of (2) converge weakly in $H^s(\mathbb{R}^N)$, and strongly in $L^2(\mathbb{R}^N)$, to a function $w \in H^s(\mathbb{R}^N)$ such that $\Omega = \{w > 0\}$. Finally, for a given s-quasi-open set $\Omega \subset \mathbb{R}^N$ of finite measure, we denote by R_Ω the *resolvent operator* of $(-\Delta)^s$, which is defined as the function $R_\Omega : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ such that $R_\Omega(f) = u$, where u is the weak solution of

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

We can now state our first main result, whose proof follows the ideas of [7].

Theorem 1.1 *Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be a sequence of s-quasi-open sets of uniformly bounded measure. Then there exists a subsequence, still denoted by the same index, such that one of the following situations occurs:*

(i) Compactness: there exists a (possibly empty) s-quasi-open set Ω, and a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$, such that $y_n + \Omega_n$ weakly γ-converges to Ω as $n \to +\infty$.

(ii) Dichotomy: there exists a sequence of subsets $\widetilde{\Omega}_n \subset \Omega_n$ such that

$$\|R_{\Omega_n} - R_{\widetilde{\Omega}_n}\|_{\mathcal{L}(L^2(\mathbb{R}^N))} \to 0, \qquad \widetilde{\Omega}_n = \Omega_n^1 \cup \Omega_n^2,$$

where dist$(\Omega_n^1, \Omega_n^2) \to +\infty$ and $\liminf_{n\to+\infty} |\Omega_n^i| > 0$ for $i = 1, 2$.

Theorem 1.1 gives, as a consequence, an existence result for optimal shapes for minimization problems, when the shape functional satisfies some structural assumptions.

Theorem 1.2 *Let*

$$\mathcal{A}(\mathbb{R}^N) := \{\Omega \subset \mathbb{R}^N \mid \Omega \ \ s\text{-quasi-open}\}$$

and let $J : \mathcal{A}(\mathbb{R}^N) \to (-\infty, +\infty]$ be a shape functional satisfying the following assumptions:

(i) J is lower semicontinuous with respect to γ-convergence;

(ii) J is decreasing with respect to set inclusion: if Ω_1, $\Omega_2 \in \mathcal{A}(\mathbb{R}^N)$, $\Omega_1 \subset \Omega_2$, then $J(\Omega_2) \leq J(\Omega_1)$;

(iii) J is invariant by translations;

(iv) J is bounded from below.

Let $c > 0$, and define

$$m := \inf\{J(\Omega) \,|\, \Omega \in \mathcal{A}(\mathbb{R}^N), |\Omega| = c\}. \tag{3}$$

Then, one of the following situations occurs:

(i) Existence of an optimal shape: there exists a s-quasi-open set $\hat{\Omega} \in \mathcal{A}(\mathbb{R}^N)$ such that $|\hat{\Omega}| = c$ and $J(\hat{\Omega}) = m$.

(ii) Dichotomy: there exists a minimizing sequence $\{\Omega_n\}_{n\in\mathbb{N}}$ with $|\Omega_n| = c$ for every $n \in \mathbb{N}$, such that $\Omega_n = \Omega_n^1 \cup \Omega_n^2$, where Ω_n^1, Ω_n^2 are such that dist$(\Omega_n^1, \Omega_n^2) \to +\infty$, $\liminf_{n\to+\infty} |\Omega_n^i| > 0$ for $i = 1, 2$, and $J(\Omega_n) \to m$ as $n \to +\infty$.

Theorem 1.2 applies in particular to spectral functionals of the kind

$$J(\Omega) := F(\lambda_1(\Omega), ..., \lambda_k(\Omega)),$$

where $k \in \mathbb{N}$, $\lambda_j(\Omega)$ is the $j-$th eigenvalue of the Dirichlet fractional Laplacian, and $F : \mathbb{R}^k \to \mathbb{R} \cup \{+\infty\}$ is a functional which is lower semicontinuous and nondecreasing in each variable.

In the local case, existence of an optimal shape and the dichotomy situation can occur at the same time. Indeed, as we have pointed out, the classical Hong-Krahn-Szego inequality asserts that among all domains of fixed volume, the disjoint union of two equal balls has the smallest second eigenvalue. However, due to the nonlocal effects of the fractional Laplacian, the mutual position of two connected components has influence over the second eigenvalue, implying nonexistence of an optimal shape. Therefore it makes sense to ask whether existence of an optimal shape and dichotomy are two mutually exclusive situations in the nonlocal case. Up to our knowledge, this remains an open question.

The manuscript is organized as follows. In section 2 we introduce some preliminary definitions and notation. Section 3 deals with the concentration-compactness principle in the fractional setting. In Section 4 we define the notion of γ- and weak γ-convergence of sets as well as some related useful result, and finally in Sections 5 and 6 we provide a proof of our main results.

2 Definitions and preliminary results

We begin this section with some definitions.

2.1 Fractional Sobolev spaces and s-capacity of sets

For $s \in (0, 1)$, the fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined as

$$H^s(\mathbb{R}^N) := \left\{u \in L^2(\mathbb{R}^N) \,|\, [u]_{H^s(\mathbb{R}^N)} < +\infty\right\},$$

endowed with the norm $\|\cdot\|_{H^s(\mathbb{R}^N)}$ defined by

$$\|u\|_{H^s(\mathbb{R}^N)} := \left(\|u\|^2_{L^2(\mathbb{R}^N)} + [u]^2_{H^s(\mathbb{R}^N)}\right)^{\frac{1}{2}},$$

where $[\cdot]_{H^s(\mathbb{R}^N)}$ is the *Gagliardo seminorm* defined as

$$[u]_{H^s(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}}\, dx\, dy\right)^{\frac{1}{2}}.$$

The Gagliardo seminorm of a function $u \in H^s(\mathbb{R}^N)$ can also be expressed in terms of its Fourier transform $\mathcal{F}u$ as

$$[u]^2_{H^s(\mathbb{R}^N)} = \frac{2}{C_{s,N}} \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 \, d\xi,$$

where $C_{s,N}$ is the normalization constant in the definition of $(-\Delta)^s$, given by

$$C_{s,N} = \left(\int_{\mathbb{R}^N} \frac{1 - \cos \zeta_1}{|\zeta|^{N+2s}} \, d\zeta \right)^{-1}$$

(see [12, Proposition 3.4]). Given a measurable set $\Omega \subset \mathbb{R}^N$, for any $s \in (0,1)$ we define the *s-capacity* of Ω as

$$\operatorname{cap}_s(\Omega) = \inf \left\{ [u]^2_{H^s(\mathbb{R}^N)} \colon u \in H^s(\mathbb{R}^N), u \geq 1 \text{ a.e. on a neighborhood of } \Omega \right\}.$$

and the *s-capacity* of $\Omega \subset \Omega'$ relative to Ω' as

$$\operatorname{cap}_s(\Omega; \Omega') = \inf \left\{ [u]^2_{H^s(\mathbb{R}^N)} \colon u \in H^s_0(\Omega'), u \geq 1 \text{ a.e. on a neighborhood of } \Omega \right\}.$$

A function realizing the infimum in $\operatorname{cap}_s(\Omega)$ (resp. $\operatorname{cap}_s(A, \Omega)$) is called the *capacitary potential* of Ω (resp. of A with respect to Ω).

We say that a property holds *s-quasi-everywhere* if it holds up to a set of null s-capacity. A measurable subset $\Omega \subset \mathbb{R}^N$ is a *s-quasi-open* set if there exists a decreasing sequence $\{\omega_n\}_{n \in \mathbb{N}}$ of open subsets of $\mathbb{R}^N$ such that $\operatorname{cap}_s(\omega_n) \to 0$, as $n \to +\infty$, and $\Omega \cup \omega_n$ is open.

A function $u \in H^s(\mathbb{R}^N)$ is said to be *s-quasi continuous* if for every $\varepsilon > 0$ there exists an open set $G \subset \mathbb{R}^N$ such that $\operatorname{cap}_s(G) < \varepsilon$ and $u|_{\mathbb{R}^N \setminus G}$ is continuous. It is well-known that cap_s is a Choquet capacity on $\mathbb{R}^N$ [1, Section 2.2] and for every $u \in H^s(\mathbb{R}^N)$ there exists a unique s-quasi continuous function $\tilde{u} : \mathbb{R}^N \to \mathbb{R}$ such that $\tilde{u} = u$ s-quasi-everywhere on $\mathbb{R}^N$. Therefore we will always consider, without loss of generality, that a function $u \in H^s(\mathbb{R}^N)$ coincides with its s-quasi continuous representative. If $u : \mathbb{R}^N \to \mathbb{R}$ is s-quasi continuous, then every superlevel set $\{u > t\}$ is s-quasi-open.

For a generic measurable set $\Omega \subset \mathbb{R}^N$, we define the fractional Sobolev space $H^s_0(\Omega)$ as

$$H^s_0(\Omega) = \{u \in H^s(\mathbb{R}^N) : u = 0 \quad s\text{-q.e. on } \mathbb{R}^N \setminus \Omega\}.$$

The space $H^s_0(\Omega)$ turns out to be a closed subspace of $H^s(\mathbb{R}^N)$.

The following Poincaré's inequality holds for measurable sets of finite measure.

Proposition 2.1 *Let $\Omega \subset \mathbb{R}^N$ be a measurable set of finite Lebesgue measure. Then, there exists a constant $C = C(s,N) > 0$ such that, for every $u \in H^s_0(\Omega)$,*

$$\|u\|_{L^2(\Omega)} \leq C |\Omega|^{\frac{2s}{N}} [u]_{H^s(\mathbb{R}^N)}.$$

Proof. Let u be a function in $H^s_0(\Omega)$ and consider the ball Ω^* such that $|\Omega^*| = |\Omega|$. Let $v := |u|^*$ be the Schwarz symmetrization of $|u|$, as defined in [19, Definition 1.3.1]. By [2, Theorem 9.2], $v \in H^s_0(\Omega^*)$, and

$$[v]_{H^s(\mathbb{R}^N)} \leq [|u|]_{H^s(\mathbb{R}^N)} \leq [u]_{H^s(\mathbb{R}^N)}.$$

By [4, Lemma 2.4], there exists $C' = C'(s, N, |\Omega|) > 0$ such that

$$\|v\|_{L^2(\Omega^*)} \leq C' [v]_{H^s(\mathbb{R}^N)}.$$

By a scaling argument, it is possible to show the existence of $C = C(s,N) > 0$ such that

$$\|v\|_{L^2(\Omega^*)} \leq C |\Omega^*|^{\frac{2s}{N}} [v]_{H^s(\mathbb{R}^N)}.$$

Since symmetrizations preserve the L^2-norm,

$$\|u\|_{L^2(\Omega)} = \|v\|_{L^2(\Omega^*)} \leq C |\Omega^*|^{\frac{2s}{N}} [v]_{H^s(\mathbb{R}^N)} \leq C |\Omega|^{\frac{2s}{N}} [u]_{H^s(\mathbb{R}^N)},$$

and the claim follows. □

The previous proposition leads to a useful compactness result.

