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Weighted inequalities for the fractional Laplacian and the existence of extremals

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In this paper, we obtain improved versions of Stein–Weiss and Caffarelli–Kohn–Nirenberg inequalities, involving Besov norms of negative smoothness. As an application of the former, we derive the existence of extremals of the Stein–Weiss inequality in certain cases, some of which are not contained in the celebrated theorem of Lieb [Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities, *Ann. of Math. (2)* **118**(2) (1983) 101–116].

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1. Introduction

In the Euclidean space $\mathbb{R}^n$, it is well-known that negative powers of the Laplacian admit the integral representation in terms of the Riesz potential or fractional integral operator:

$$(-\Delta)^{-s/2} f(x) = I_s(f) = c(n,s)\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-s}}dy, \quad 0<s<n.$$

One basic result for this operator is the Stein–Weiss inequality, which gives its behavior in Lebesgue spaces with power weights:

Theorem 1.1 ([19, Theorem B^*]). *Let $n \geq 1$, $0<s<n$, $1<p\leq r<\infty$, $\alpha < \frac{n}{p'}$, $\gamma > -\frac{n}{r}$, $\alpha \geq \gamma$, and*

$$\frac{1}{r} = \frac{1}{p} + \frac{\alpha-\gamma-s}{n}. \tag{1.1}$$

Then,

$$\||x|^\gamma (-\Delta)^{-s/2} f\|_{L^r} \leq C\||x|^\alpha f\|_{L^p}, \quad \forall f \in L^p(\mathbb{R}^n, |x|^{\alpha p}). \tag{1.2}$$

Equivalently, we can rewrite this result as fractional Sobolev inequality, namely,

$$\||x|^\gamma u\|_{L^r} \leq C\||x|^\alpha (-\Delta)^{s/2} u\|_{L^p}, \quad \forall u \in \dot{H}^{s,p}_\alpha(\mathbb{R}^n) \tag{1.3}$$

meaning that we have a continuous embedding

$$\dot{H}^{s,p}_\alpha(\mathbb{R}^n) \subset L^r(\mathbb{R}^n, |x|^{\gamma r})$$

where

$$\dot{H}^{s,p}_\alpha(\mathbb{R}^n) = \{u = (-\Delta)^{-s/2} f : f \in L^p(\mathbb{R}^n, |x|^{\alpha p})\} \tag{1.4}$$

is the weighted homogeneous Sobolev space of potential type, which is a Banach space with the norm $\|u\|_{\dot{H}^{s,p}_\alpha} = \|f|x|^\alpha\|_{L^p}$.

We remark that this embedding is not compact due to the scaling invariance of the Stein–Weiss inequality. In other words, the scaling condition (1.1) means that r plays the role of the critical Sobolev exponent in the weighted setting.

Our first aim in this work is to obtain an improved version of (1.2) and (1.3). More precisely, for suitable values of the parameters we will prove that there holds:

$$\||x|^\gamma (-\Delta)^{-s/2} f\|_{L^r} \leq C\||x|^\alpha f\|^\theta_{L^p} \|f\|^{1-\theta}_{\dot{B}^{-\mu-s}_{\infty,\infty}} \tag{1.5}$$

for every $f \in L^p(\mathbb{R}^n, |x|^{\alpha p}) \cap \dot{B}^{-\mu-s}_{\infty,\infty}$, or, equivalently,

$$\||x|^\gamma u\|_{L^r} \leq C\||x|^\alpha (-\Delta)^{s/2} u\|^\theta_{L^p} \|u\|^{1-\theta}_{\dot{B}^{-\mu}_{\infty,\infty}} \tag{1.6}$$

for every $u \in \dot{H}^{s,p}_\alpha(\mathbb{R}^n) \cap \dot{B}^{-\mu}_{\infty,\infty}$, where the Besov norm of negative smoothness is defined in terms of the heat kernel (see Sec. 2 for a precise definition).

The reader will observe that inequality (1.6) is reminiscent of the well-known Caffarelli–Kohn–Nirenberg first-order interpolation inequality:

Theorem 1.2 ([3]). *Assume*

$$p,q \geq 1, \quad r>0, \quad 0 \leq \theta \leq 1, \quad \frac{1}{p}+\frac{\alpha}{n}, \quad \frac{1}{q}+\frac{\beta}{n}, \quad \frac{1}{r}+\frac{\gamma}{n} > 0,$$

where

$$\gamma = \theta\sigma + (1-\theta)\beta.$$

Then, there exists a positive constant C such that the following inequality holds for all $u \in C_0^\infty(\mathbb{R}^n)$

$$\||x|^\gamma u\|_{L^r} \le C\||x|^\alpha \nabla u\|_{L^p}^\theta \||x|^\beta u\|_{L^q}^{1-\theta} \tag{1.7}$$

if and only if the following relations hold:

$$\frac{1}{r} + \frac{\gamma}{n} = \theta\left(\frac{1}{p} + \frac{\alpha-1}{n}\right) + (1-\theta)\left(\frac{1}{q} + \frac{\beta}{n}\right)$$

$$0 \le \alpha - \sigma \quad \text{if } \theta > 0,$$

and

$$\alpha - \sigma \le 1 \quad \text{if } \theta > 0 \quad \text{and} \quad \frac{1}{p} + \frac{\alpha-1}{n} = \frac{1}{r} + \frac{\gamma}{n}.$$

Indeed, in the local case $s = 1$ we will also obtain an improvement of this inequality in some cases, namely, that

$$\||x|^\gamma u\|_{L^r} \le C\||x|^\alpha \nabla u\|_{L^p}^\theta \|u\|_{\dot{B}^{-\mu}_{\infty,\infty}}^{1-\theta} \tag{1.8}$$

holds for every $u \in \dot{H}^{1,p}_\alpha(\mathbb{R}^n) \cap \dot{B}^{-\mu}_{\infty,\infty}$ for an appropriate range of parameters.

Our second aim in this paper is to prove the existence of extremals of inequality (1.2) by means of a rearrangement-free technique, that allows us to obtain some previously unknown cases, namely for $p = 2$ and $0 < \gamma < \alpha$. Let us recall that, by definition, the best constant S in (1.2) is

$$S = \sup \frac{\||x|^\gamma (-\Delta)^{-s/2} f\|_{L^r}}{\||x|^\alpha f\|_{L^p}}, \tag{1.9}$$

where the supremum is taken over all the non-vanishing functions $f \in L^p(\mathbb{R}^n, |x|^{\alpha p})$.

Best constants and the existence of minimizers/maximizers for the Stein–Weiss, Sobolev and related inequalities have been studied extensively in the literature, and it would be impossible to cite all the references, but we mention some of them that are closely related to our work. In a celebrated paper, Lieb proved [14, Theorem 5.1] the existence of minimizers of the Stein–Weiss inequality (in an equivalent formulation), under the extra assumptions

$$p < r, \quad \alpha \ge 0, \quad \gamma \le 0.$$

The sign restriction on the exponents in his result comes from the fact that his argument is based on a symmetrization (rearrangement) technique.

