Research Article

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A class of shape optimization problems for some nonlocal operators

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Abstract: In this work we study a family of shape optimization problem where the state equation is given in terms of a nonlocal operator. Examples of the problems considered are monotone combinations of fractional eigenvalues. Moreover, we also analyze the transition from nonlocal to local state equations.

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1 Introduction

In this article, we consider shape optimization problems that in the most general form can be stated as follows: Given a *cost functional* F, and a class of *admissible domains* $\mathcal{A}$, solve the minimization problem

$$\min_{A\in\mathcal{A}} F(A). \tag{1.1}$$

These types of problems have extensively been considered, and they arise in many fields and in many applications. The literature is very wide, from the classical cases of isoperimetrical problems to the most recent applications including elasticity and spectral optimization. Only to mention some references, we refer the reader to the books of Allaire [2], Bucur and Buttazzo [7], Henrot [19], Pironneau [27] and Sokołowski and Zolésio [34], where a huge amount of shape optimization problems are tackled.

In most of the existing references, the cost functional F is given in terms of a function u_A which is the solution of a *state equation* to be solved on A of the form

$$F(A) = \int_A j(\nabla u_A, u_A, x)\,dx.$$

Typically, this state equation is an elliptic PDE.

In recent years there has been an increasing amount of interest in nonlocal problems due to several interesting applications that include some physical models [14, 15, 17, 23, 26, 36], finance [1, 24, 31], fluid dynamics [10], ecology [20, 25, 30] and image processing [18].

However, there are only a handful of results of shape optimization problems of the form (1.1), where the state equation involves a nonlocal operator instead of an elliptic PDE.

For instance, in [33], the authors extend the well-known Faber–Krahn inequality to the fractional case and as a simple corollary, they solve problem (1.1) in the case when $F(A) = \lambda_1^s(A)$, where $\lambda_1^s(A)$ is the first

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eigenvalue of the Dirichlet fractional laplacian and the class $\mathcal{A}$ is the class of open sets of fixed measure. (See the next section for precise definitions.)

In [4] the authors consider again the class $\mathcal{A}$ of open sets of fixed measure and $F(A) = \lambda_2^s(A)$ and prove that problem (1.1) does not have a solution. In fact, a minimization sequence of domains consists of a sequence of balls of the same measure where the distance of the centers diverges.

Finally, in [16], the authors take the class $\mathcal{A}$ of measurable sets of fixed measure contained in a fixed open set Ω and the cost functional $F(A) = \lambda_1^s(\Omega \setminus A)$, where in this case, $\lambda_1^s(\Omega \setminus A)$ is the first eigenvalue of the fractional laplacian with Dirichlet condition on A and Neumann condition in $\mathbb{R}^n \setminus \Omega$.

For other recent shape optimization problems where the state equation is nonlocal, see [8, 11, 21, 22, 29], and references therein.

The purpose of this article is to consider a family of minimization problems of the form (1.1) for costs functions F under some natural assumptions that includes the particular cases mentioned above. These natural assumptions are similar to those considered in [9] where the authors addressed this problem when the state equation is given in terms of an elliptic PDE. Roughly speaking, these assumptions are:

- monotonicity with respect to the inclusion,
- lower semicontinuity with respect to a suitable defined notion of convergence of domains.

Observe that the results of [4] put a restriction on the classes of admissible domains that one needs to consider if you want to obtain a positive result. This is mainly due to the fact that taking a domain with two connected components and making these components go far away from each other makes the nonlocal energy decrease. So, in the spirit of [9] we restrict ourselves to the class $\mathcal{A}$ of open sets of fixed measure that are contained in a fixed box $Q \subset \mathbb{R}^n$.

Under these conditions, we are able to recover the results of [9] in the fractional setting and, moreover, we analyze the transition from the fractional case to the classical elliptic PDE case proving convergence of the minima and of the optimal shapes.