Proposition 2.2 *Let $\Omega \subset \mathbb{R}^N$ be a measurable set of finite Lebesgue measure. Then, for every bounded sequence $\{u_n\}_{n\in\mathbb{N}}$ in $H^s_0(\Omega)$, there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ and a function $u \in H^s_0(\Omega)$ such that $u_{n_k} \to u$ in $L^2(\Omega)$.*

Proof. The proof can be performed as in [4, Theorem 2.7], using the Poincaré inequality stated in Proposition 2.1. □

Proposition 2.3 *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $H^s(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ weakly in $H^s(\mathbb{R}^N)$ as $n \to +\infty$. Then, for every function $\varphi \in W^{1,\infty}(\mathbb{R}^N)$, it holds that $\varphi u_n \in H^s(\mathbb{R}^N)$ for every $n \in \mathbb{N}$, and $\varphi u_n \rightharpoonup \varphi u$ weakly in $H^s(\mathbb{R}^N)$ as $n \to +\infty$.*

Proof. The sequence $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $H^s(\mathbb{R}^N)$. Moreover, since the embedding $H^s(B_r) \hookrightarrow L^2(B_r)$ is compact for every $r > 0$, it follows that $u_n \to u$ strongly in $L^2(B_r)$ for every $r > 0$. Arguing as in [12, Lemma 5.3], we have that the sequence $\{\varphi u_n\}_{n\in\mathbb{N}}$ is also bounded in $H^s(\mathbb{R}^N)$. Therefore, every subsequence $\{\varphi u_{n_k}\}$ admits a subsequence $\{\varphi u_{n_{k_j}}\}$ which converges weakly in $H^s(\mathbb{R}^N)$, and almost everywhere in $\mathbb{R}^N$, to some $v \in H^s(\mathbb{R}^N)$. But $u_{n_{k_j}}$ must converge to u almost everywhere in $\mathbb{R}^N$. Therefore, $\varphi u_{n_{k_j}} \to \varphi u$ a.e. in $\mathbb{R}^N$, and thus $v = \varphi u$. Hence all the sequence φu_n converges weakly in $H^s(\mathbb{R}^N)$ to φu. □

For an s-quasi-open set Ω of finite Lebesgue measure and $f \in L^2(\mathbb{R}^N)$, the weak solution u of

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{4}$$

is defined as the function $u \in H^s_0(\Omega)$ satisfying

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy = \int_\Omega f(x)v(x)\,dx \qquad \text{for all } v \in H^s(\Omega).$$

Similarly, $u \in H^s_0(\Omega)$ is a weak sub-solution (resp. super-solution) of (4) if

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy \le (\text{resp. } \ge) \int_\Omega f(x)v(x)\,dx \qquad \text{for all } v \in H^s(\Omega),\ v \ge 0 \text{ in } \Omega.$$

It is easy to verify that $u \in H^s_0(\Omega)$ is a weak solution if and only if it is a weak sub- and super-solution.

Proposition 2.4 *Let Ω be a measurable set of finite Lebesgue measure. Then, if $f \in L^2(\mathbb{R}^N)$ satisfies $f \ge 0$ in Ω, then the weak solution $u \in H^s_0(\Omega)$ of (4) satisfies $u \ge 0$ in Ω.*

Proof. Let $u^+ := \max\{u, 0\}$, $u^- := \max\{-u, 0\}$ be the positive and the negative parts of u respectively. It holds $u^+, u^- \in H^s_0(\Omega)$. Testing the equation with u^- we have

$$\begin{aligned} 0 &\le \int_\Omega f u^- \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{N+2s}}\,dx\,dy \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{((u^+(x)-u^+(y))-(u^-(x)-u^-(y)))(u^-(x)-u^-(y))}{|x-y|^{N+2s}}\,dx\,dy \\ &= -2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}}\,dx\,dy - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u^-(x)-u^-(y))^2}{|x-y|^{N+2s}}\,dx\,dy \le 0 \end{aligned}$$

which implies $u^- \equiv 0$ and hence $u \ge 0$ in Ω. □

Proposition 2.5 *Let Ω, $\Omega' \subset \mathbb{R}^N$ be two measurable sets of finite Lebesgue measure, satisfying $\Omega \subset \Omega'$. Then, the function $w_\Omega \in H^s_0(\Omega)$, extended by zero in $\Omega' \setminus \Omega$, satisfies*

$$(-\Delta)^s w_\Omega \le 1 \quad \text{in } \Omega'.$$

Proof. Let us define the convex set $K \subset H^s_0(\Omega')$ as

$$K := \{u \in H^s_0(\Omega') \,|\, u \le 0 \ s-\text{q.e. on } \Omega' \setminus \Omega\}.$$

Let $u_\Omega \in K$ be the unique minimizer on K of the functional

$$v \mapsto \frac{1}{2}[v]^2_{H^s(\mathbb{R}^N)} - \int_{\Omega'} v.$$

It is easy to verify that

$$(-\Delta)^s u_\Omega \le 1 \quad \text{in } \Omega' \tag{5}$$

and

$$(-\Delta)^s u_\Omega = 1 \quad \text{in } \Omega. \tag{6}$$

u_Ω satisfies the variational inequality

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u_\Omega(x) - u_\Omega(y)][(v-u_\Omega)(x) - (v-u_\Omega)(y)]}{|x-y|^{N+2s}}\,dx\,dy \ge \int_{\Omega'} (v - u_\Omega) \quad \text{for every } v \in K.$$

Testing the variational inequality with $v = u^+_\Omega \in K$ (note that $u^+_\Omega = 0$ in $\Omega' \setminus \Omega$) we obtain

$$0 \ge -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u^-_\Omega(x) - u^-_\Omega(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \ge \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u_\Omega(x) - u_\Omega(y)][u^-_\Omega(x) - u^-_\Omega(y)]}{|x-y|^{N+2s}}\,dx\,dy \ge \int_{\Omega'} u^-_\Omega$$

so that $u^-_\Omega \equiv 0$, which implies $u_\Omega = 0$ on $\Omega' \setminus \Omega$, and hence $u_\Omega \in H^s_0(\Omega)$. Equation (6) then implies $u_\Omega = w_\Omega$, so that, from (5),

$$(-\Delta)^s w_\Omega \le 1 \quad \text{in } \Omega'.$$

□

The following useful propositions can be proven as in [18, Proposition 3.3.44].

Proposition 2.6 *Let $A \subset \mathbb{R}^N$ be a measurable set. There exists a unique s-quasi-open set $\Omega \subset \mathbb{R}^N$ such that $H^s_0(A) = H^s_0(\Omega)$.*

Proof. Since $H^s_0(A)$ is separable, there exists a dense sequence $\{u_n\}_{n\in\mathbb{N}} \subset H^s_0(A)$. Let us define $\Omega := \cup_{n\in\mathbb{N}}\{\tilde{u}_n \ne 0\}$, where $\tilde{u}_n$ is the s-quasi continuous representative of u_n. Then Ω is s-quasi-open, as a countable union of s-quasi-open sets. Moreover, since $\{\tilde{u}_n \ne 0\} \subset A$ s-q.e. for every $n \in \mathbb{N}$, it holds $\Omega \subset A$ and hence $H^s_0(\Omega) \subset H^s_0(A)$. Conversely, every $u \in H^s_0(A)$ is the H^s and s-q.e. limit of a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$. Hence, $\{\tilde{u} \ne 0\} \subset \{\tilde{u}_{n_k} \ne 0\} \subset \Omega$ s-q.e. for every $k \in \mathbb{N}$, which implies $H^s_0(A) \subset H^s_0(\Omega)$. In conclusion, $H^s_0(A) = H^s_0(\Omega)$.

To prove uniqueness, we will prove that, for any Ω_1, Ω_2 s-quasi-open sets, $H^s_0(\Omega_1) \subset H^s_0(\Omega_2)$ implies $\Omega_1 \subset \Omega_2$ s-q.e. Indeed, suppose by contradiction that $\operatorname{cap}_s(\Omega_1 \setminus \Omega_2) > 0$. There exists a ball B such that $\operatorname{cap}_s(B \cap (\Omega_1 \setminus \Omega_2)) > 0$. Let $\{\omega_n\}_{n\in\mathbb{N}}$ be a non-increasing sequence of open sets contained in B such that $\operatorname{cap}_s(\omega_n) \to 0$ as $n \to +\infty$, and $(B \cap \Omega_1) \cup \omega_n$ is an open set. Let u_{ω_n} be the capacitary potential of ω_n in B. We fix $n_0 \in \mathbb{N}$ sufficiently large, such that $\operatorname{cap}_s(B \cap \{u_{\omega_{n_0}} < 1\} \cap (\Omega_1 \setminus \Omega_2)) > 0$. Let $K_m \subset (B \cap \Omega_1) \cup \omega_{n_0}$ be an increasing sequence of compact sets exhausting $(B \cap \Omega_1) \cup \omega_{n_0}$, and let u_{K_m} be the capacitary potential of K_m with respect to $(B \cap \Omega_1) \cup \omega_{n_0}$. For m large enough, the function $u_{K_m}(1 - u_{\omega_{n_0}})$ belongs to $H^s_0(\Omega_1)$, but not to $H^s_0(\Omega_2)$, a contradiction. Therefore, $\Omega_1 \subset \Omega_2$ s-q.e. □

We denote by R_Ω the resolvent operator of the fractional Laplacian with Dirichlet boundary conditions, that is, $R_\Omega : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ and $R_\Omega(f) = u$, where u is the weak solution of (4). In particular, $w_\Omega = R_\Omega(1)$. It is easy to check that R_Ω defines a continuous, compact, self-adjoint linear operator from $L^2(\mathbb{R}^N)$ in itself. We denote by $\|\cdot\|_{\mathcal{L}(L^2(\mathbb{R}^N))}$ the corresponding operator norm. The resolvent operator is positivity preserving, which means that $f \ge 0$ implies $R_\Omega(f) \ge 0$, as a consequence of the weak maximum principle.

Proposition 2.7 *Let $\Omega \subset \mathbb{R}^N$ be a s-quasi-open set. Then, the set $Z(\Omega) := \{R_\Omega(f) \in H_0^s(\Omega) \,|\, f \in C_c^\infty(\mathbb{R}^N)\}$ is dense in $H_0^s(\Omega)$.*

Proof. Let $u \in H_0^s(\Omega)$, and $g := (-\Delta)^s u \in H^{-s}(\mathbb{R}^N)$ its fractional Laplacian, defined in distributional sense. Then, $u = R_\Omega(g)$. By density of $C_c^\infty(\mathbb{R}^N)$ in $H^{-s}(\mathbb{R}^N)$ (see [23, Lemma 15.10]), there exists a sequence $\{g_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\Omega)$ converging in $H^{-s}(\mathbb{R}^N)$ to g. Then, $R_\Omega(g_n) \in Z(\Omega)$ for every $n \in \mathbb{N}$, and $R_\Omega(g_n) \to R_\Omega(g) = u$ in $H_0^s(\Omega)$ as $n \to +\infty$. □

Proposition 2.8 *Let $\Omega \subset \mathbb{R}^N$ be a s-quasi-open set. Then, $\Omega = \{w_\Omega > 0\}$ s-quasi-everywhere.*

Proof. By definition, $\{w_\Omega > 0\} \subset \Omega$ up to a set of null capacity. Let $u \in H_0^s(\Omega)$. By Proposition 2.7, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset W$ such that $u_n = R_\Omega(g_n)$ with $g_n \in C_c^\infty(\mathbb{R}^N)$, and $u_n \to u$ in $H_0^s(\Omega)$. By the weak maximum principle, $|u_n| \le \|g_n\|_\infty w_\Omega$ in Ω. Therefore, $u_n(x) = 0$ for every $n \in \mathbb{N}$ s-q.e. on $\mathbb{R}^N \setminus \{w_\Omega > 0\}$. Passing to the limit, we obtain that $u \in H_0^s(\{w_\Omega > 0\})$. Therefore, $H_0^s(\Omega) = H_0^s(\{w_\Omega > 0\})$, and by Proposition 2.6 we obtain the claim. □

Given an s-quasi-open set Ω, we say that λ is an *eigenvalue* of the fractional Laplacian if there exists a nontrivial function $u \in H_0^s(\Omega)$, called *eigenfunction*, which is a weak solution of

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{7}$$

By linear operator theory, for every s-quasi-open set $\Omega \subset \mathbb{R}^N$ of finite Lebesgue measure there exists a sequence $\{\lambda_k(\Omega)\}_{k\in\mathbb{N}}$ of eigenvalues of the fractional Laplacian, satisfying

$$0 < \lambda_1(\Omega) < \lambda_2(\Omega) \le \cdots \le \lambda_k(\Omega) \to +\infty \qquad \text{as } k \to +\infty.$$

The first eigenvalue $\lambda_1(\Omega)$ is characterized as

$$\lambda_1(\Omega) = \inf_{u \in H_0^s(\Omega)\setminus\{0\}} \frac{[u]^2_{H^s(\mathbb{R}^N)}}{\|u\|^2_{L^2(\mathbb{R}^N)}}$$

and the associated first eigenfunction is unique (up to multiplicative constant) and strictly positive (or negative) in Ω.