Recently, new rearrangement-free techniques for dealing with these inequalities in the unweighted case have been introduced in [6, 7, 17]. Avoiding the use of rearrangements could be useful to extend the results to settings where this technique is not available, for instance when mixed norms are considered (see, for instance, [5]), or in the setting of stratified Lie groups like in [4, 7]. Indeed, we believe that our results can be extended to the latter setting without essential modifications (replacing n by the homogeneous dimension of the group, and the Euclidean norm by an homogeneous norm in the group). However, we have chosen to work in the Euclidean space $\mathbb{R}^n$, in order to make our paper accessible to a broader audience.

Improved Sobolev inequalities play an important role for the proofs of existence of maximizers via concentration-compactness arguments. We can mention, for instance, [7, Lemma 4.4] where Frank and Lieb obtain an (unweighted) improved inequality with a Besov norm in the context of the Heisenberg group, which they use derive sharp constants for analogues to the Hardy–Littlewood–Sobolev inequality in that group. More recently, the work of Palatucci and Pisante [17], deals with the existence of maximizers in the unweighted case ($\alpha = \gamma = 0$) in $\mathbb{R}^n$, also using and improved Sobolev inequality involving a Morrey norm [17, Theorem 1.1], of which they give two different proofs. One of them, related to our work, is based on the refined Sobolev inequality of Gérard, Meyer and Oru [9] involving a Besov norm of negative smoothness, and an embedding result between Besov and Morrey spaces [17, Lemma 3.4].

Along these lines, the existence of maximizers of (1.9) in the case $\alpha = 0$ was considered in [22]. However, we believe that the argument in that paper is not correct. Indeed, we could not check the validity of inequality (3.2) in [22], as the application of the invoked rearrangement inequality would require a decreasing function, that is, a negative exponent in the previous inequality. Hence, we do not know whether it is possible to perform the argument using the refined Sobolev inequality with the Morrey norm in the presence of weights. For this reason, we choose to work directly with the Besov norm and exploit some properties of the heat semigroup. With our improved inequality (Theorem 3.1), a weighted compactness result (Proposition 2.2), and the so-called "method of missing mass" (invented by Lieb in [14]), we can prove the existence of minimizers of the Stein–Weiss inequality only in the case $p = 2$ but, in turn, we can have any γ in the range $-\frac{n}{r} < \gamma < \alpha$, thus extending the range $\gamma \geq 0$ of [14].

It should be mentioned that there is increasing literature devoted to the study of improved versions of the Sobolev–Gagliardo–Nirenberg and related inequalities for their own sake. Besides the above mentioned papers [9, 17], we can mention [1, 4, 11–13, 21], among others. In the proof of our improved (1.5) inequality, instead of using the Littlewood–Paley characterization of the Besov space (as in [1, 9]), we use a simpler approach inspired by [4], which is based on the thermal definition of the Besov spaces and the representation of the negative powers of the Laplacian in terms of the heat semigroup. Besides that, we use the boundedness of the Hardy–Littlewood maximal function with Muckenhoupt weights, and the Stein–Weiss inequality. Moreover, our method does not involve truncations (as in [13]), a technique which seems not to work in our context due to the non-local character of the fractional Laplacian; and makes no use of rearrangements (as in [12]).

The rest of the paper is organized as follows: in Sec. 2, we recall the definition of the Besov spaces of negative smoothness and some results on the heat semigroup that will be used in the rest of the paper; in Sec. 3, we obtain the improved Stein–Weiss inequality (1.5) and its rewritten form (1.6), as well as the improved Caffarelli–Kohn–Nirenberg inequality (1.8); in Sec. 4, we prove that the embedding given by (1.6) is locally compact; and finally in Sec. 5, we use the method of missing mass and the results of the previous sections to prove the existence of extremals of the Stein–Weiss inequality.

2. Weighted Estimates for the Heat Semigroup

In this section, we recall the definition of the Besov spaces of negative smoothness and collect some auxiliary results for the heat semigroup, which will play a central role in our approach.

We recall that the heat semigroup $e^{\Delta t}$ in $\mathbb{R}^n$ is given by

$$e^{t\Delta}f(x_0) = f * h_t(x_0) = \int_{\mathbb{R}^n} f(x)h_t(x_0 - x)dx, \tag{2.1}$$

where

$$h_t(x) = \frac{1}{(4\pi t)^{n/2}} \exp\left\{-\frac{|x|^2}{4t}\right\}$$

is the heat kernel.

We shall use the following thermic definition of the Besov spaces, which goes back to the work of Flett [8]:

Definition 2.1. For any real $\delta > 0$ one can define the homogeneous Besov space $\dot{B}^{-\delta}_{\infty,\infty}$ as the space of tempered distributions u on $\mathbb{R}^n$ (possibly modulo polynomials) for which the following norm

$$\|u\|_{\dot{B}^{-\delta}_{\infty,\infty}} := \sup_{t>0} t^{\delta/2}\|e^{\Delta t}f\|_{L^\infty}$$

is finite.

We shall also need the following result, which is a particular case of [15, Proposition 3.2]. We include a proof for the sake of completeness:

Proposition 2.1. *Let* $n \geq 1$, $1 \leq p \leq +\infty$. *Assume further that*

$$0 \leq \beta < \alpha < \frac{n}{p'}.$$

Then the following estimates hold:

$$|x|^\beta |e^{t\Delta}f(x)| \leq Ct^{-\frac{1}{2}(\frac{n}{p}+\alpha)}\||x|^\alpha f\|_{L^p}, \tag{2.2}$$

and

$$|x|^\beta |\partial_{x_j} e^{t\Delta}f(x)| \leq Ct^{-\frac{1}{2}(\frac{n}{p}+\alpha+1)}\||x|^\alpha f\|_{L^p}, \quad j = 1, 2, \dots, n \tag{2.3}$$

for any $x \in \mathbb{R}^n$, *any* $t > 0$ *and any* $f \in L^p(\mathbb{R}^n, |x|^{\alpha p})$ *with* C *independent from* x, t *and* f.

Proof. First observe that $e^{t\Delta}f = S_{\sqrt{t}} e^{1.\Delta} S_{\frac{1}{\sqrt{t}}} f$ with $S_\lambda f(x) = f(\frac{x}{\lambda})$. Hence, it suffices to prove the result for $t = 1$.

By definition,

$$e^{1.\Delta} f(x) = (f * h_1)(x) = \frac{1}{(4\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4} f(y) dy.$$

By Hölder's inequality, we have that

$$\begin{aligned} |e^{1.\Delta} f(x)| &\leq \frac{1}{(4\pi)^{n/2}} \| |x|^\alpha f \|_{L^p} \left(\int_{\mathbb{R}^n} e^{-p'|x-y|^2/4} |y|^{-\alpha p'} dy \right)^{\frac{1}{p'}} \\ &= \frac{1}{(4\pi)^{n/2}} \| |x|^\alpha f \|_{L^p} I(x)^{\frac{1}{p'}}. \end{aligned}$$

So that, in order to prove (2.2), we need to show that

$$I(x) \leq C|x|^{-\beta p'}.$$

To this end, we split the integral $I(x)$ into two parts:

$$\begin{aligned} I(x) &= \int_{|x-y|\leq 1} e^{-p'|x-y|^2/4} |y|^{-\alpha p'} dy + \int_{|x-y|>1} e^{-p'|x-y|^2/4} |y|^{-\alpha p'} dy \\ &:= I_{\text{loc}}(x) + I_\infty(x) \end{aligned}$$

and bound each one of them separately.