2 Setting of the problem

2.1 Some preliminaries and notation

Given $s \in (0, 1)$ we consider the fractional laplacian, that for smooth functions u (C^2 and bounded are enough, see [13]) is defined as

$$(-\Delta)^s u(x) := c(n, s)\,\text{p.v.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}}\,dy = -\frac{c(n, s)}{2} \int_{\mathbb{R}^n} \frac{u(x + z) - 2u(x) + u(x - z)}{|z|^{n+2s}}\,dz.$$

where

$$c(n, s) := \left(\int_{\mathbb{R}^n} \frac{1 - \cos \zeta_1}{|\zeta|^{n+2s}}\,d\zeta \right)^{-1}$$

is a normalization constant. The constant $c(n, s)$ is chosen in such a way that the following identity holds:

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}(u)),$$

for u in the Schwarz class of rapidly decreasing and infinitely differentiable functions, where $\mathcal{F}$ denotes the Fourier transform. See [13, Proposition 3.3].

The natural functional setting for this operator is the fractional Sobolev space $H^s(\mathbb{R}^n)$ defined as

$$\begin{aligned} H^s(\mathbb{R}^n) &:= \left\{ u \in L^2(\mathbb{R}^n) : \frac{u(x) - u(y)}{|x - y|^{\frac{n}{2}+s}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n) \right\} \\ &= \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^{2s})|\mathcal{F}(u)(\xi)|^2\,d\xi < \infty \right\}, \end{aligned}$$

which is a Banach space endowed with the norm $\|u\|_s^2 := \|u\|_2^2 + [u]_s^2$, where the term

$$[u]_s^2 := \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}}\, dx\, dy$$

is the so-called Gagliardo semi-norm of u.

Due to the nonlocal nature of the operator $(-\Delta)^s$, when dealing with Dirichlet-type problems in a open bounded set $\Omega \subset \mathbb{R}^n$, it is necessary to contemplate the "boundary condition" not only on $\partial\Omega$ but in the whole $\mathbb{R}^n \setminus \Omega$. The natural space to work with is denoted as $H_0^s(\Omega)$ and it is defined by the closure of $C_c^\infty(\Omega)$ in the norm $\|\cdot\|_s$. When Ω is a Lipschitz domain, $H_0^s(\Omega)$ coincides with the space of functions vanishing outside Ω, i.e.,

$$H_0^s(\Omega) = \{u \in H^s(\mathbb{R}^n) : u = 0 \text{ in } \mathbb{R}^n \setminus \Omega\}.$$

Aimed at our purposes in this paper, it is suitable to analyze the behavior of the normalization constant $c(n, s)$ as $s \uparrow 1$. In [35], E. Stein studied the relation between negative powers of the Laplace operator and Riesz potentials. In this context it is proved that

$$\lim_{s \uparrow 1} \frac{c(n, s)}{1 - s} = \frac{4n}{\omega_{n-1}},$$

where ω_{n-1} denotes the $(n-1)$-dimensional measure of the unit sphere S^{n-1}. That choice of the constant is consistent in order to recover the usual laplacian in the sense that

$$\lim_{s \uparrow 1} (-\Delta)^s u = -\Delta u \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n). \tag{2.1}$$

For a direct proof of these facts, we refer to the article [13].

Moreover, in [13, Remark 4.3] it is shown that

$$\lim_{s \uparrow 1} \frac{c(n, s)}{2} [u]_s^2 = \|\nabla u\|_2^2.$$

2.2 Statements of the main results

We begin with some definitions.

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be an open set. Given $A \subset \Omega$, for any $0 < s < 1$, we define the Gagliardo s-capacity of A relative to Ω as

$$\operatorname{cap}_s(A, \Omega) = \inf\{[u]_s^2 : u \in C_c^\infty(\Omega),\ u \geq 0,\ A \subset \{u \geq 1\}^\circ\}.$$

In this context, we say that a subset A of Ω is a *s-quasi open* set if there exists a decreasing sequence $\{\omega_k\}_{k \in \mathbb{N}}$ of open subsets of Ω such that $\operatorname{cap}_s(\omega_k, \Omega) \to 0$, as $k \to \infty$, and $A \cup \omega_k$ is an open set for all $k \in \mathbb{N}$. We denote by $\mathcal{A}_s(\Omega)$ the class of all s-quasi open subsets of Ω.

In the case $s = 1$ the definitions are completely analogous with $\|\nabla u\|_2$ instead of $[u]_s^2$.

Remark 2.2. From Hölder's inequality is easy to see that $\mathcal{A}_s(\Omega) \subset \mathcal{A}_t(\Omega)$ when $0 < t < s \leq 1$.

For further properties of the s-capacity we refer the reader, for instance, to [32].