Eigenfunctions satisfy the following regularity property.

Proposition 2.9 *Let $\Omega \subset \mathbb{R}^N$ be a s-quasi-open set of finite Lebesgue measure, and let $u \in H_0^s(\Omega)$ be an eigenfunction of the fractional Laplacian. Then, $u \in L^\infty(\Omega)$.*

Proof. The proof can be performed as in [17, Theorem 3.2] taking into account Theorems 6.5 and 6.9 from [12]. □

3 The concentration-compactness principle

This section deals with a nonlocal version of Lions' concentration-compactness principle. More particularly, we provide a proof of relation (8) which relies on some computations performed in [6], and which differs from other results that can be found in the literature such as [15].

Proposition 3.1 *Let $\{u_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $H^s(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} |u_n|^2 \to \lambda$ for $n \to +\infty$. Then there exists a subsequence $\{n_k\}_{k\in\mathbb{N}}$ such that one of the following three cases occur:*

(i) *Compactness: there exists $\{y_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^N$ such that*

$$\forall \varepsilon > 0,\ \exists R < +\infty \text{ s.t. } \int_{y_k + B_R} |u_{n_k}|^2 \ge \lambda - \varepsilon.$$

(ii) *Vanishing:*

$$\lim_{k\to+\infty}\sup_{y\in\mathbb{R}^N}\int_{y+B_R}|u_{n_k}|^2=0\qquad \forall R>0.$$

(iii) *Dichotomy: there exists $\alpha\in(0,\lambda)$, such that for all $\varepsilon>0$, there exist $k_0\in\mathbb{N}$, $\{v_k\}_{k\in\mathbb{N}}$, $\{w_k\}_{k\in\mathbb{N}}\subset H^s(\mathbb{R}^N)$ such that, for $k\geq k_0$:*

$$\|u_{n_k}-v_k-w_k\|_{L^2(\mathbb{R}^N)}\leq\delta(\varepsilon)\to 0\qquad \textit{for } \varepsilon\to 0;$$

$$\left|\int_{\mathbb{R}^N}|v_k|^2-\alpha\right|\leq\varepsilon,\qquad \left|\int_{\mathbb{R}^N}|w_k|^2-(\lambda-\alpha)\right|\leq\varepsilon;$$

$$\operatorname{dist}(\operatorname{supp} v_k,\operatorname{supp} w_k)\to+\infty\qquad \textit{for } k\to+\infty;$$

$$[u_{n_k}]^2_{H^s(\mathbb{R}^N)}-[v_k]^2_{H^s(\mathbb{R}^N)}-[w_k]^2_{H^s(\mathbb{R}^N)}\geq-2\varepsilon. \tag{8}$$

Proof. All the assertions of this theorem, with exception of (8), follow from the classical concentration-compactness lemma [21, Lemma I.1]. To prove (8), we suitably modify [21, Lemma III.1]. Let $\varepsilon>0$, and let $R_0>0$ be chosen as in [21, Lemma III.1]. Let us define two cut-off functions $\varphi,\psi\in C^\infty(\mathbb{R}^N)$ satisfying $0\leq\varphi,\ \psi\leq 1$, $\varphi\equiv 1$ on B_1, $\varphi\equiv 0$ on $\mathbb{R}^N\setminus B_2$ and $\psi\equiv 0$ on B_1, $\psi\equiv 1$ on $\mathbb{R}^N\setminus B_2$. Denote by φ_R, ψ_R the functions defined by

$$\varphi_R(x):=\varphi\left(\frac{x}{R}\right),\qquad \psi_R(x):=\psi\left(\frac{x}{R}\right). \tag{9}$$

For any function $u\in H^s(\mathbb{R}^N)$ with $[u]_{H^s(\mathbb{R}^N)}\leq M$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi_R(x)u(x)-\varphi_R(y)u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\
&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi_R(x)u(x)+\varphi_R(x)u(y)-\varphi_R(x)u(y)-\varphi_R(y)u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\
&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\varphi_R(x)|^2\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u(y)|^2\frac{|\varphi_R(x)-\varphi_R(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\
&\quad+2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\varphi_R(x)u(y)[\varphi_R(x)-\varphi_R(y)][u(x)-u(y)]}{|x-y|^{N+2s}}\,dx\,dy.
\end{aligned}$$

By the computations in [6, Lemma A.2], it is possible to estimate

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u(y)|^2\frac{|\varphi_R(x)-\varphi_R(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\leq\frac{C}{R^{2s}},$$

where C only depends on $\|\nabla\varphi\|_\infty$ and $\|u\|_{L^2(\mathbb{R}^N)}$. Moreover, the Cauchy-Schwarz inequality together with the last inequality gives that

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\varphi_R(x)u(y)[\varphi_R(x)-\varphi_R(y)][u(x)-u(y)]}{|x-y|^{N+2s}}\,dx\,dy\\
&\leq\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^2|\varphi_R(x)-\varphi_R(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\right)^{\frac12}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi_R(x)|^2|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\right)^{\frac12}\\
&\leq\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^2|\varphi_R(x)-\varphi_R(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\right)^{\frac12}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\right)^{\frac12}\\
&\leq\frac{C}{R^s},
\end{aligned}$$

where C only depends on $\|\nabla\varphi\|_\infty$, $\|u\|_{L^2(\mathbb{R}^N)}$, and $[u]_{H^s(\mathbb{R}^N)}$.

Similar computations hold true for the quantity

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\psi_R(x)u(x)-\psi_R(y)u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy.$$

Therefore it is possible to choose $R_1 \geq R_0$ such that, for $R \geq R_1$, and for every $n \in \mathbb{N}$,

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi_R(x)u_n(x)-\varphi_R(y)u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi_R(x)|^2|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\right|\leq\varepsilon,$$

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\psi_R(x)u_n(x)-\psi_R(y)u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\psi_R(x)|^2|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\right|\leq\varepsilon.$$

The claim follows defining

$$v_k(x)=\varphi_{R_1}(x-y_k)u_{n_k}(x),\qquad w_k(x)=\psi_{R_k}(x-y_k)u_{n_k}(x),$$

where y_k and $R_k \to +\infty$ are defined as in [21, pp 136-137] and observing that

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_{n_k}(x)-u_{n_k}(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\ &\geq\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi_{R_1}(x)|^2|u_{n_k}(x)-u_{n_k}(y)|^2}{|x-y|^{N+2s}}\,dx\,dy+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\psi_{R_k}(x)|^2|u_{n_k}(x)-u_{n_k}(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\end{aligned}$$

since φ_{R_1} and ψ_{R_k} have disjoint support for k big enough, and therefore

$$|\varphi_{R_1}(x)|^2+|\psi_{R_k}(x)|^2\leq 1\qquad\text{for every } x\in\mathbb{R}^N.$$

□

Corollary 3.2 *In the dichotomy case, it is possible to find sequences* $\{u_k^{(1)}\}_{k\in\mathbb{N}}$, $\{u_k^{(2)}\}_{k\in\mathbb{N}} \subset H^s(\mathbb{R}^N)$ *such that*

$$\|u_{n_k}-u_k^{(1)}-u_k^{(2)}\|_{L^2(\mathbb{R}^N)}\to 0\qquad\textit{for } k\to+\infty;$$

$$\int_{\mathbb{R}^N}|u_k^{(1)}|^2\to\alpha,\qquad\int_{\mathbb{R}^N}|u_k^{(2)}|^2\to\lambda-\alpha\qquad\textit{for } k\to+\infty;$$

$$\operatorname{dist}(\operatorname{supp}u_k^{(1)},\operatorname{supp}u_k^{(2)})\to+\infty\qquad\textit{for } k\to+\infty;$$

$$\liminf_{k\to+\infty}\left([u_{n_k}]^2_{H^s(\mathbb{R}^N)}-[u_k^{(1)}]^2_{H^s(\mathbb{R}^N)}-[u_k^{(2)}]^2_{H^s(\mathbb{R}^N)}\right)\geq 0.\tag{10}$$

4 γ-convergence of sets

In this section we introduce the notions of γ-convergence and weak γ-convergence of sets, and we prove some useful results leading to our main theorem.

4.1 Convergence of sets

In this subsection we prove that a functional J defined in $\mathcal{A}(\mathbb{R}^N)$ which is l.s.c. with respect to the γ-convergence is also l.s.c. with respect to the weak γ-convergence if it is assumed to be decreasing with respect to the inclusion of sets.

Definition 4.1 Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be a sequence of s-quasi-open sets such that $|\Omega_n| \le c$ for every $n \in \mathbb{N}$. We say that $\{\Omega_n\}_{n\in\mathbb{N}}$ *γ-converges* to the s-quasi-open set Ω if the solutions $w_{\Omega_n} \in H_0^s(\Omega_n)$ of the problems

$$\begin{cases} (-\Delta)^s w_{\Omega_n} = 1 & \text{in } \Omega_n, \\ w_{\Omega_n} = 0 & \text{in } \mathbb{R}^N \setminus \Omega_n, \end{cases} \tag{11}$$

strongly converge in $L^2(\mathbb{R}^N)$ to the solution $w_\Omega \in H_0^s(\Omega)$ of the problem

$$\begin{cases} (-\Delta)^s w_{\Omega} = 1 & \text{in } \Omega, \\ w_{\Omega} = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Definition 4.2 Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be a sequence of s-quasi-open sets. We say that $\{\Omega_n\}_{n\in\mathbb{N}}$ *weakly γ-converges* to the s-quasi-open set Ω if the solutions $w_{\Omega_n} \in H_0^s(\Omega_n)$ of the problems

$$\begin{cases} (-\Delta)^s w_{\Omega_n} = 1 & \text{in } \Omega_n, \\ w_{\Omega_n} = 0 & \text{in } \mathbb{R}^N \setminus \Omega_n, \end{cases} \tag{12}$$

converge weakly in $H^s(\mathbb{R}^N)$, and strongly in $L^2(\mathbb{R}^N)$, to a function $w \in H^s(\mathbb{R}^N)$ such that $\Omega = \{w > 0\}$.