We begin by considering I_{loc}. We consider two cases:

First case: When $|x| < 2$. In this case,

$$|x - y| \leq 1, \quad |x| < 2 \Rightarrow |y| \leq |x| + |x - y| \leq 2 + 1 = 3.$$

Then,

$$I_{\text{loc}}(x) \leq \int_{|y|\leq 3} |y|^{-\alpha p'} dy \leq C \quad \text{if } |x| < 2. \tag{2.4}$$

Second case: When $|x| \geq 2$. In this case, we observe that

$$|x - y| \leq 1, \quad |x| \geq 2 \Rightarrow |y| \geq |x| - |x - y| \geq |x| - 1 \geq |x| - \frac{1}{2}|x| = \frac{1}{2}|x|.$$

As a consequence, we obtain that

$$I_{\text{loc}}(x) \leq \int_{|x-y|\leq 1} \left(\frac{|x|}{2} \right)^{-\alpha p'} dy \leq C|x|^{-\alpha p'} \quad \text{if } |x| \geq 2. \tag{2.5}$$

Since $0 \leq \beta < \alpha$, (2.4) and (2.5) imply that

$$I_{\text{loc}}(x) \leq C|x|^{-\beta p'} \quad \text{for all } x. \tag{2.6}$$

Now we can proceed to bound I_∞. Again, we consider two cases:

First case: When $|x| < 1/2$. In this case,

$$|x-y| > 1, \quad |x| < \frac{1}{2} \Rightarrow |y| \geq |x-y| - |x| > 1 - \frac{1}{2} = \frac{1}{2}$$

and, therefore,

$$I_\infty(x) \leq 2^{\alpha p'} \int_{|x-y|>1} e^{-p'|x-y|^2/4} dy = 2^{\alpha p'} \int_{|z|>1} e^{-p'|z|^2/4} dz$$

which gives

$$I_\infty(x) \leq C \quad \text{if } |x| < \frac{1}{2}. \tag{2.7}$$

Second case: When $|x| \geq 1/2$. Let us observe that, for any $\gamma > 0$ to be chosen later, there exists a constant C_γ such that

$$e^{-u} \leq \frac{C_\gamma}{u^\gamma} \quad \text{for all } u > 0.$$

Hence,

$$I_\infty(x) \leq C \int_{\mathbb{R}^n} |x-y|^{-2\gamma} |y|^{-\alpha p'} dy.$$

The last integral can be computed explicitly using the well-known identity

$$\int_{\mathbb{R}^n} |x-y|^{-2\gamma} |y|^{-\alpha p'} dy = C|x|^{n-\alpha p' - 2\gamma}$$

provided that $0 < \alpha < \frac{n}{p'}$ and that $\frac{n-\alpha p'}{2} < \gamma < \frac{n}{2}$ (see, e.g. [18, Chap. V, § 1.1]). Therefore, for such a γ we obtain that

$$I_\infty(x) \leq C|x|^{n-\alpha p' - 2\gamma}$$

where the exponent is negative. Since

$$n - \alpha p' - 2\gamma \leq -\beta p' \Leftrightarrow \gamma \leq \frac{n - (\alpha-\beta)p'}{2}$$

and now $|x| \geq 1/2$, we obtain

$$I_\infty(x) \leq C|x|^{-\beta p'} \quad \text{if } |x| \geq \frac{1}{2} \tag{2.8}$$

provided that we choose γ to satisfy $\frac{n-(\alpha-\beta)p'}{2} < \gamma < \frac{n}{2}$, as we may under the conditions of the theorem.

Recalling (2.7), we obtain that

$$I_\infty(x) \leq C|x|^{-\beta p'} \quad \text{for all } x. \tag{2.9}$$

From (2.6) and (2.9), we can finally conclude that

$$I(x) \leq C|x|^{-\beta p'} \quad \text{for all } x$$

as announced, and hence, we obtain the desired bound (2.2) for $|x|^\beta |e^{t\Delta} f(x)|$. The bound (2.3) for its derivatives can be obtained analogously. □

Corollary 2.1. *We have an embedding*

$$L^p(\mathbb{R}^n, |x|^{\alpha p}) \subset \dot{B}^{-\mu-s}_{\infty,\infty}$$

with

$$\mu = \frac{n}{p} + \alpha - s$$

provided that $\mu > 0$ *and* $0 < \alpha < \frac{n}{p'}$. *By the lifting property of Besov spaces* (*see* [20, *Sec.* 5.2.3, *Theorem* 1]), *this implies that*

$$\dot{H}^{s,p}_{\alpha}(\mathbb{R}^n) \subset \dot{B}^{-\mu}_{\infty,\infty}.$$

It follows that, with this choice of μ, *our inequality* (1.5) *is indeed a refinement of the Stein–Weiss inequality* (1.2), *that* (1.6) *is a refinement of the weighted fractional Sobolev inequality* (1.3) *and that* (1.8) *is in some cases a refinement of the Caffarelli–Kohn–Nirenberg inequality* (1.7), *in the sense of* [17].

Proposition 2.2. *Let* $n \geq 2$, $1 \leq p \leq +\infty$. *Then, for any fixed* $t > 0$, *the operator* $e^{t\Delta}$ *is compact from* $L^p(\mathbb{R}^n, |x|^{\alpha p})$ *to* $L^\infty(\mathbb{R}^n)$ *provided that*

$$0 < \alpha < \frac{n}{p'}. \tag{2.10}$$

Proof. Let $\{u_j\}_{j\in\mathbb{N}}$ be a bounded sequence in $L^p(\mathbb{R}^n, |x|^{\alpha p})$, so that

$$\||x|^\alpha u_j\|_{L^p} \leq C.$$

Let $v_j = e^{t\Delta}u_j$. Then by Proposition 2.1 (with $\beta = 0$), $\{v_j\}_{j\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}^n)$.

For each $k \in \mathbb{N}$, let us consider the compact set

$$C_k = \{x \in \mathbb{R}^n : 0 \leq |x| \leq k\}.$$

The estimates of Proposition 2.1 also imply that $\{v_j\}_{j\in\mathbb{N}}$ is equibounded in C_k, and so are their first-order derivatives, hence $\{v_j\}_{j\in\mathbb{N}}$ is also equicontinuous in C_k. Using the Arzelá–Ascoli theorem and Cantor's diagonal argument, we conclude that passing again to a subsequence we may assume that

$$v_j \to v \quad \text{uniformly in each } C_k.$$

Since $\alpha > 0$ we can choose $\beta > 0$ such that $0 < \beta < \alpha$. Then, by Proposition 2.1 we get

$$\begin{aligned}\sup_{|x|>k} |v_j| &\leq \sup_{|x|>k} \left(\frac{|x|}{k}\right)^\beta |v_j| \\ &\leq \frac{1}{k^\beta}\||x|^\beta v_j\|_{L^\infty} \leq \frac{C}{k^\beta}\||x|^\alpha u_j\|_{L^p} \leq \frac{C_1}{k^\beta}\end{aligned}$$

which tends to 0 as $k \to \infty$, uniformly in j. A standard argument gives that $v_j \to v$ strongly in $L^\infty(\mathbb{R}^n)$. □

3. Improved Inequalities

This section is devoted to establish the "improved" Stein–Weiss and Caffarelli–Kohn–Nirenberg inequalities. Recall that, as discussed in Corollary 2.1, they are indeed refinements of the original inequalities, which justifies that name.