Given $A \in \mathcal{A}_s(\Omega)$, we denote by $u_A^s \in H_0^s(A)$ the unique (weak) solution to

$$(-\Delta)^s u_A^s = 1 \quad \text{in } A, \qquad u_A^s = 0 \quad \text{in } \mathbb{R}^n \setminus A.$$

Remark 2.3. Observe also that u_A^s is the unique minimizer of

$$I_s(u) := \frac{c(n, s)}{2} [u]_s^2 - \int_A u\, dx \tag{2.2}$$

in $H_0^s(A)$.

With this notation, we define the following notion of set convergence.

Definition 2.4. Let $\{A_k\}_{k \in \mathbb{N}} \subset \mathcal{A}_s(\Omega)$ and $A \in \mathcal{A}_s(\Omega)$. We say that $A_k \xrightarrow{\gamma_s} A$ if $u_{A_k}^s \to u_A^s$ strongly in $L^2(\Omega)$.

Remark 2.5. This is the fractional version of the γ-convergence of sets defined in [9].

Now, take $0 < s < 1$ be fixed and let $F_s : \mathcal{A}_s(\Omega) \to \mathbb{R}$ be such that
(H_1^s) F_s is lower semicontinuous with respect to the γ_s-convergence; that is,

$$A_k \xrightarrow{\gamma_s} A \quad \text{implies} \quad F_s(A) \le \liminf_{k\to\infty} F_s(A_k).$$

(H_2^s) F_s is decreasing with respect to set inclusion; that is $F_s(A) \ge F_s(B)$ whenever $A \subset B$.
So, the problem that we address in this paper is the following:

$$\min\{F_s(A) : A \in \mathcal{A}_s(\Omega),\ |A| \le c\}, \tag{2.3}$$

where F_s satisfies (H_1^s)–(H_2^s).

Remark 2.6. Observe that from the monotonicity assumption (H_2^s) on F_s, this problem is equivalent to minimize in the class of s-quasi open sets A of measure $|A| = c$. In fact, assume that a minimizer $A_0 \in \mathcal{A}_s$ for (2.3) verifies that $|A_0| < c$. Then, for any $\tilde{A}_0 \supset A_0$ such that $|\tilde{A}_0| = c$, we have

$$F_s(\tilde{A}_0) \le F_s(A_0) = \inf_{A\in\mathcal{A}_s} F_s(A)$$

and so $\tilde{A}_0$ is a minimizer of F_s.

Following the same approach and ideas of [9], problem (2.3) can be analyzed and that is the content of our first result.

Theorem 2.7. *Let $0 < s < 1$ be fixed and let $\Omega \subset \mathbb{R}^n$ be open and bounded. Let $F_s : \mathcal{A}_s(\Omega) \to \mathbb{R}$ be such that (H_1^s) and (H_2^s) are satisfied. Then, for every $0 < c < |\Omega|$, problem (2.3) has a solution.*

As we mentioned, the proof of Theorem 2.7 follows the ideas developed in [9] and that is carried out in Section 3.

Next, we want to analyze the behavior of these minimum problems and its minimizers when $s \uparrow 1$. In order to perform such an analysis we need to assume some asymptotic behavior on the cost functionals F_s. In order to do this, we need to define a notion of convergence for sets when s varies.

Definition 2.8. Let $0 < s_k \uparrow 1$ and let $A_k \in \mathcal{A}_{s_k}(\Omega)$ and $A \in \mathcal{A}_1(\Omega)$. We say that $A_k \xrightarrow{\gamma} A$ if $u_{A_k}^{s_k} \to u_A^1$ strongly in $L^2(\Omega)$.

Remark 2.9. Observe that the notion of γ-convergence of sets given in [9] is denoted in this paper by γ_1-convergence. This should not cause any confusion.

Now we can give the assumptions of the functionals F_s:
(H_1) Continuity with respect to A: if $A \in \mathcal{A}_1(\Omega)$, then

$$F_1(A) = \lim_{s\uparrow 1} F_s(A).$$

(H_2) Liminf inequality: for every $0 < s_k \uparrow 1$ and $A_k \xrightarrow{\gamma} A$, then

$$F_1(A) \le \liminf_{k\to\infty} F_{s_k}(A_k),$$

Under these assumptions, we obtain the following result.