Proposition 4.3 *Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be a sequence of s-quasi-open sets of uniformly bounded measure which γ-converges to the s-quasi-open set Ω. Then $\{\Omega_n\}_{n\in\mathbb{N}}$ weakly γ-converges to Ω.*

P r o o f. By definition of γ-convergence, $w_{\Omega_n} \to w_\Omega$ in $L^2(\mathbb{R}^N)$. By Proposition 2.8, $\Omega = \{w_\Omega > 0\}$ s-q.e., which means that $\{\Omega_n\}_{n\in\mathbb{N}}$ weakly γ-converges to Ω. □

Proposition 4.4 *Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be a sequence of s-quasi-open sets of uniformly bounded measure, which weakly γ-converges to the s-quasi-open set Ω. Then,*

$$|\Omega| \le \liminf_{n\to+\infty} |\Omega_n|.$$

P r o o f. Let $m := \liminf_{n\to+\infty} |\Omega_n|$. Up to extracting a subsequence, we can suppose that $m = \lim_{n\to+\infty} |\Omega_n|$. Let $w_{\Omega_n} \in H_0^s(\Omega_n)$ be the sequence of torsion functions defined in (12). Since $w_{\Omega_n} \to w$ strongly in $L^2(\mathbb{R}^N)$, there exists a subsequence $w_{\Omega_{n_k}}$ such that $w_{\Omega_{n_k}}$ converges almost everywhere in $\mathbb{R}^N$ to w. Since $\Omega = \{w > 0\}$, it holds $\chi_\Omega \le \liminf_{k\to+\infty} \chi_{\Omega_{n_k}}$ almost everywhere in $\mathbb{R}^N$. By Fatou's Lemma,

$$|\Omega| = \int_{\mathbb{R}^N} \chi_\Omega \le \liminf_{k\to+\infty} \int_{\mathbb{R}^N} \chi_{\Omega_{n_k}} = m$$

as required. □

Remark 4.5 We observe that, if $\{\Omega_n\}_{n\in\mathbb{N}}$ are s-quasi-open sets, with $|\Omega_n| \le c$, which γ-converge to Ω, then $w_{\Omega_n} \to w_\Omega$ strongly in $H^s(\mathbb{R}^N)$. Indeed, by Propositions 4.3 and 4.4, one has $|\Omega| \le c$. Therefore

$$\begin{aligned} \int_{\Omega_n} w_{\Omega_n} - \int_\Omega w_\Omega &\le \int_{\Omega_n\setminus\Omega} w_{\Omega_n} + \int_{\Omega_n\cap\Omega} |w_{\Omega_n} - w_\Omega| + \int_{\Omega\setminus\Omega_n} w_\Omega \\ &\le \int_{\Omega_n\cup\Omega} |w_{\Omega_n} - w_\Omega| \le (2c)^{\frac12} \|w_{\Omega_n} - w_\Omega\|_{L^2(\mathbb{R}^N)} \end{aligned}$$

and therefore

$$\lim_{n\to+\infty} \int_{\Omega_n} w_{\Omega_n} = \int_\Omega w_\Omega.$$

Passing to the limit in the weak formulation, we obtain

$$[w_{\Omega_n}]^2_{H^s(\mathbb{R}^N)} = \int_{\Omega_n} w_{\Omega_n} \to \int_\Omega w_\Omega = [w_\Omega]^2_{H^s(\mathbb{R}^N)}$$

and therefore, by reflexivity of $H^s(\mathbb{R}^N)$, $w_{\Omega_n} \to w_\Omega$ strongly in $H^s(\mathbb{R}^N)$.

Lemma 4.6 *Suppose that $\{\Omega_n\}_{n\in\mathbb{N}}$ is a sequence of s-quasi-open sets of uniformly bounded measure which weakly γ-converges to the s-quasi-open set Ω. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $H^s(\mathbb{R}^N)$ such that $u_n \in H_0^s(\Omega_n)$ for every $n \in \mathbb{N}$, and $u_n \rightharpoonup u$ weakly in $H^s(\mathbb{R}^N)$. Then, $u \in H_0^s(\Omega)$.*

Proof. The proof can be performed as in [18, Lemma 4.7.10]. Let $w \in H^s(\mathbb{R}^N)$ be such that w_{Ω_n} converge to w weakly in $H^s(\mathbb{R}^N)$, and strongly in $L^2(\mathbb{R}^N)$. Since it is enough to show that $\widetilde{u} := \min\{|u|, k\} \in H_0^s(\Omega)$ for every $k > 0$, and $\widetilde{u}$ is the weak limit of $\widetilde{u}_n := \min\{|u_n|, k\} \in H_0^s(\Omega_n)$, we may assume that the functions u_n are nonnegative, and such that $\|u_n\|_{L^\infty}$ is uniformly bounded by a constant $k > 0$. For fixed $\lambda > 0$, let v_n^λ be the weak solution of

$$\begin{cases} \lambda(-\Delta)^s v_n^\lambda + v_n^\lambda &= u_n \quad \text{in } \Omega_n, \\ v_n^\lambda &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega_n, \end{cases} \tag{13}$$

namely, the function $v_n^\lambda \in H_0^s(\Omega_n)$ satisfying, for every $\varphi \in H_0^s(\Omega_n)$,

$$\lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(v_n^\lambda(x) - v_n^\lambda(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}}\,dx\,dy + \int_\Omega v_n^\lambda \varphi = \int_\Omega u_n \varphi.$$

By the weak maximum principle given in Proposition 2.4, $v_n^\lambda \geq 0$. Choosing $\varphi = v_n^\lambda - u_n$, we obtain

$$\begin{aligned} &\lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(v_n^\lambda - u_n)(x) - (v_n^\lambda - u_n)(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} |v_n^\lambda - u_n|^2 \\ &= -\lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u_n(x) - u_n(y)][(v_n^\lambda - u_n)(x) - (v_n^\lambda - u_n)(y)]}{|x-y|^{N+2s}}\,dx\,dy \\ &\leq \frac{\lambda}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(v_n^\lambda - u_n)(x) - (v_n^\lambda - u_n)(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \frac{\lambda}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \end{aligned} \tag{14}$$

which implies

$$\begin{aligned} &\frac{\lambda}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(v_n^\lambda - u_n)(x) - (v_n^\lambda - u_n)(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} |v_n^\lambda - u_n|^2 \\ &\leq \frac{\lambda}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \end{aligned}$$

and hence the boundedness of $\|v_n^\lambda\|_{H^s(\mathbb{R}^n)}$ by a constant C depending only on $\sup_{n\in\mathbb{N}} \|u_n\|_{H^s(\mathbb{R}^N)}$ in light of Proposition 2.1. After subtracting k from both sides of (13), and choosing $\varphi = (v_n^\lambda - k)^+$, we obtain

$$\begin{aligned} &\lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(v_n^\lambda - k)^+(x) - (v_n^\lambda - k)^+(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} |(v_n^\lambda - k)^+|^2 \\ &\leq \lambda \int_{\{v_n^\lambda(x)>k\}}\int_{\{v_n^\lambda(y)>k\}} \frac{|(v_n^\lambda - k)^+(x) - (v_n^\lambda - k)^+(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \\ &- 2\lambda \int_{\{v_n^\lambda(x)<k\}}\int_{\{v_n^\lambda(y)>k\}} \frac{(v_n^\lambda - k)(x)(v_n^\lambda - k)^+(y)}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} |(v_n^\lambda - k)^+|^2 \\ &= \int_{\mathbb{R}^N} (u_n - k)(v_n^\lambda - k) \leq 0 \end{aligned}$$

which implies that $\|v_n^\lambda\|_{L^\infty} \leq k$ for every $n \in \mathbb{N}$ and every $\lambda > 0$. Since v_n^λ solves (13), we obtain the bound $0 \leq v_n^\lambda \leq \frac{2k}{\lambda} w_{\Omega_n}$. If v^λ is a weak limit in $H^s(\mathbb{R}^N)$ of $\{v_n^\lambda\}_{n\in\mathbb{N}}$, we obtain $0 \leq v^\lambda \leq \frac{2k}{\lambda} w$, which implies that $v^\lambda \in H_0^s(\Omega)$. From (14) we have $\|v_n^\lambda - u_n\|^2_{L^2(\mathbb{R}^N)} \leq C\lambda$, which implies $\|v^\lambda - u\|^2_{L^2(\mathbb{R}^N)} \leq C\lambda$. Passing to the limit for $\lambda \to 0$, we get that $v^\lambda \to u$ in $L^2(\mathbb{R}^N)$; since $\{v_\lambda\}_{\lambda>0}$ is uniformly bounded in $H^s(\mathbb{R}^N)$, the convergence is also weak in $H^s(\mathbb{R}^N)$, which lastly implies that $u \in H_0^s(\Omega)$. □

4.2 γ-convergence and continuity of the spectrum

Here we prove that γ-convergence of s-quasi-open sets implies the convergence of their resolvent operators in the $\mathcal{L}(L^2(\mathbb{R}^N))$ norm. In particular we obtain continuity of the spectrum with respect to the γ-convergence.

Proposition 4.7 *Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be a sequence of s-quasi-open sets of uniformly bounded measure, which γ-converges to the s-quasi-open set Ω. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $H^s(\mathbb{R}^N)$ such that $u_n \in H_0^s(\Omega_n)$ for every $n \in \mathbb{N}$, and $u_n \rightharpoonup u$ weakly in $H^s(\mathbb{R}^N)$. Then, $u_n \to u$ strongly in $L^2(\mathbb{R}^N)$.*

Proof. The proof goes as in [7, Theorem 2.1]. Denoting by $\mathcal{F}u_n$, $\mathcal{F}u$ the Fourier transforms of u_n and u respectively, for $R > 0$ we have that

$$\begin{aligned}\|u_n - u\|^2_{L^2(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} |\mathcal{F}u_n(\xi) - \mathcal{F}u(\xi)|^2 \, d\xi \\ &= \int_{|\xi|\geq R} (1+|\xi|^{2s})^{-1}(1+|\xi|^{2s})|\mathcal{F}u_n(\xi) - \mathcal{F}u(\xi)|^2 \, d\xi + \int_{|\xi|<R} |\mathcal{F}u_n(\xi) - \mathcal{F}u(\xi)|^2 \, d\xi \\ &\leq \frac{C_{s,N}}{1+R^{2s}} \|u_n - u\|^2_{H^s(\mathbb{R}^N)} + \int_{|\xi|<R} |\mathcal{F}u_n(\xi) - \mathcal{F}u(\xi)|^2 \, d\xi,\end{aligned}$$

where the constant $C_{s,N}$ is the equivalence norm constant given [12, Proposition 3.4]. Let $\varepsilon > 0$ be fixed. Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H^s(\mathbb{R}^N)$, there exists $R > 0$ such that, for every $n \in \mathbb{N}$,

$$\frac{C_{s,N}}{1+R^{2s}} \|u_n - u\|^2_{H^s(\mathbb{R}^N)} < \frac{\varepsilon}{2}.$$

It remains to prove that

$$\int_{|\xi|<R} |\mathcal{F}u_n(\xi) - \mathcal{F}u(\xi)|^2 \, d\xi \to 0$$

as $n \to +\infty$. For $\xi \in B_R$, define the complex-valued function $g_\xi : \mathbb{R}^N \to \mathbb{C}$ as $g_\xi(x) = e^{2\pi i \langle x, \xi\rangle}$. By Proposition 2.3 applied to the real and imaginary parts of g_ξ, it holds that $ug_\xi \in H_0^s(\Omega; \mathbb{C})$ and $u_n g_\xi \in H_0^s(\Omega_n; \mathbb{C})$ for every $n \in \mathbb{N}$, and $u_n g_\xi \rightharpoonup u g_\xi$ weakly in $H^s(\mathbb{R}^N; \mathbb{C})$ as $n \to +\infty$.