Theorem 3.1. *Let* $n \geq 2$, $0 < s < n$, $1 < p \leq r$, $\alpha < \frac{n}{p'}$, $-\gamma < \frac{n}{r}$, $\alpha - \frac{\gamma}{\theta} \geq 0$, $\mu > 0$, $\max\{\frac{p}{r}, \frac{\mu}{\mu+s}\} \leq \theta \leq 1$, *and*

$$\gamma + \frac{n}{r} = \theta\left(\alpha + \frac{n}{p} - s\right) + (1-\theta)\mu. \tag{3.1}$$

Then, for every $f \in L^p(\mathbb{R}^n, |x|^{\alpha p}) \cap \dot{B}^{-\mu-s}_{\infty,\infty}$ *there holds*:

$$\||x|^{\gamma}(-\Delta)^{-s/2}f\|_{L^r} \leq C\||x|^{\alpha}f\|^{\theta}_{L^p}\|f\|^{1-\theta}_{\dot{B}^{-\mu-s}_{\infty,\infty}}$$

or, equivalently, for every $u \in \dot{H}^{s,p}_{\alpha}(\mathbb{R}^n) \cap \dot{B}^{-\mu}_{\infty,\infty}$

$$\||x|^{\gamma}u\|_{L^r} \leq C\||x|^{\alpha}(-\Delta)^{s/2}u\|^{\theta}_{L^p}\|u\|^{1-\theta}_{\dot{B}^{-\mu}_{\infty,\infty}}.$$

Proof. Notice that the case $\theta = 1$ corresponds to Theorem 1.1, so we may restrict ourselves to the case $\theta < 1$.

Let $u := (-\Delta)^{-s/2}f$, where $f \in L^p(\mathbb{R}^n, |x|^{\alpha p}dx)$. Hence, we write

$$u = \frac{1}{\Gamma\left(\frac{s}{2}\right)}\int_0^{\infty} t^{s/2-1}e^{t\Delta}f\,dt$$

and, for fixed $T > 0$ to be chosen later, we split the above integral in high and low frequencies, setting

$$Hf(x) := \frac{1}{\Gamma\left(\frac{s}{2}\right)}\int_0^{T} t^{s/2-1}e^{t\Delta}f\,dt$$

and

$$Lf(x) := \frac{1}{\Gamma\left(\frac{s}{2}\right)}\int_T^{\infty} t^{s/2-1}e^{t\Delta}f\,dt.$$

We will obtain pointwise bounds for Lf and Hf.

To bound Lf, we proceed as in [4], using the thermal definition of Besov spaces (Definition 2.1) to deduce that

$$|Lf(x)| \leq C\int_T^{\infty} t^{-\mu/2-1}\|f\|_{\dot{B}^{-\mu-s}_{\infty,\infty}}dt = CT^{-\mu/2}\|f\|_{\dot{B}^{-\mu-s}_{\infty,\infty}}. \tag{3.2}$$

To bound Hf, we have to consider different cases, according to whether $\theta = \frac{\mu}{\mu+s}$ or $\theta > \frac{\mu}{\mu+s}$.

First case: $\theta = \frac{\mu}{\mu+s}$.

Observe that in this case, we must also have $\theta = \frac{p}{r}$. Indeed, replacing $\theta = \frac{\mu}{\mu+s}$ in (3.1) and rearranging terms we have

$$\frac{\mu+s}{r} - \frac{\mu}{p} = \frac{\alpha\mu}{n} - \frac{\gamma(\mu+s)}{n} \geq 0,$$

where the last inequality follows from the condition $\alpha - \frac{\gamma}{\theta} \geq 0$ and the fact that $\mu > 0$. This immediately implies that $\frac{p}{r} \geq \frac{\mu}{\mu+s}$ but, since the reverse inequality holds by hypothesis, we obtain $\frac{p}{r} = \frac{\mu}{\mu+s} = \theta$.

Now, replacing $\theta = \frac{\mu}{\mu+s} = \frac{p}{r}$ in (3.1), we obtain $\alpha p = \gamma r$. This will be useful in what follows.

Having established the relations between the parameters, we remark that this case is contained in [4, Annexe C], but we outline the result here for the sake of completeness. Following [4], we obtain

$$|Hf(x)| \leq CT^{s/2}Mf(x), \tag{3.3}$$

where Mf is the Hardy–Littlewood maximal function.

Now we choose T to optimize the sum of (3.2) and (3.3), namely

$$T = \left(\frac{\|f\|_{\dot{B}^{-\mu-s}_{\infty,\infty}}}{Mf}\right)^{\frac{2}{\mu+s}},$$

and we arrive at the pointwise bound

$$|(-\Delta)^{-s/2}f| \leq C(Mf)^{\theta}\|f\|^{1-\theta}_{\dot{B}^{-\mu-s}_{\infty,\infty}}.$$

This implies

$$\begin{aligned}\|(-\Delta)^{-s/2}f|x|^{\gamma}\|_{L^r} &\leq C\|(Mf)^{\theta}|x|^{\gamma}\|_{L^r}\|f\|^{1-\theta}_{\dot{B}^{-\mu-s}_{\infty,\infty}} \\ &= C\|Mf|x|^{\gamma r/p}\|^{\theta}_{L^p}\|f\|^{1-\theta}_{\dot{B}^{-\mu-s}_{\infty,\infty}} \\ &\leq C\|f|x|^{\alpha}\|^{\theta}_{L^p}\|f\|^{1-\theta}_{\dot{B}^{-\mu-s}_{\infty,\infty}},\end{aligned}$$

where we have used the relations between the parameters and the fact that $|x|^{\alpha p} = |x|^{\gamma r}$ belongs to the Muckenhoupt class A_p since $-n < \gamma r = \alpha p < n(p-1)$ by hypothesis, and hence the Hardy–Littlewood maximal function is continuous in L^p with that weight (see [16, Theorem 9]).

Second case: $\theta > \frac{\mu}{\mu+s}$.

In this case, observe that

$$Hf(x) = (K_{s,T} * f)(x),$$

where

$$K_{s,T}(x) = \frac{1}{\Gamma\left(\frac{s}{2}\right)}\int_0^T t^{s/2-1}h_t(x)dt$$

and

$$h_t(x) = \frac{1}{(4\pi t)^{n/2}} \exp\left\{ -\frac{|x|^2}{4t} \right\}$$

is the heat kernel.