Theorem 2.10. *For any $0 < s \le 1$, let $F_s : \mathcal{A}_s(\Omega) \to \mathbb{R}$ be such that (H_1^s) and (H_2^s) are satisfied. Assume moreover that (H_1) and (H_2) are satisfied. Then*

$$\min\{F_1(A) : A \in \mathcal{A}_1(\Omega),\ |A| \le c\} = \lim_{s\uparrow 1} \min\{F_s(A) : A \in \mathcal{A}_s(\Omega),\ |A| \le c\}$$

and, moreover, if $A_s \in \mathcal{A}_s(\Omega)$ is a minimizer for (2.3), then there exists a sequence $0 < s_k \uparrow 1$, sets $\tilde{A}_{s_k} \supset A_{s_k}$ and a set $A_1 \in \mathcal{A}_1(\Omega)$ such that $\tilde{A}_{s_k} \xrightarrow{\gamma} A_1$ and A_1 is a minimizer for (2.3) with $s = 1$.

The proof of Theorem 2.10 is carried out in Section 4 and also uses ideas developed in [9]. However, in this case nontrivial modifications need to be made in order to consider the varying spaces where one is working.

2.3 Examples

Let first establish some notations. Given a bounded domain $A \in \mathcal{A}_s(\Omega)$, consider the problem

$$(-\Delta)^s u = \lambda^s u \quad \text{in } A, \qquad u \in H_0^s(A), \tag{2.4}$$

where $\lambda^s \in \mathbb{R}$ is the eigenvalue parameter. It is well known that there exists a discrete sequence $\{\lambda_k^s(A)\}_{k\in\mathbb{N}}$ of positive eigenvalues of (2.4) approaching $+\infty$ whose corresponding eigenfunctions $\{u_k^s\}_{k\in\mathbb{N}}$ form an orthogonal basis in $L^2(A)$. These fact follows directly from the spectral theorem for compact and self adjoints operators, see [6]. Moreover, the following variational characterization holds for the eigenvalues:

$$\lambda_k^s(A) = \min_{u \perp W_{k-1}} \frac{c(n,s)}{2} \frac{[u]_s^2}{\|u\|_2^2},$$

where W_k is the space spanned by the first k eigenfunctions $u_1^s, \dots, u_k^s$.

Functions F_s satisfying hypothesis (H_1^s) and (H_2^s) include a large family of examples. For instance, if we consider the application $A \mapsto \lambda_k^s(A)$, Theorem 2.7 claims that for every $k \in \mathbb{N}$ and $0 < c < |\Omega|$, the minimum

$$\min\{\lambda_k^s(A) : A \in \mathcal{A}_s(\Omega),\ |A| \le c\}$$

is achieved. More generally, the minimum

$$\min\{\Phi_s(\lambda_{k_1}^s(A), \dots, \lambda_{k_N}^s(A)) : A \in \mathcal{A}_s(\Omega),\ |A| \le c\}$$

is achieved, where $\Phi_s : \mathbb{R}^N \to \bar{\mathbb{R}}$ is lower semicontinuous and increasing in each coordinate.

Moreover, if $\Phi_s(t_1, \dots, t_N) \to \Phi_1(t_1, \dots, t_N)$ for every $(t_1, \dots, t_N) \in \mathbb{R}^N$ and

$$\Phi_1(t_1, \dots, t_N) \le \liminf_{k\to\infty} \Phi_{s_k}(t_1^k, \dots, t_N^k),$$

for every $(t_1^k, \dots, t_N^k) \to (t_1, \dots, t_N)$, then Theorem 2.10 together with the result of [5] imply that

$$\min\{\Phi_1(\lambda_{k_1}(A), \dots, \lambda_{k_N}(A)) : A \in \mathcal{A}_1(\Omega),\ |A| \le c\} = \lim_{s\uparrow 1} \min\{\Phi_s(\lambda_{k_1}^s(A), \dots, \lambda_{k_N}^s(A)) : A \in \mathcal{A}_s(\Omega),\ |A| \le c\}.$$

3 Proof of Theorem 2.7

This section is devoted to proving Theorem 2.7. The arguments follow essentially the lines of [9] with some modifications for the nonlocal setting.