Let $w_{\Omega_n} \in H_0^s(\Omega_n)$ be the solution of (11). Testing this equation with $u_n g_\xi$, we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(w_{\Omega_n}(x) - w_{\Omega_n}(y))(u_n(x)g_\xi(x) - u_n(y)g_\xi(y))}{|x-y|^{N+2s}} \, dx \, dy = \int_{\mathbb{R}^N} u_n(x) g_\xi(x) \, dx.$$

Letting $n \to +\infty$ and observing that $w_{\Omega_n} \to w_\Omega$ strongly in $H^s(\mathbb{R}^N)$ by Remark 4.5, we obtain

$$\int_{\mathbb{R}^N} u_n(x) g_\xi(x) \, dx \to \int_{\mathbb{R}^N} u(x) g_\xi(x) \, dx$$

as $n \to +\infty$. Observing that

$$\mathcal{F}u_n(\xi) = \int_{\Omega_n} u_n(x) g_\xi(x) \, dx$$

and

$$\mathcal{F}u(\xi) = \int_{\Omega} u(x) g_\xi(x) \, dx,$$

we have $|\mathcal{F}u_n(\xi) - \mathcal{F}u(\xi)| \to 0$ as $n \to +\infty$. Moreover,

$$|\mathcal{F}u_n(\xi)| \leq \int_{\Omega_n} |u_n(x)| \, dx \leq |\Omega_n|^{\frac{1}{2}} \|u_n\|_{L^2(\mathbb{R}^N)},$$

and a similar relation holds for $\mathcal{F}u$. Therefore, $\mathcal{F}u_n$ and $\mathcal{F}u$ are uniformly bounded in L^∞. Applying Lebesgue's dominated convergence Theorem we get

$$\int_{|\xi|<R} |\mathcal{F}u_n(\xi) - \mathcal{F}u(\xi)|^2 \, d\xi \to 0$$

and hence the claim. □

Proposition 4.8 *Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be a sequence of s-quasi-open sets such that $|\Omega_n| \le c$ for every $n \in \mathbb{N}$. Suppose that $\{\Omega_n\}_{n\in\mathbb{N}}$ γ-converges to the s-quasi-open set Ω. Then, for every sequence $f_n \in L^2(\Omega_n)$ converging weakly in $L^2(\mathbb{R}^N)$ to $f \in L^2(\Omega)$, the solutions $u_n \in H^s(\mathbb{R}^N)$ of the problems*

$$\begin{cases} (-\Delta)^s u_n = f_n & \text{in } \Omega_n, \\ u_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega_n, \end{cases}$$

strongly converge in $L^2(\mathbb{R}^N)$ to the solution $u \in H^s(\mathbb{R}^N)$ of the problem

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{15}$$

Proof. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(\mathbb{R}^N)$. Since $|\Omega_n| \le c$ for every $n \in \mathbb{N}$, it easily follows from Cauchy-Schwarz inequality and Poincaré's inequality (see Proposition 2.1) that $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $H^s(\mathbb{R}^N)$. Let $v \in H^s(\mathbb{R}^N)$ be a weak limit of a subsequence of $\{u_n\}_{n\in\mathbb{N}}$. We will prove that $v = u$. Let $\varphi \in Z(\Omega)$, as defined in Proposition 2.7, a nonnegative function. Let $\varphi_n \in H_0^s(\Omega_n)$ be defined as $\varphi_n = \min\{\varphi, m w_{\Omega_n}\}$, where $m \ge \|g\|_{L^\infty(\mathbb{R}^N)}$ and $g = (-\Delta)^s \varphi \in L^\infty(\mathbb{R}^N)$. By Remark 4.5, $w_{\Omega_n} \to w_\Omega$ strongly in $H^s(\mathbb{R}^N)$, and therefore $\varphi_n \to \varphi$ strongly in $H^s(\mathbb{R}^N)$, since $0 \le \varphi \le m w_\Omega$. Exploiting the weak formulation of the equation, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_n(x)-u_n(y))(\varphi_n(x)-\varphi_n(y))}{|x-y|^{N+2s}} \, dx \, dy = \int_\Omega f_n(x)\varphi_n(x) \, dx.$$

Passing to the limit as $n \to +\infty$, we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} \, dx \, dy = \int_\Omega f(x)\varphi(x) \, dx. \tag{16}$$

and since the functions $\varphi \in W$ are dense in $H_0^s(\Omega)$, it follows that (16) holds true for every $\varphi \in H_0^s(\Omega)$, with $\varphi \ge 0$. This means that v is a weak sub- and supersolution, and hence the weak solution, of (15), which implies $v = u$. Therefore, the whole sequence $\{u_n\}_{n\in\mathbb{N}}$ converges weakly in $H^s(\mathbb{R}^N)$ to u. Finally, by Proposition 4.7, $u_n \to u$ strongly in $L^2(\mathbb{R}^N)$. □

Proposition 4.9 *Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be a sequence of s-quasi-open sets such that $|\Omega_n| \le c$ for every $n \in \mathbb{N}$. Suppose that $\{\Omega_n\}_{n\in\mathbb{N}}$ γ-converges to the s-quasi-open set Ω. Then, the resolvents R_{Ω_n} converge to R_Ω in $\mathcal{L}(L^2(\mathbb{R}^N))$. In particular, for every $k \ge 1$,*

$$\lambda_k(\Omega_n) \to \lambda_k(\Omega) \qquad \text{as } n \to +\infty.$$

Proof. We have to show that

$$\lim_{n\to+\infty} \sup\left\{\|R_{\Omega_n}(f) - R_\Omega(f)\|_{L^2(\mathbb{R}^N)} \,\middle|\, f \in L^2(\mathbb{R}^N),\ \|f\|_{L^2(\mathbb{R}^N)} \le 1\right\} = 0.$$

It is equivalent to prove that, for every sequence $\{f_n\}_{n\in\mathbb{N}}$ such that $\|f_n\|_{L^2(\mathbb{R}^N)} = 1$, the following limit holds

$$\lim_{n\to+\infty} \|R_{\Omega_n}(f_n) - R_\Omega(f_n)\|_{L^2(\mathbb{R}^N)} = 0.$$

Let $\{f_n\}_{n\in\mathbb{N}}$ be such a sequence. Without loss of generality, we can suppose that there exists $f \in L^2(\mathbb{R}^N)$ such that $f_n \rightharpoonup f$ in $L^2(\mathbb{R}^N)$. By the triangular inequality we get

$$\limsup_{n\to+\infty} \|R_{\Omega_n}(f_n) - R_\Omega(f_n)\|_{L^2(\mathbb{R}^N)} \leq \limsup_{n\to+\infty} \|R_{\Omega_n}(f_n) - R_\Omega(f)\|_{L^2(\mathbb{R}^N)} + \limsup_{n\to+\infty} \|R_\Omega(f_n) - R_\Omega(f)\|_{L^2(\mathbb{R}^N)}.$$

The first term in the previous inequality is equal to zero by Proposition 4.8, while the second term is also zero since the injection $H_0^s(\Omega) \to L^2(\Omega)$ is compact due to Proposition 2.1. By [13, Corollary XI.9.4], we have, for every $k \geq 1$,

$$\left| \frac{1}{\lambda_k(\Omega_n)} - \frac{1}{\lambda_k(\Omega)} \right| \leq \|R_{\Omega_n} - R_\Omega\|_{\mathcal{L}(L^2(\mathbb{R}^N))} \tag{17}$$

and hence

$$\lambda_k(\Omega_n) \to \lambda_k(\Omega) \qquad \text{as } n \to +\infty,$$

concluding the proof. □

Remark 4.10 When $\Omega = \emptyset$ s-quasi-everywhere, by definition $H_0^s(\Omega) = \{0\}$, R_Ω is the null operator, and formally $\lambda_k(\Omega) = +\infty$ for every $k \geq 1$. In this case, (17) becomes

$$0 \leq \frac{1}{\lambda_k(\Omega_n)} \leq \|R_{\Omega_n}\|_{\mathcal{L}(L^2(\mathbb{R}^N))}. \tag{18}$$

In other words, if Ω_n γ-converges to the empty set, then $\lambda_k(\Omega_n) \to +\infty$ for every $k \geq 1$. Conversely, if Ω is a s-quasi-open set such that $w_\Omega = 0$, then $(-\Delta)^s w_\Omega = 0$ in Ω, and therefore $\Omega = \emptyset$ s-quasi-everywhere.

5 Proof of Theorem 1.1

In the following, $\{\Omega_n\}_{n\in\mathbb{N}}$ will be a sequence of s-quasi-open sets of uniformly bounded measure. The proof of Theorem 1.1, which will be performed in several steps, is based on the behavior of the sequence $\{w_{\Omega_n}\}_{n\in\mathbb{N}}$ according to the concentration-compactness principle stated in Proposition 3.1. Without loss of generality, we can suppose that $\int_{\mathbb{R}^N} |w_{\Omega_n}|^2 \to \lambda$ as $n \to +\infty$ for some $\lambda > 0$.

5.1 Compactness for w_{Ω_n}

Assume that $\{w_{\Omega_n}\}_{n\in\mathbb{N}}$ is in the compactness case, that is, up to some subsequence still denoted with the same index, one can find a sequence $\{y_n\}_{n\in\mathbb{N}}$ such that the sequence $\{w_{y_n+\Omega_n}\}_{n\in\mathbb{N}}$ converges strongly in $L^2(\mathbb{R}^N)$ to some $w \in H^s(\mathbb{R}^N)$. Then, by definition, $y_n + \Omega_n$ weakly γ-converges to the set $\Omega := \{w > 0\}$.

5.2 Vanishing for w_{Ω_n}

In the spirit of [20] we prove the following lemma.

Lemma 5.1 *Let A and B be two measurable sets. Then there exists $z \in \mathbb{R}^N$ such that, if $A_z = z + A$,*

$$\lambda_1(A_z \cap B) \leq 2(\lambda_1(A) + \lambda_1(B)).$$

Proof. Let $z \in \mathbb{R}^N$ be arbitrary and let u and v be positive first eigenfunctions on A and B respectively, normalized such that $\|u\|_{L^2(A)} = \|v\|_{L^2(B)} = 1$. By regularity, the function u_z defined by $u_z(x) = u(z - x)$ satisfies $u_z \in H_0^s(A_z) \cap L^\infty(A_z)$, and $v \in H_0^s(B) \cap L^\infty(B)$. The function w_z defined as $w_z(x) = u(x-z)v(x)$ belongs to $H_0^s(A_z \cap B) \cap L^\infty(A_z \cap B)$. Define

$$T(z) := [w_z]^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w_z(x) - w_z(y)|^2}{|x-y|^{N+2s}}\, dx\, dy, \qquad D(z) := \int_{\mathbb{R}^N} |w_z(x)|^2\, dx.$$

It holds that

$$\int_{\mathbb{R}^N} D(z)\,dz = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |w_z(x)|^2\,dx\,dz = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x-z)v(x)|^2\,dx\,dz = 1.$$

Moreover,

$$\begin{aligned}
&|w_z(x) - w_z(y)|^2 \\
&= |u(x-z)v(x) - u(y-z)v(y)|^2 \\
&= |u(x-z)v(x) - u(x-z)v(y) + u(x-z)v(y) - u(y-z)v(y)|^2 \\
&= |u(x-z)|^2|v(x)-v(y)|^2 + |v(y)|^2|u(x-z)-u(y-z)|^2 \\
&\quad + 2u(x-z)v(y)[v(x)-v(y)][u(x-z)-u(y-z)].
\end{aligned}$$

Using the elementary inequality $2ab \le a^2 + b^2$, the last term in the inequality above can be bounded as

$$|u(x-z)|^2|v(x)-v(y)|^2 + |v(y)|^2|u(x-z)-u(y-z)|^2,$$

and from the last two expressions we get

$$|w_z(x) - w_z(y)|^2 \le 2\left(|u(x-z)|^2|v(x)-v(y)|^2 + |v(y)|^2|u(x-z)-u(y-z)|^2\right).$$

Thus

$$T(z) \le 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x-z)|^2|v(x)-v(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(y)|^2|u(x-z)-u(y-z)|^2}{|x-y|^{N+2s}}\,dx\,dy.$$