Now, setting $2\varepsilon = \mu/\theta - \mu > 0$ and noting that $(n-s)/2 + \varepsilon > 0$, we have that

$$e^{-x} \le \frac{C}{x^{(n-s)/2+\varepsilon}} \quad \text{for } x > 0$$

whence

$$\begin{aligned} 0 \le K_{s,T}(x) &\le C \int_0^T t^{(s-n)/2-1} \left(\frac{4t}{|x|^2} \right)^{(n-s)/2+\varepsilon} dt \\ &\le C \frac{1}{|x|^{n-s+2\varepsilon}} \int_0^T t^{-1+\varepsilon} dt \\ &\le C \frac{1}{|x|^{n-s+2\varepsilon}} T^{\varepsilon}. \end{aligned}$$

Hence,

$$|Hf(x)| \le CT^{\varepsilon} I_{s-2\varepsilon} f(x) \tag{3.4}$$

and to optimize the sum of (3.2) and (3.4), we choose

$$T = \left(\frac{\|f\|_{\dot{B}^{-\mu-s}_{\infty,\infty}}}{I_{s-2\varepsilon} f(x)} \right)^{1/(\varepsilon+\mu/2)}.$$

Hence, in this case we have the pointwise bound

$$|(-\Delta)^{-s/2} f(x)| \le C I_{s-2\varepsilon} f(x)^{\theta} \|f\|^{1-\theta}_{\dot{B}^{-\mu-s}_{\infty,\infty}}.$$

Setting $\tilde{r} = \theta r$, taking r-norm and using Theorem 1.1, we have

$$\begin{aligned} \||x|^{\gamma} (-\Delta)^{-s/2} f\|_{L^r} &\le C \||x|^{\gamma/\theta} I_{s-2\varepsilon} f\|^{\theta}_{L^{\tilde{r}}} \|f\|^{1-\theta}_{\dot{B}^{-\mu-s}_{\infty,\infty}} \\ &\le C \||x|^{\alpha} f\|^{\theta}_{L^p} \|f\|^{1-\theta}_{\dot{B}^{-\mu-s}_{\infty,\infty}}. \end{aligned} \tag{3.5}$$

It remains to check the conditions of Theorem 1.1: $\alpha < \frac{n}{p'}$ and $\alpha - \frac{\gamma}{\theta} \ge 0$ are immediate, while $-\frac{\gamma}{\theta} < \frac{n}{\tilde{r}}, p \le \tilde{r}$ and $\frac{1}{\tilde{r}} = \frac{1}{p} + \frac{\alpha - \gamma/\theta - (s-2\varepsilon)}{n}$ follow by our choice of $\tilde{r}$ and ε and the hypotheses of our theorem.

This proves our first inequality. To prove the equivalence with the second one we need to use the lifting property of Besov spaces

$$\|f\|_{\dot{B}^{-\mu-s}_{\infty,\infty}} = \|(-\Delta)^{s/2} u\|_{\dot{B}^{-\mu-s}_{\infty,\infty}} = \|u\|_{\dot{B}^{-\mu}_{\infty,\infty}}$$

and we arrive at the desired inequality. □

As announced, an analogous result can also be obtained in the local case $s = 1$, with the gradient instead of the fractional Laplacian. Indeed, Theorem 3.1 implies the following refined weighted Sobolev inequality, which is an improvement of the Caffarelli–Kohn–Nirenberg inequalities [3] in some cases:

Theorem 3.2. *Let* $n > 1$, $1 < p \le r$, $\alpha < \frac{n}{p'}$, $-\frac{n}{r} < \gamma < \frac{n}{r'}$, $\alpha - \frac{\gamma}{\theta} \ge 0$, $\mu > 0$, $\max\{\frac{p}{r}, \frac{\mu}{\mu+1}\} \le \theta \le 1$, *and*

$$\gamma + \frac{n}{r} = \theta\left(\alpha + \frac{n}{p} - 1\right) + (1-\theta)\mu.$$

Then, for every $u \in \dot{H}^{1,p}_\alpha(\mathbb{R}^n) \cap \dot{B}^{-\mu}_{\infty,\infty}$

$$\||x|^\gamma u\|_{L^r} \le C\||x|^\alpha \nabla u\|^\theta_{L^p} \|u\|^{1-\theta}_{\dot{B}^{-\mu}_{\infty,\infty}}.$$

Proof. We consider the classical Riesz transforms R_j,

$$R_j u = \frac{\partial}{\partial x_j}(-\Delta)^{-1/2} u.$$

Since R_j is a Calderón–Zygmund operator, it is bounded in $L^r(\mathbb{R}^n, |x|^{\gamma r})$, because the weight $|x|^{\gamma r}$ belongs to the Muckenhoupt class A_r by hypothesis. Moreover, using the Fourier transform, it is easy to check that

$$\sum_{j=1}^n R_j^2 = -I.$$

Then,

$$\||x|^\gamma u\|_{L^r} \le \sum_{j=1}^n \||x|^\gamma R_j^2 u\|_{L^r} \le C \sum_{j=1}^n \||x|^\gamma R_j u\|_{L^r}.$$

We apply Theorem 3.1 (with $s = 1$) in order to obtain

$$\begin{aligned}
\||x|^\gamma R_j u\|_{L^r} &\le C\||x|^\alpha (-\Delta)^{1/2} R_j u\|^\theta_{L^p} \|R_j u\|^{1-\theta}_{\dot{B}^{-\mu}_{\infty,\infty}} \\
&\le C \left\| |x|^\alpha \frac{\partial u}{\partial x_j} \right\|^\theta_{L^p} \|R_j u\|^{1-\theta}_{\dot{B}^{-\mu}_{\infty,\infty}} \\
&\le C \left\| |x|^\alpha \frac{\partial u}{\partial x_j} \right\|^\theta_{L^p} \|u\|^{1-\theta}_{\dot{B}^{-\mu}_{\infty,\infty}}
\end{aligned}$$

since R_j is a bounded operator in the Besov space $\dot{B}^{-\mu}_{\infty,\infty}$ (see [10]). Hence, we conclude that

$$\||x|^\gamma u\|_{L^r} \le C \sum_{j=1}^n \left\| |x|^\alpha \frac{\partial u}{\partial x_j} \right\|^\theta_{L^p} \|u\|^{1-\theta}_{\dot{B}^{-\mu}_{\infty,\infty}} \le C\||x|^\alpha \nabla u\|^\theta_{L^p} \|u\|^{1-\theta}_{\dot{B}^{-\mu}_{\infty,\infty}}$$

as announced. □

4. Local Compactness of the Embedding

In this section, we prove a version of the Rellich-Kondrachov theorem for the weighted homogeneous Sobolev space $\dot{H}^{s,p}_\alpha(\mathbb{R}^n)$. We shall need it in the proof of

Theorem 5.1, but we believe that it could be of independent interest for the study of other fractional elliptic problems. It is worth noting that this result does not seem to follow directly from the standard unweighted version of the compactness theorem, since it is not easy to perfom truncation arguments due to the non-local nature of the fractional Laplacian operator in the definition (1.4) of the space $\dot{H}^{s,p}_{\alpha}(\mathbb{R}^n)$. Instead, we work directly with the definition of the weighted space.