Then, integrating over z and performing a change of variables, since u and v are normalized in L^2 norm, we get

$$\int_{\mathbb{R}^N} T(z)\,dz \le 2(\lambda_1(A) + \lambda_1(B)) := \Lambda.$$

Therefore, $\int_{\mathbb{R}^N}[T(z) - \Lambda D(z)]\,dz \le 0$, hence $0 \le T(z) \le \Lambda D(z)$ on a set of positive measure. From the definitions of T, D and Λ the lemma follows. □

Assume that $\{w_{\Omega_n}\}_{n\in\mathbb{N}}$ is in the vanishing case, that is, for all $R > 0$ it holds that

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{y+B_R} |w_{\Omega_n}|^2 = 0.$$

Since the sequence $\{w_{\Omega_n}\}_{n\in\mathbb{N}} \subset H^s(\mathbb{R}^N)$ is bounded, we can assume that $w_{\Omega_n} \rightharpoonup w$ weakly in $H^s(\mathbb{R}^N)$. Fix $\varepsilon > 0$. By Lemma 5.1, there exists $R > 0$ and a sequence $\{y_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ such that

$$\lambda_1((y_n + \Omega_n) \cap B_R) \le 2\lambda_1(\Omega_n) + \varepsilon. \tag{19}$$

From the weak maximum principle (Proposition 2.4) it follows that $w_{y_n+\Omega_n} \ge w_{(y_n+\Omega_n)\cap B_R} \ge 0$, and then, the vanishing assumption on w_{Ω_n} gives that

$$\lim_{n\to+\infty} \int_{B_R} |w_{(y_n+\Omega_n)\cap B_R}|^2 = 0.$$

This means that $w_{(y_n+\Omega_n)\cap B_R} \to 0$ strongly in $L^2(\mathbb{R}^N)$, and therefore $(y_n + \Omega_n) \cap B_R$ $\gamma-$converges to the empty set. By Remark 4.10,

$$\lambda_1((y_n + \Omega_n) \cap B_R) \to +\infty \qquad \text{as } n \to +\infty.$$

By (19) we obtain that

$$\lambda_1(\Omega_n) \to +\infty \quad \text{as } n \to +\infty.$$

From the Poincaré inequality given in Proposition 2.1 we find that

$$\|w_{\Omega_n}\|_{L^2(\Omega_n)} \le \frac{1}{\lambda_1(\Omega_n)}[w_{\Omega_n}]_{H^s(\mathbb{R}^N)} \to 0 \quad \text{as } n \to +\infty$$

since $\{w_{\Omega_n}\}_{n\in\mathbb{N}}$ is bounded in $H^s(\mathbb{R}^N)$. Finally, by Proposition 4.9 and Remark 4.10 we obtain that $\|R_{\Omega_n}\|_{\mathcal{L}(L^2(\mathbb{R}^N))} \to 0$. By definition, the sequence $\{\Omega_n\}_{n\in\mathbb{N}}$ γ-converges, and hence weakly γ-converges, to the empty set.

5.3 Dichotomy for w_{Ω_n}

Finally, suppose that w_{Ω_n} is in the dichotomy case. That means that it is possible to find two sequences $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ of functions in $H^s_0(\Omega_n)$ and a number $\alpha\in(0,\lambda)$ such that, up to a subsequence,

$$\|w_{\Omega_n}-u_n-v_n\|_{L^2(\mathbb{R}^N)}\to 0 \qquad \text{as } n\to+\infty;$$

$$\int_{\mathbb{R}^N}u_n^2\to\alpha, \qquad \int_{\mathbb{R}^N}v_n^2\to\lambda-\alpha \qquad \text{for } n\to+\infty;$$

$$\operatorname{dist}(\operatorname{supp}u_n,\operatorname{supp}v_n)\to+\infty \qquad \text{for } n\to+\infty;$$

$$\liminf_{n\to+\infty}\Big([w_{\Omega_n}]^2_{H^s(\mathbb{R}^N)}-[u_n]^2_{H^s(\mathbb{R}^N)}-[v_n]^2_{H^s(\mathbb{R}^N)}\Big)\geq 0. \tag{20}$$

Looking at the proof of Proposition 3.1, we observe that by construction the functions u_n and v_n are nonnegative, since w_{Ω_n} is nonnegative by the weak maximum principle. We define the following sets

$$\Omega^1_n:=\{u_n>0\}, \quad \Omega^2_n:=\{v_n>0\}, \quad \widetilde{\Omega}_n:=\Omega^1_n\cup\Omega^2_n, \tag{21}$$

and then $\widetilde{\Omega}_n$ is a s-quasi-open set contained in Ω_n.

The proof of the claims in the dichotomy case will be a consequence of the following results.

Lemma 5.2 *The sequence of sets* (21) *satisfies*

$$\liminf_{n\to+\infty}|\Omega^i_n|>0 \qquad \textit{for } i=1,2.$$

P r o o f. Suppose by contradiction that, for instance, $\liminf_{n\to+\infty}|\Omega^1_n|=0$. The functions w_{Ω_n} are uniformly bounded in L^∞ by [5, Theorem 3.1], and therefore, by construction, also the functions u_n are uniformly bounded in L^∞. But then, $\int_{\mathbb{R}^N}u_n^2\to0$, which contradicts the fact that $\int_{\mathbb{R}^N}u_n^2\to\alpha>0$. □

Lemma 5.3 *With the previous notation, we have that*

$$\|w_{\Omega_n}-w_{\widetilde{\Omega}_n}\|_{H^s(\mathbb{R}^N)}\to0 \quad \textit{as } n\to+\infty.$$

P r o o f. We observe that $w_{\widetilde{\Omega}_n}$ is the orthogonal projection of w_{Ω_n} on the space $H^s_0(\widetilde{\Omega}_n)$. Indeed, let us consider the functional $F: H^s_0(\tilde{\Omega}_n)\to\mathbb{R}$ defined by

$$F(v)=\frac12[w_{\Omega_n}-v]^2_{H^s(\mathbb{R}^N)}.$$

Observe that

$$F(v)=\frac12[w_{\Omega_n}]^2_{H^s(\mathbb{R}^N)}+\frac12[v]^2_{H^s(\mathbb{R}^N)}-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(w_{\Omega_n}(x)-w_{\Omega_n}(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy.$$

Using the weak formulation of w_{Ω_n} we have that

$$F(v)=\frac12[w_{\Omega_n}]^2_{H^s(\mathbb{R}^N)}+\frac12[v]^2_{H^s(\mathbb{R}^N)}-\int_{\widetilde{\Omega}_n}v.$$

Then, the functional F will be minimized for $v=w_{\tilde{\Omega}_n}$, since $w_{\tilde{\Omega}_n}$ minimizes the functional

$$v\mapsto\frac12[v]^2_{H^s(\mathbb{R}^N)}-\int_{\widetilde{\Omega}_n}v.$$

Hence,

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w_{\Omega_n}(x)-w_{\Omega_n}(y)-w_{\widetilde{\Omega}_n}(x)+w_{\widetilde{\Omega}_n}(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\
&\qquad\le\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w_{\Omega_n}(x)-w_{\Omega_n}(y)-(u_n+v_n)(x)+(u_n+v_n)(y))|^2}{|x-y|^{N+2s}}\,dx\,dy\\
&\qquad=[w_{\Omega_n}]^2_{H^s(\mathbb{R}^N)}+[u_n+v_n]^2_{H^s(\mathbb{R}^N)}\\
&\qquad-2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[w_{\Omega_n}(x)-w_{\Omega_n}(y)][(u_n+v_n)(x)-(u_n+v_n)(y)]}{|x-y|^{N+2s}}\,dx\,dy\\
&\qquad=\int_{\mathbb{R}^N}w_{\Omega_n}+[u_n+v_n]^2_{H^s(\mathbb{R}^N)}-2\int_{\mathbb{R}^N}(u_n+v_n)\\
&\qquad=2\left(\int_{\mathbb{R}^N}w_{\Omega_n}-\int_{\mathbb{R}^N}(u_n+v_n)\right)+[u_n+v_n]^2_{H^s(\mathbb{R}^N)}-[w_{\Omega_n}]^2_{H^s(\mathbb{R}^N)}.
\end{aligned}$$

Observe that

$$\left|\int_{\mathbb{R}^N}w_{\Omega_n}-\int_{\mathbb{R}^N}(u_n+v_n)\right|\le|\Omega_n|^{\frac12}\|w_{\Omega_n}-(u_n+v_n)\|_{L^2(\mathbb{R}^N)}\to 0$$

as $n\to+\infty$. Moreover, using the fact that $[u_n+v_n]^2_{H^s(\mathbb{R}^N)}\le[u_n]^2_{H^s(\mathbb{R}^N)}+[v_n]^2_{H^s(\mathbb{R}^N)}$ since they are nonnegative functions, we obtain from (20) that

$$\limsup_{n\to+\infty}\left([u_n+v_n]^2_{H^s(\mathbb{R}^N)}-[w_{\Omega_n}]^2_{H^s(\mathbb{R}^N)}\right)\le 0$$

and therefore

$$[w_{\Omega_n}-w_{\widetilde{\Omega}_n}]_{H^s(\mathbb{R}^N)}\to 0\quad\text{as } n\to+\infty.$$

By Proposition 2.1, there exists $C>0$ such that, for every $n\in\mathbb{N}$,

$$\|w_{\Omega_n}-w_{\widetilde{\Omega}_n}\|_{L^2(\mathbb{R}^N)}=\|w_{\Omega_n}-w_{\widetilde{\Omega}_n}\|_{L^2(\Omega_n)}\le C[w_{\Omega_n}-w_{\widetilde{\Omega}_n}]_{H^s(\mathbb{R}^N)}$$

and hence

$$\|w_{\Omega_n}-w_{\widetilde{\Omega}_n}\|_{H^s(\mathbb{R}^N)}\to 0\quad\text{as } n\to+\infty.$$

□

Lemma 5.4 *Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be a sequence of s-quasi-open sets of uniformly bounded measure, such that*

$$\Omega_n=\Omega_n^{(1)}\cup\Omega_n^{(2)},\qquad dist(\Omega_n^{(1)},\Omega_n^{(2)})\to+\infty \text{ as } n\to+\infty.$$

Then,

$$\|w_{\Omega_n}-(w_{\Omega_n^{(1)}}+w_{\Omega_n^{(2)}})\|_{H^s(\mathbb{R}^N)}\to 0\qquad \text{as } n\to+\infty.$$

P r o o f. We observe that, by [11, Theorem 1], w_{Ω_n} and $w_{\Omega_n^{(1)}}$ are uniformly bounded in L^∞. Moreover, w_{Ω_n} satisfies

$$\begin{cases}(-\Delta)^s w_{\Omega_n} = 1 & \text{in } \Omega_n^{(1)},\\ w_{\Omega_n} = g_n & \text{in } \mathbb{R}^N\setminus\Omega_n^{(1)},\end{cases}\tag{22}$$

where g_n coincides with w_{Ω_n} on $\Omega_n^{(2)}$, and is equal to zero in $\mathbb{R}^N\setminus\Omega_n^{(2)}$. The function $z_n:=w_{\Omega_n}-w_{\Omega_n^{(1)}}$ satisfies

$$\begin{cases}(-\Delta)^s z_n = 0 & \text{in } \Omega_n^{(1)},\\ z_n = g_n & \text{in } \mathbb{R}^N\setminus\Omega_n^{(1)}.\end{cases}\tag{23}$$

Let $\hat{w}_n$ be the function which coincides with w_{Ω_n} on $\Omega_n^{(1)}$, and is equal to zero in $\mathbb{R}^N \setminus \Omega_n^{(1)}$. By testing (23) with $\varphi_n = \hat{w}_n - w_{\Omega_n^{(1)}} \in H_0^s(\Omega_n^{(1)})$, and observing that $z_n = \varphi_n + g_n$, we obtain

$$
\begin{aligned}
0 &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(z_n(x)-z_n(y))(\varphi_n(x)-\varphi_n(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varphi_n(x)-\varphi_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\varphi_n(x)-\varphi_n(y))(g_n(x)-g_n(y))}{|x-y|^{N+2s}}\,dx\,dy.
\end{aligned}
$$

It holds

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\varphi_n(x)-\varphi_n(y))(g_n(x)-g_n(y))}{|x-y|^{N+2s}}\,dx\,dy\right| \\
&= 2\left|\int_{\Omega_n^{(1)}}\int_{\Omega_n^{(2)}} \frac{\varphi_n(x)g_n(y)}{|x-y|^{N+2s}}\,dx\,dy\right| \le C\mathrm{dist}(\Omega_n^{(1)},\Omega_n^{(2)})^{-N-2s} \to 0
\end{aligned}
$$

as $n\to+\infty$. This implies

$$
[\hat{w}_n - w_{\Omega_n^{(1)}}]_{H^s(\mathbb{R}^N)} \to 0 \qquad \text{as } n\to+\infty.
$$

This reasoning can be repeated for the sets $\Omega_n^{(2)}$. Finally, by the triangle inequality, and Poincaré's inequality,

$$
\|w_{\Omega_n} - (w_{\Omega_n^{(1)}} + w_{\Omega_n^{(2)}})\|_{H^s(\mathbb{R}^N)} \to 0 \qquad \text{as } n\to+\infty.
$$

□

Lemma 5.5 *Let $\tilde{\Omega} \subset \Omega \subset \mathbb{R}^N$ two sets of finite measure. There exists a constant $C = C(|\Omega|, N) > 0$ and $\alpha = \alpha(N,s) > 0$ such that*

$$
\|R_\Omega - R_{\tilde{\Omega}}\|_{\mathcal{L}(L^2(\mathbb{R}^N))} \le C\|w_\Omega - w_{\tilde{\Omega}}\|^\alpha_{L^2(\mathbb{R}^N)}.
$$

P r o o f. Let $0<s<1$ be fixed. Observe that if $u, v \in H_0^1(\Omega)$ are the unique solutions of $(-\Delta)^s u = f$ in Ω, $(-\Delta)^s v = 1$ in Ω, respectively, using v and u as test functions in the weak formulation of the two previous equations, respectively, we get

$$
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy = \int_\Omega fw = \int_\Omega u,
$$

that is, $\int_\Omega f w_\Omega = \int_\Omega R(f)$. The previous computation gives that

$$
\int_\Omega R_\Omega(f) - R_{\tilde{\Omega}}(f) = \int_\Omega f(w_\Omega - w_{\tilde{\Omega}}).
$$

By [5, Theorem 3.1], for $N<4s$ we have

$$
\|R_\Omega(f)\|_{L^\infty(\Omega)} \le C(N,|\Omega|)\|f\|_{L^2(\Omega)}, \tag{24}
$$

and then, by using (24) and Hölder's inequality we get

$$
\begin{aligned}
\|R_\Omega(f) - R_{\tilde{\Omega}}(f)\|^2_{L^2(\Omega)} &\le \|R_\Omega(f) - R_{\tilde{\Omega}}(f)\|_{L^\infty(\Omega)}\|R_\Omega(f) - R_{\tilde{\Omega}}(f)\|_{L^1(\Omega)} \\
&\le C\|f\|_{L^2(\Omega)}\|f(w_\Omega - w_{\tilde{\Omega}})\|_{L^1(\Omega)} \\
&\le C\|f\|^2_{L^2(\Omega)}\|w_\Omega - w_{\tilde{\Omega}}\|_{L^2(\Omega)}.
\end{aligned}
$$

The case $N \ge 4s$ will follow by an interpolation argument. For that end, consider $p>2$, $N\ge 4s$ and $f\in L^p(\Omega)$, $f\ge 0$. By using again [5, Theorem 3.1] and Hölder's inequality we get

$$
\|R_\Omega(f) - R_{\tilde{\Omega}}(f)\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)}\|w_\Omega - w_{\tilde{\Omega}}\|^{\frac{1}{p}}_{L^{p'}(\Omega)}
$$

for a suitable constant C depending only on p, N and $|\Omega|$, that is,

$$\|R_\Omega - R_{\tilde\Omega}\|_{\mathcal{L}(L^p(\mathbb{R}^N))} \le C\|w_\Omega - w_{\tilde\Omega}\|_{L^{p'}(\Omega)}^{\frac{1}{p}}.$$

Now, let R^*_Ω and $R^*_{\tilde\Omega}$ be the adjoint operators of R_Ω and $\Omega_{\tilde\Omega}$, respectively, which are defined from $L^{p'}(\Omega)$ in itself. Since the $L^{p'}$ norm of $R^*_\Omega - R^*_{\tilde\Omega}$ coincides with the L^p norm of $R_\Omega - R_{\tilde\Omega}$, we get

$$\|R^*_\Omega - R^*_{\tilde\Omega}\|_{\mathcal{L}(L^{p'}(\mathbb{R}^N))} \le C\|w_\Omega - w_{\tilde\Omega}\|_{L^{p'}(\Omega)}^{\frac{1}{p}}.$$

Since R_Ω and $R_{\tilde\Omega}$ are self-adjoint on $L^2(\Omega)$, keeping the same notation for R_A, $R_{\tilde\Omega}$ and their extension on $L^{p'}(\Omega)$, we obtain that $R_\Omega - R_{\tilde\Omega} : L^{p'}(\Omega) \to L^{p'}(\Omega)$ and

$$\|R_\Omega - R_{\tilde\Omega}\|_{\mathcal{L}(L^{p'}(\mathbb{R}^N))} \le C\|w_\Omega - w_{\tilde\Omega}\|_{L^{p'}(\Omega)}^{\frac{1}{p}}.$$

Finally, from the Riesz-Thorin interpolation theorem and since $1 < p' < 2$, we obtain that

$$\begin{aligned}\|R_\Omega - R_{\tilde\Omega}\|_{\mathcal{L}(L^2(\mathbb{R}^N))} &\le \|R_\Omega - R_{\tilde\Omega}\|_{\mathcal{L}(L^p(\mathbb{R}^N))}^{\frac{1}{2}}\|R_\Omega - R_{\tilde\Omega}\|_{\mathcal{L}(L^{p'}(\mathbb{R}^N))}^{\frac{1}{2}} \\ &\le C\|w_\Omega - w_{\tilde\Omega}\|_{L^{p'}(\Omega)}^{\frac{1}{p}} \\ &\le C|\Omega|^{\frac{2-p'}{p'p}}\|w_\Omega - w_{\tilde\Omega}\|_{L^2(\Omega)}^{\frac{1}{p}}\end{aligned}$$

which ends the proof. □

6 Proof of Theorem 1.2

Lemma 6.1 *Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be a sequence of s-quasi-open sets of uniformly bounded measure, which weakly γ-converges to the s-quasi-open set Ω. Then, there exists an increasing sequence of positive integers $\{n_k\}_{k\in\mathbb{N}}$ and a sequence of s-quasi-open sets $\{C_k\}_{k\in\mathbb{N}}$ such that $\Omega_{n_k} \subset C_k$ for every $k \in \mathbb{N}$, and $\{C_k\}_{k\in\mathbb{N}}$ γ-converges to Ω.*

P r o o f. The proof can be performed by following the ideas of [18, Lemma 4.7.11]. By definition of weak γ-convergence, the torsion functions w_{Ω_n} converge in $L^2(\mathbb{R}^N)$ to a function $w \in H^s(\mathbb{R}^N)$ such that $\Omega = \{w > 0\}$. For $\varepsilon > 0$, we introduce the s-quasi-open sets $\Omega^\varepsilon := \{w_\Omega > \varepsilon\}$ and $\Omega^\varepsilon_n := \Omega_n \cup \Omega^\varepsilon$. We can suppose that ε is sufficiently small so that Ω^ε is not empty. The sequence $\{w_{\Omega^\varepsilon_n}\}_{n\in\mathbb{N}}$ is uniformly bounded in $H^s(\mathbb{R}^N)$, so that we can suppose that it converges weakly in $H^s(\mathbb{R}^N)$ to a function $w^\varepsilon \in H^s(\mathbb{R}^N)$. By the weak maximum principle, $w_{\Omega^\varepsilon_n} \ge w_{\Omega_n}$ and $w_{\Omega^\varepsilon_n} \ge w_{\Omega^\varepsilon}$ for every $n \in \mathbb{N}$. Let us apply the concentration-compactness principle to the sequence $\{w_{\Omega^\varepsilon_n}\}_{n\in\mathbb{N}}$. The inequality $w_{\Omega^\varepsilon_n} \ge w_{\Omega^\varepsilon}$ implies that the sequence can not have vanishing subsequences. On the other hand, suppose that the sequence admits a subsequence (not relabeled) in the dichotomy case. We can suppose that $\|w_{\Omega^\varepsilon_n}\|_{L^2(\mathbb{R}^N)} \to \lambda$ for some $\lambda > 0$. Then, there exists $\alpha \in (0, \lambda)$, and two sequences $\{u_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}} \subset H^s(\mathbb{R}^N)$ such that

$$\|w_{\Omega^\varepsilon_n} - u_n - v_n\|_{L^2(\mathbb{R}^N)} \to 0 \qquad \text{for } n \to +\infty;$$

$$\left|\int_{\mathbb{R}^N} |u_n|^2 - \alpha\right| \to 0, \qquad \left|\int_{\mathbb{R}^N} |v_n|^2 - (\lambda - \alpha)\right| \to 0;$$

$$\operatorname{dist}(\operatorname{supp} u_n, \operatorname{supp} v_n) \to +\infty \qquad \text{for } n \to +\infty.$$

Set $A_n := \operatorname{supp} u_n$ and $B_n := \operatorname{supp} v_n$. We have $A_n, B_n \subset \Omega_n \cup \Omega^\varepsilon$. Arguing as in Lemma 5.2, we observe that

$$\liminf_{n\to+\infty} |A_n| > 0, \qquad \liminf_{n\to+\infty} |B_n| > 0.$$

By Proposition A.1, Ω^ε is a bounded set, therefore the relation

$$A_n \cap \Omega^\varepsilon \neq \emptyset \quad \text{and} \quad B_n \cap \Omega^\varepsilon \neq \emptyset$$

can hold true only for a finite number of indices n. Without loss of generality, suppose $A_n \subset \Omega_n$ for every n. By Lemma 5.4, Lemma 5.3 and the inequality $w_{A_n} \leq w_{\Omega_n}$, we would have a contradiction with the fact that $w_{\Omega_n} \to w$ in $L^2(\mathbb{R}^N)$. Therefore, $w_{\Omega_n^\varepsilon} \to w^\varepsilon$ in $L^2(\mathbb{R}^N)$. We are going to prove the inequalities

$$(w_\Omega - \varepsilon)^+ \leq w^\varepsilon \leq w_\Omega. \tag{25}$$

Indeed, since $\Omega^\varepsilon \subset \Omega_n^\varepsilon$, by the weak maximum principle we have

$$w_{\Omega_n^\varepsilon} \geq w_{\Omega^\varepsilon} = (w_\Omega - \varepsilon)^+.$$

On the other hand, define $v^\varepsilon := \frac{1}{\varepsilon}(\varepsilon - w_\Omega)^+$ and $v_n := \inf\{w_{\Omega_n^\varepsilon}, v^\varepsilon\}$. By definition, $v_n \in H_0^s(\Omega_n)$, and v_n converges weakly in $H^s(\mathbb{R}^N)$ to $v := \inf\{w^\varepsilon, v^\varepsilon\}$. By Lemma 4.6, $v \in H_0^s(\Omega)$. Since $v^\varepsilon = 1$ on $\mathbb{R}^N \setminus \Omega$, it must hold $w^\varepsilon \in H_0^s(\Omega)$. Since $(-\Delta)^s w_{\Omega_n^\varepsilon} \leq 1$ in weak sense in Ω by Proposition 2.5, we have $(-\Delta)^s w^\varepsilon \leq 1$ in Ω, and therefore, by the weak maximum principle, $w^\varepsilon \leq w_\Omega$ in Ω.