Theorem 4.1. *Let $n \geq 1$, $0 < s < n$, $1 < p \leq q < \infty$. Assume further that α, β and s satisfy the set of conditions*

$$\beta > -\frac{n}{q}, \quad \alpha < \frac{n}{p'}, \quad \alpha \geq \beta \tag{4.1}$$

and

$$\alpha + \frac{n}{p} > s > \frac{n}{p} - \frac{n}{q} + \alpha - \beta > 0. \tag{4.2}$$

Then, for any compact set $\mathcal{K} \subset \mathbb{R}^n$, we have the compact embedding

$$\dot{H}^{s,p}_{\alpha}(\mathbb{R}^n) \subset L^q(\mathcal{K}, |x|^{\beta q}). \tag{4.3}$$

We observe that (4.2) is a subcriticality condition. Indeed, under the hypotheses of the theorem

$$s > \frac{n}{p} - \frac{n}{q}$$

and if $s < \frac{n}{p}$, this is equivalent to

$$q < p^* = \frac{np}{n - sp}.$$

We first prove the continuity of the embedding (4.3). Let $u \in \dot{H}^{s,p}_{\alpha}(\mathbb{R}^n)$. Then

$$u = (-\Delta)^{-s/2} f \quad \text{with } f \in L^p(\mathbb{R}^n, |x|^{\alpha p})$$

and

$$\|u\|_{\dot{H}^{s,p}_{\alpha}} = \||x|^{\alpha} f\|_{L^p(\mathbb{R}^n)}.$$

We define a new exponent $\tilde{q}$ satisfying the Stein–Weiss scaling condition

$$\frac{n}{\tilde{q}} := \frac{n}{p} + \alpha - \beta - s.$$

From (4.2) it follows that $\tilde{q} > q$. We define

$$r := \frac{\tilde{q}}{q} > 1, \quad \tilde{\beta} := \frac{\beta q}{\tilde{q}} = \frac{\beta}{r}$$

so that

$$\beta > -\frac{n}{q} \Leftrightarrow \tilde{\beta} > -\frac{n}{\tilde{q}}$$

and apply Hölder's inequality with exponents r and r' to obtain

$$\begin{aligned}\int_{\mathcal{K}} |u|^q |x|^{\beta q} dx &= \int_{\mathcal{K}} |u|^q |x|^{\beta q/r} |x|^{\beta q/r'} dx \\ &\leq \left(\int_{\mathcal{K}} |u|^{\tilde{q}} |x|^{\tilde{\beta}\tilde{q}} dx \right)^{1/r} \left(\int_{\mathcal{K}} |x|^{\beta q} dx \right)^{1/r'} \\ &\leq C_{\mathcal{K}} \left(\int_{\mathbb{R}^n} |u|^{\tilde{q}} |x|^{\tilde{\beta}\tilde{q}} dx \right)^{1/r}\end{aligned}$$

since $\beta > -\frac{n}{q}$. Then, using Theorem 1.1

$$\begin{aligned}\left(\int_{\mathcal{K}} |u|^q |x|^{\beta q} dx \right)^{1/q} &\leq C_{\mathcal{K}}^{1/q} \left(\int_{\mathbb{R}^n} |u|^{\tilde{q}} |x|^{\tilde{\beta}\tilde{q}} dx \right)^{1/\tilde{q}} \\ &\leq C \left(\int_{\mathbb{R}^n} |f|^p |x|^{\alpha p} \right)^{1/p} \\ &= C \|u\|_{\dot{H}^{s,p}_{\alpha}}.\end{aligned}$$

This shows that (4.3) is a continuous embedding.

The main difficulty in the proof of the local compactness is that the kernel

$$K_s^t(x) = C(n,s)|x|^{-(n-s)}$$

of the Riesz potential is not in the dual space $L^{p'}(\mathbb{R}^n, |x|^{-\alpha p'})$. For this reason, we introduce for $t > 0$ the truncated kernels

$$K_s^t(x) = C(n,s)|x|^{-(n-s)} \chi_{\{|x|>t\}}.$$

The following lemma gives a kind of pseudo-Poincaré inequality using these kernels.

Lemma 4.1. *Under the conditions of Theorem* 4.1, *set*

$$\delta = s - \left(\frac{n}{p} - \frac{n}{q} + \alpha - \beta \right).$$

Then, for any function, $u \in L^p(\mathbb{R}^n, |x|^{\alpha p})$ *and any* $t > 0$, *we have that*

$$\|(K_s^t * u - K_s * u)|x|^{\beta}\|_{L^q} \leq C t^{\delta} \||x|^{\alpha} u\|_{L^p}.$$

Proof. Notice that

$$K_s^t * f - K_s * f(x) = C(n,s) \int_{|x-y|\leq t} \frac{f(y)}{|x-y|^{n-s}} dy.$$

Hence, since $\delta > 0$ by (4.2),

$$
\begin{aligned}
|K_s^t * f - K_s * f(x)| &\leq C(n,s) \int_{|x-y|\leq t} \frac{|f(y)|}{|x-y|^{n-s}} dy \\
&\leq C(n,s) \int_{|x-y|\leq t} \frac{|f(y)|}{|x-y|^{n-s}} \left(\frac{t}{|x-y|}\right)^{\delta} dy \\
&\leq C(n,s) t^{\delta} \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-(s-\delta)}} dy
\end{aligned}
$$

and the lemma follows from the Stein–Weiss inequality since $s-\delta > 0$ (the definition of δ means that the required scaling-condition holds with $s-\delta$ in place of s). $\square$

Now we are ready to prove the compactness of the embedding (4.3): let (u_k) be a bounded sequence $\dot{H}^{s,p}_{\alpha}(\mathbb{R}^n)$. We may write it as

$$u_k = (-\Delta)^{-s/2} f_k = K_s * f_k,$$

where f_k is a bounded sequence in $L^p(\mathbb{R}^n, |x|^{\alpha p})$. By reflexivity, passing to a subsequence, we may assume that

$$f_k \rightharpoonup f \quad \text{weakly in } L^p(\mathbb{R}^n, |x|^{\alpha p}).$$

We consider the functions

$$u_k^t = K_s^t * f_k, \quad u^t = K_s^t * f$$

and we write

$$
\begin{aligned}
&\|(u_k - u)|x|^{\beta}\|_{L^q(\mathcal{K})} \\
&\quad \leq \|(u_k - u_k^t)|x|^{\beta}\|_{L^q(\mathcal{K})} + \|(u_k^t - u^t)|x|^{\beta}\|_{L^q(\mathcal{K})} + \|(u^t - u)|x|^{\beta}\|_{L^q(\mathcal{K})}.
\end{aligned}
$$

We observe that

$$\|(u_k - u_k^t)|x|^{\beta}\|_{L^q(\mathcal{K})} \leq t^{\delta} \|f_k|x|^{\alpha}\|_{L^p} \leq C t^{\delta}$$

and that

$$\|(u^t - u)|x|^{\beta}\|_{L^q(\mathcal{K})} \leq t^{\delta} \|f|x|^{\alpha}\|_{L^p} \leq C t^{\delta}.$$

Hence, given $\varepsilon > 0$ we can make this two terms less than $\frac{\varepsilon}{3}$ for all k, provided that we fix t small enough.

We check that $K_s^t(x_0 - \cdot)$ is in $L^{p'}(\mathbb{R}^n, |x|^{-\alpha p'})$. For that, we consider the integral.

$$
\begin{aligned}
I(x) &:= \int_{\mathbb{R}^n} |K_s^t(x-y)|^{p'} |y|^{-\alpha p'} dy \\
&= C(n,s)^{p'} \int_{|x-y|>t} \frac{1}{|x-y|^{(n-s)p'}} |y|^{-\alpha p'} dy.
\end{aligned}
$$

The integrability condition at zero is

$$-\alpha p' + n > 0 \Leftrightarrow \alpha < \frac{n}{p'}$$

and at infinity is

$$-\alpha p' - (n-s)p' + n < 0 \Leftrightarrow s < \alpha + \frac{n}{p}.$$

Hence $I(x)$ is finite. We conclude that, that for any fixed $t > 0$, and any fixed x_0 we have that

$$u_k^t(x_0) = K_s^t * f_k(x_0) \to K_s^t * f(x_0) = u^t(x_0).$$

Moreover, we have that for any x_0 in the compact set $\mathcal{K}$

$$\int_{\mathbb{R}^n} |K_s^t(x_0 - x)|^{p'} |x|^{-\alpha p'} dx \leq C_{\mathcal{K}}. \tag{4.4}$$

Indeed, if we write

$$I(x) = I_0(x) + I_\infty(x)$$

where $\mathcal{K} \subset B(0,R)$ and

$$I_0(x) = C(n,s)^{p'} \int_{|x-y|>t, |y|\leq 2R} \frac{1}{|x-y|^{(n-s)p'}} |y|^{-\alpha p'} dy,$$

$$I_\infty(x) = C(n,s)^{p'} \int_{|x-y|>t, |y|> 2R} \frac{1}{|x-y|^{(n-s)p'}} |y|^{-\alpha p'} dy,$$

then,

$$I_0(x) \leq C(n,s)^{p'} \int_{y|\leq 2R} \frac{1}{t^{(n-s)p'}} |y|^{-\alpha p'} dy = C(n,s)^{p'} \frac{1}{t^{(n-s)p'}} (2R)^{-\alpha p' + n}.$$

On the other hand, when $x \in \mathcal{K}$ and y is in the integration region of I_∞

$$|x| \leq R < \frac{1}{2}|y|$$

and

$$|x-y| \geq |y| - |x| \geq |y| - \frac{1}{2}|y| = \frac{1}{2}|y|.$$

Hence,

$$\begin{aligned} I_\infty(x) &\leq C(n,s)^{p'} \int_{|y|>2R} \frac{1}{\left(\frac{1}{2}|y|\right)^{(n-s)p)'}} |y|^{-\alpha p'} dy \\ &= \tilde{C}(s,n,p) R^{-(n-s)p' - \alpha p' + n} \end{aligned}$$

which implies (4.4). Thus, (u_k^t) is uniformly bounded on $\mathcal{K}$, and by the bounded convergence theorem,

$$\|(u_k^t - u^t)|x|^\beta\|_{L^q(\mathcal{C})} \to 0$$

as $k \to \infty$ (since the condition $\beta > -\frac{n}{q}$ means that the weight $|x|^{\beta q}$ is integrable on $\mathcal{K}$). Therefore, we can make it less than $\varepsilon/3$ for $k \geq k_0(\varepsilon)$.

We conclude that $u_k \to u$ strongly in $L^q(\mathbb{R}^n, |x|^{\beta q})$ as we wanted.

5. Existence of Maximizers of the Stein–Weiss Inequality

In this section, we prove our main theorem, which extends the result of Lieb [14, Theorem 5.1] to some previously unknown cases when $p = 2$.

The proof uses a well-known strategy, but the results are new thanks to our improved Stein–Weiss inequality (Theorem 3.1), and the weighted compactness results (Proposition 2.2 and Theorem 4.1). First, we show that from any maximizing sequence we can extract — after a suitable rescaling — a subsequence with a nonzero weak limit. In the second part, we use the so-called "method of missing mass" (invented by Lieb in [14]) to prove that such a limit is actually an optimizer.

Theorem 5.1. *Assume that $n \geq 2$, $0 < s < \frac{n}{2}$, $2 < r < \infty$, $0 < \alpha < \frac{n}{2}$, $-\frac{n}{r} < \gamma < \alpha$ and that the relation*

$$\frac{1}{r} - \frac{1}{2} = \frac{\alpha - \gamma - s}{n}$$

holds. Then, there exists a maximizer for S.

Remark 5.1. Notice that condition $0 < s < \frac{n}{2}$ does not appear explicitly in [14, Theorem 5.1] but is implied by the other conditions on the parameters. Indeed, since $\alpha \geq \gamma$, we have that $1 < r \leq \frac{2n}{n-2s}$ and, in particular, there must hold $n - 2s > 0$.

Proof of Theorem 5.1. Let $\{f_k\}_{k\in\mathbb{N}}$ be a maximizing sequence of S, that is,

$$\||x|^\alpha f_k\|_{L^2} = 1 \quad \text{and} \quad \||x|^\gamma(-\Delta)^{-s/2} f_k\|_{L^r} \to S. \tag{5.1}$$

By Corollary 2.1, if we set

$$\mu = \frac{n}{2} + \alpha - s, \tag{5.2}$$

it holds that

$$\|f_k\|_{\dot{B}^{-\mu-s}_{\infty,\infty}} \leq C\||x|^\alpha f_k\|_{L^2} = C.$$

On the other hand, Theorem 3.1 gives that

$$\|f_k\|_{\dot{B}^{-\mu-s}_{\infty,\infty}} \geq C > 0$$

provided we choose θ such that

$$\max\left\{\frac{2}{r}, \frac{\mu}{\mu+s}, \frac{\gamma}{\alpha}\right\} < \theta < 1$$

which is possible by the hypotheses of the Theorem. This means that

$$\sup_{t>0} t^{\frac{\mu+s}{2}} \|e^{t\Delta} f_k\|_{L^\infty} \geq C > 0.$$

Consequently, for each $k \in \mathbb{N}$ we can find $t_k > 0$ such that

$$t_k^{\frac{\mu+s}{2}} \|e^{t_k\Delta} f_k\|_{L^\infty} \geq \frac{C}{2} > 0.$$

Now we set

$$\tilde{f}_k(x) = t_k^{\frac{1}{2}(\frac{n}{2}+\alpha)} f_k(t_k^{\frac{1}{2}} x), \tag{5.3}$$

and observe that, by parabolic scaling,

$$e^{1.\Delta} \tilde{f}_k(x) = t^{\frac{1}{2}(\frac{n}{2}+\alpha)} e^{\Delta t_k} f_k(t_k^{\frac{1}{2}} x).$$

Then,

$$\|e^{1.\Delta} \tilde{f}_k\|_{L^\infty} = t_k^{\frac{1}{2}(\frac{n}{2}+\alpha)} \|e^{t_k \Delta} f_k(t_k^{\frac{1}{2}} x)\|_{L^\infty} = t_k^{\frac{\mu+s}{2}} \|e^{t_k\Delta} f_k\|_{L^\infty} \geq \frac{C}{2} > 0 \tag{5.4}$$

since relation (5.2) holds.