Inequality (25) now implies that the functions w^ε converge, as $\varepsilon \to 0^+$, to w_Ω weakly in $H_0^s(\Omega)$ and strongly in $L^2(\Omega)$. Given a sequence $\varepsilon_k \to 0^+$, we can find a subsequence n_k such that $w_{n_k}^{\varepsilon_k}$ converges to w_Ω weakly in $H^s(\mathbb{R}^N)$ and strongly in $L^2(\mathbb{R}^N)$. Therefore, $C_k := \Omega_{n_k}^{\varepsilon_k}$ γ-converges to Ω. □

Proposition 6.2 *Let $J : \mathcal{A}(\mathbb{R}^N) \to (-\infty, +\infty]$ be a functional satisfying:*

(i) J is decreasing with respect to the inclusion of sets;

(ii) J is lower semicontinuous with respect to the γ-convergence.

Then J is lower semicontinuous with respect to the weak γ-convergence.

Proof. Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be a sequence of s-quasi-open sets of uniformly bounded measure, which weakly γ-converges to the s-quasi-open set Ω. By Lemma 6.1, there exists an increasing sequence of positive integers $\{n_k\}_{k\in\mathbb{N}}$ and a sequence of s-quasi-open sets $\{C_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{n\to+\infty} J(\Omega_{n_k}) = \liminf_{n\to+\infty} J(\Omega_n),$$

$\Omega_{n_k} \subset C_k$ for every $k \in \mathbb{N}$, and $\{C_k\}_{k\in\mathbb{N}}$ γ-converges to Ω. Since J is decreasing with respect to the inclusion of sets,

$$J(\Omega) \leq \liminf_{k\to+\infty} J(C_k) \leq \liminf_{k\to+\infty} J(\Omega_{n_k}) = \liminf_{n\to+\infty} J(\Omega_n).$$

The proof is concluded. □

We are now ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $\{\Omega_n\}_{n\in\mathbb{N}} \subset \mathcal{A}(\mathbb{R}^N)$ be a minimizing sequence for Problem (3), satisfying $|\Omega_n| = c$ for every $n \in \mathbb{N}$, and $J(\Omega_n) \to m$ as $n \to +\infty$. By Theorem 1.1, we have two possible cases:

(i) there exists a subsequence, still denoted by $\{\Omega_n\}_{n\in\mathbb{N}}$, and a set $\Omega \in \mathcal{A}(\mathbb{R}^N)$, such that, up to some translations, $\{\Omega_n\}_{n\in\mathbb{N}}$ weakly γ-converges to Ω. Since J is invariant by translations, the sequence will be again a minimizing sequence for J. By Proposition 4.4, $|\Omega| \leq c$. Let $\hat{\Omega} \in \mathcal{A}(\mathbb{R}^N)$ be such that $\Omega \subset \hat{\Omega}$ and $|\hat{\Omega}| = c$. Since J is decreasing with respect to set inclusion, and by Propositions 4.9 and 6.2,

$$m \leq J(\hat{\Omega}) \leq J(\Omega) \leq \liminf_{n\to+\infty} J(\Omega_n) = m.$$

Therefore, $\hat{\Omega}$ is a minimizing set.

(ii) there exists a subsequence, still denoted by $\{\Omega_n\}_{n\in\mathbb{N}}$, such that we can define $\widetilde{\Omega}_n = \Omega_n^1 \cup \Omega_n^2 \subset \Omega_n$, where Ω_n^1, Ω_n^2 are such that $\mathrm{dist}(\Omega_n^1, \Omega_n^2) \to +\infty$, $\liminf_{n\to+\infty} |\Omega_n^i| > 0$ for $i = 1, 2$, and $J(\widetilde{\Omega}_n) \to m$ as $n \to +\infty$. If $|\widetilde{\Omega}_n| < c$, it is possible to modify suitably the sequence in order to respect the volume constraint as well, since the functional J is decreasing with respect to set inclusion.

□

A A decay estimate for the torsion function

In this appendix we prove a useful estimate for the torsion function in s-quasi open sets of finite measure, in the spirit of [3, Lemma 5.1] (see also [9, Theorem 3.1]).

Proposition A.1 *Let w_Ω be the torsion function on a s-quasi open set of finite measure. Then there exists a constant C, depending only on s, N and $|\Omega|$, such that*

$$\|w_\Omega\|_{L^\infty(\Omega\setminus B_{R+1})} \le C|\Omega\setminus B_R|^{\frac{s}{N}}.$$

Proof. Let w_Ω be the torsion function on Ω. By [11, Theorem 1], we have

$$\|w_\Omega\|_{L^\infty(\Omega)} \le \|w_{\Omega^*}\|_{L^\infty(\Omega^*)},$$

where Ω^* is the ball having the same Lebesgue measure as Ω. From [14, Table 3],

$$w_{\Omega^*}(x) = C(r^2-|x|^2)_+^s,$$

where $C = C(s,N)$ and $|\Omega| = |\Omega^*| = N\omega_n r^N$, so that $\|w_{\Omega^*}\|_{L^\infty(\Omega^*)} \le C = C(s,N,|\Omega|)$. Set $R_k := R+1-2^{-k}$. Let $\varphi_k \in W^{1,\infty}(\mathbb{R}^N)$ be a radial cutoff function, such that $\varphi_k \equiv 1$ on $B^c_{R_k}$, $\varphi_k \equiv 0$ on $B_{R_{k-1}}$ and $|\nabla\varphi| \le 2^k$ in $B_{R_k}\setminus \overline{B}_{R_{k-1}}$. We set

$$t_k := M|\Omega\setminus B_R|^{s/N}(1-2^{-k}),$$

where the constant $M>0$ will be determined later. The function $v = \varphi_k^2(w_\Omega - t_k)_+$ belongs to $H^s_0(\Omega)$ by [6, Lemma A.1] By notational simplicity, we write $w_k := (w_\Omega - t_k)_+$. We have

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varphi_k(x)w_k(x)-\varphi_k(y)w_k(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(w_k(x)-w_k(y))(\varphi_k^2(x)w_k(x)-\varphi_k^2(y)w_k(y))}{|x-y|^{N+2s}}\,dx\,dy\\
&+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varphi_k(x)-\varphi_k(y)|^2 w_k(x)w_k(y)}{|x-y|^{N+2s}}\,dx\,dy\\
&\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(w_\Omega(x)-w_\Omega(y))(\varphi_k^2(x)w_k(x)-\varphi_k^2(y)w_k(y))}{|x-y|^{N+2s}}\,dx\,dy\\
&+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varphi_k(x)-\varphi_k(y)|^2 w_k(x)w_k(y)}{|x-y|^{N+2s}}\,dx\,dy\\
&= \int_\Omega \varphi_k^2 w_k + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varphi_k(x)-\varphi_k(y)|^2 w_k(x)w_k(y)}{|x-y|^{N+2s}}\,dx\,dy\\
&= \int_\Omega \varphi_k^2 w_k + \int_\Omega\int_\Omega \frac{|\varphi_k(x)-\varphi_k(y)|^2 w_k(x)w_k(y)}{|x-y|^{N+2s}}\,dx\,dy\\
&= \int_\Omega \varphi_k^2 w_k + \int_{\Omega\setminus B_{R_{k-1}}}\int_{\Omega\setminus B_{R_{k-1}}} \frac{|\varphi_k(x)-\varphi_k(y)|^2 w_k(x)w_k(y)}{|x-y|^{N+2s}}\,dx\,dy\\
&+ 2\int_{\Omega\cap B_{R_{k-1}}}\int_{\Omega\setminus B_{R_{k-1}}} \frac{|\varphi_k(x)-\varphi_k(y)|^2 w_k(x)w_k(y)}{|x-y|^{N+2s}}\,dx\,dy\\
&\le \int_\Omega \varphi_k^2 w_k + 4^k C\int_{\Omega\setminus B_{R_{k-1}}} w_k(x)\left(\int_{\Omega\setminus B_{R_{k-1}}} \frac{1}{|x-y|^{N+2s-2}}\,dy\right)dx\\
&+ 2\cdot 4^k C\int_{\Omega\setminus B_{R_{k-1}}} w_k(x)\left(\int_{\Omega\cap B_{R_{k-1}}} \frac{1}{|x-y|^{N+2s-2}}\,dy\right)dx\\
&\le \int_\Omega \varphi_k^2 w_k + 2\cdot 4^k C\int_{\Omega\setminus B_{R_{k-1}}} w_k(x)\left(\int_\Omega \frac{1}{|x-y|^{N+2s-2}}\,dy\right)dx.
\end{aligned}$$

Since Ω has finite measure, the quantity

$$\int_\Omega \frac{1}{|x-y|^{N+2s-2}}\,dy$$

is finite for every x, and uniformly bounded from above, from instance by

$$\int_\Omega \frac{1}{|x-y|^{N+2s-2}}\,dy = \int_{\Omega\cap B_1(x)} \frac{1}{|x-y|^{N+2s-2}}\,dy + \int_{\Omega\setminus B_1(x)} \frac{1}{|x-y|^{N+2s-2}}\,dy \le \frac{N\omega_N}{2-2s} + |\Omega|.$$

By Proposition 2.1, we obtain

$$\begin{aligned}\int_\Omega (\varphi_k w_k)^2 &\le C\,|\{\varphi_k w_k > 0\}|^{\frac{2s}{N}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varphi_k(x)w_k(x) - \varphi_k(y)w_k(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \\ &\le C\cdot 4^k\,|\{\varphi_k w_k > 0\}|^{\frac{2s}{N}} \left(\int_\Omega \varphi_k^2 w_k + \int_{\Omega\setminus B_{R_{k-1}}} w_k\right) \\ &\le C\cdot 4^k\,|\{\varphi_k w_k > 0\}|^{1+\frac{2s}{N}}.\end{aligned}$$

Arguing as in [3, Lemma 5.1], we obtain the recursive relation

$$a_{k+1} \le \frac{C}{M^2} 16^k a_k^{1+\frac{2s}{N}} \quad \text{for every } k \ge 1,$$

where

$$a_k = \frac{|\{(w_\Omega - t_k)_+ > 0\} \cap (\Omega \setminus B_{R_{k-1}})|}{|\Omega\setminus B_R|}.$$

Choosing M such that $\frac{C}{M^2} = 16^{-\frac{N}{s}}$, one can prove by induction that

$$a_k \le \left(\frac{1}{16}\right)^{\frac{N}{2s}(k-1)},$$

which implies $a_k \to 0$ as $k \to +\infty$. Therefore,

$$|\{(w_\Omega - t_\infty)_+ > 0\} \cap (\Omega\setminus B_R)| = 0,$$

where $t_\infty = M|\Omega\setminus B_R|^{\frac{s}{N}} = C^{\frac{1}{2}} 4^{\frac{N}{s}} |\Omega\setminus B_R|^{\frac{s}{N}}$. This proves the claim. □

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