Observe that, in view of the scaling invariance of the $L^2(\mathbb{R}^n, |x|^{2\alpha})$ norm, the sequence $\{\tilde{f}_k\}_k$ is bounded in $L^2(\mathbb{R}^n, |x|^{2\alpha})$ and, that the maximization problem (1.9) is invariant under the rescaling given by $\tilde{f}_k$ as long as

$$\gamma = \frac{n}{2} - \frac{n}{r} + \alpha - s,$$

which holds by our assumptions. Indeed,

$$\||x|^\alpha \tilde{f}_k\|_{L^2} = \||x|^\alpha f_k\|_{L^2} = 1. \tag{5.5}$$

Consequently, $\{\tilde{f}_k\}_k$ is also a minimizing sequence of S, that is,

$$\||x|^\alpha \tilde{f}_k\|_{L^2} = 1 \quad \text{and} \quad \||x|^\gamma (-\Delta)^{-s/2} \tilde{f}_k\|_{L^r} \to S. \tag{5.6}$$

It remains to show that there exists $g \not\equiv 0$ such that $\tilde{f}_k \to g$ strongly in $L^2(\mathbb{R}^n, |x|^{2\alpha})$. The last requirement will allow us to deduce that g is also a minimizer for S, i.e.

$$\||x|^\alpha g\|_{L^2} = 1 \quad \text{and} \quad \||x|^\gamma (-\Delta)^{-s/2} g\|_{L^r} = S. \tag{5.7}$$

By reflexivity, from (5.5) there exists $g \in L^2(\mathbb{R}^n, |x|^{2\alpha})$ and a subsequence still denoted by $\tilde{f}_k$ such that

$$\tilde{f}_k \rightharpoonup g \quad \text{weakly in } L^2(\mathbb{R}^n, |x|^{2\alpha}). \tag{5.8}$$

We set

$$u_k := (-\Delta)^{-s/2} \tilde{f}_k, \quad w := (-\Delta)^{-s/2} g.$$

From Proposition 2.2, the compactness of the operator $e^{1.\Delta}$ implies that

$$e^{1.\Delta} \tilde{f}_k \to e^{1.\Delta} g \quad \text{strongly in } L^\infty(\mathbb{R}^n)$$

and then $g \not\equiv 0$, since by (5.4) we have

$$\|e^{1.\Delta} g\|_{L^\infty} \geq \frac{C}{2} > 0.$$

By Theorem 4.1 with $2 < q < r$ and $\alpha = \beta$, for any compact set K we have the compact embedding

$$\dot{H}^{s,2}(\mathbb{R}^n) \subset L^q(K)$$

which implies that passing to a subsequence we may assume that

$$u_k \to w \quad \text{strongly in } L^q(K)$$

and, therefore, up to a subsequence, $u_k \to w$ a.e. in K. By using a diagonal argument we obtain that, again, up to a subsequence,

$$u_k \to w \quad \text{a.e. } \mathbb{R}^n. \tag{5.9}$$

Let us prove that (5.7) holds. Since $u_k \to w$ a.e. $\mathbb{R}^n$, the Brezis–Lieb Lemma ([14, Lemma 2.6; 2]) claims that

$$\lim_{k\to\infty} \left(\int_{\mathbb{R}^n} |u_k|^r |x|^{r\gamma} dx - \int_{\mathbb{R}^n} |u_k - w|^r |x|^{r\gamma} dx \right) = \int_{\mathbb{R}^n} |w|^r |x|^{r\gamma} dx,$$

but, from (5.6),

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} |u_k|^r |x|^{r\gamma} dx = \int_{\mathbb{R}^n} |w|^r |x|^{r\gamma} dx + \lim_{k\to\infty} \int_{\mathbb{R}^n} |u_k - w|^r |x|^{r\gamma} dx. \tag{5.10}$$

Combining (5.10) with (5.6) and the elementary inequality

$$a^{\frac{r}{2}} + b^{\frac{r}{2}} \le (a+b)^{\frac{r}{2}} \tag{5.11}$$

for $a, b \ge 0$ and $r > 2$, we have

$$\begin{aligned}
S^r &= \lim_{k\to\infty} \int_{\mathbb{R}^n} |u_k|^r |x|^{r\gamma} dx \\
&= \int_{\mathbb{R}^n} |w|^r |x|^{r\gamma} dx + \lim_{k\to\infty} \int_{\mathbb{R}^n} |u_k - w|^r |x|^{r\gamma} dx \\
&\le S^r \left(\int_{\mathbb{R}^n} |g|^2 |x|^{2\alpha} dx \right)^{\frac{r}{2}} + S^r \left(\limsup_{k\to\infty} \int_{\mathbb{R}^n} |\tilde{f}_k - g|^2 |x|^{2\alpha} dx \right)^{\frac{r}{2}} \\
&\le S^r \left(\int_{\mathbb{R}^n} |g|^2 |x|^{2\alpha} dx + \limsup_{k\to\infty} \int_{\mathbb{R}^n} |\tilde{f}_k - g|^2 |x|^{2\alpha} dx \right)^{\frac{r}{2}} = S^r,
\end{aligned}$$

where we have used that

$$\int_{\mathbb{R}^n} |g|^2 |x|^{2\alpha} dx + \limsup_{k\to\infty} \int_{\mathbb{R}^n} |\tilde{f}_k - g|^2 |x|^{2\alpha} dx = \limsup_{k\to\infty} \int_{\mathbb{R}^n} |\tilde{f}_k|^2 |x|^{2\alpha} dx = 1 \tag{5.12}$$

since $\tilde{f}_k \rightharpoonup g$ weakly in $L^2(\mathbb{R}^n, |x|^{2\alpha})$.

Observe that (5.11) is a strict inequality unless $a = 0$ or $b = 0$. Hence, since all the previous inequalities are in fact equalities, we obtain that $\| |x|^\alpha g \|_{L^2} = 1$ and $\tilde{f} \to g$ strongly in $L^2(\mathbb{R}^n, |x|^{2\alpha})$.

By Theorem 1.1, $(-\Delta)^{-s/2}$ is a continuous operator from $L^2(\mathbb{R}^n, |x|^{2\alpha})$ into $L^r(\mathbb{R}^n, |x|^{\gamma r})$ and then

$$u_k \to w \quad \text{strongly in } L^r(\mathbb{R}^n, |x|^{\gamma r})$$

from where (5.7) follows. The proof is now complete. $\square$

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