The sketch of the argument is as follows: Given $A \in \mathcal{A}_s(\Omega)$, we first prove that u_A^s is the solution to

$$\max\{w \in H_0^s(\Omega) : w \le 0 \text{ in } \mathbb{R}^n \setminus A,\ (-\Delta)^s w \le 1 \text{ in } \Omega\}. \tag{3.1}$$

Moreover, as a consequence of Lemma 3.2 below, u_A^s belongs to the convex closed set $\mathcal{K}_s$ defined as

$$\mathcal{K}_s = \{w \in H_0^s(\Omega) : w \ge 0,\ (-\Delta)^s w \le 1 \text{ in } \Omega\}. \tag{3.2}$$

It will be convenient to also consider $\mathcal{K}_1$ defined as in (3.2) with $s = 1$, where $(-\Delta)^1 = -\Delta$.

Finally, one defines a functional G_s on $\mathcal{K}_s$ satisfying that

(G_1) G_s is decreasing on $\mathcal{K}_s$,

(G_2) G_s is lower semicontinuous on $\mathcal{K}_s$ with respect to the strong topology on $L^2(\Omega)$,

(G_3) $G_s(u_A^s) = F_s(A)$ for every $A \in \mathcal{A}_s(\Omega)$,

to conclude that the problem

$$\min\{G_s(w) : w \in \mathcal{K}_s,\ |\{w > 0\}| \le c\} \tag{3.3}$$

has a solution w_0. If we denote $A_0 = \{w_0 > 0\}$, then $u_{A_0}^s$ is also a minimum point of G_s over the whole $\mathcal{K}_s$ subject to the condition $|\{w > 0\}| \le c$ and hence, A_0 is a minimizer for F_s in $\mathcal{A}_s(\Omega)$ subject to the condition $|A| \le c$.

We start by proving (3.1). Let us define

$$K_A = \{w \in H_0^s(\Omega) : w \le 0 \text{ in } \mathbb{R}^n \setminus A\},$$

and $w_A \in K_A$ the (unique) minimizer of

$$I_s : K_A \to \mathbb{R}, \quad I_s(w) = \frac{c(n,s)}{2}[w]_s^2 - \int_\Omega w\,dx.$$

Observe that, by Stampacchia's theorem, w_A is characterized by the variational inequality

$$\mathcal{E}(w_A, v - w_A) \ge \int_\Omega (v - w_A)\,dx \quad \text{for all } v \in K_A, \tag{3.4}$$

where we denote

$$\mathcal{E}(u,v) := c(n,s) \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy. \tag{3.5}$$

Next, we prove that both functions u_A^s and w_A agree.

Lemma 3.1. *With the previous notation we have that $w_A = u_A^s$.*

Proof. The proof is standard. We will use the standard notations of $w^+ = \max\{w, 0\}$ and $w^- = \max\{-w, 0\}$. Take w_A^+ as test function in the variational inequality (3.4) and obtain

$$0 \le \int_\Omega w_A^-\,dx \le \mathcal{E}(w_A, w_A^-) \le -c(n,s) \iint_{\{w_A \le 0\} \times \{w_A \le 0\}} \frac{(w_A^-(x) - w_A^-(y))^2}{|x-y|^{n+2s}}\,dx\,dy.$$

From this inequality one easily conclude that $w_A^- = 0$ and so, since $w_A \in K_A$, $w_A \in H_0^s(A)$. Therefore, since, by Remark 2.3, u_A^s is the unique minimum of I_s over $H_0^s(A)$ and, since also $u_A^s \in K_A$, $I_s(w_A) \le I_s(u_A^s)$ the lemma follows. □

Using the previous lemma, we prove the following properties on u_A^s.

Lemma 3.2. *With the above notations, $u_A^s \ge 0$ on Ω. Moreover, u_A^s is the solution to* (3.1).

Proof. First observe that from the maximum principle it follows that $u_A^s \ge 0$ in Ω. Given $v \in H_0^s(\Omega)$ such that $v \ge 0$, we have that $-v \in K_A$. Using it as a test function in (3.4), we obtain that

$$\mathcal{E}(u_A^s, -v - u_A^s) = -c(n,s)[u_A^s]_s^2 - \mathcal{E}(u_A^s, v) \ge -\int_\Omega v\,dx - \int_\Omega u_A^s\,dx.$$

Using that $(-\Delta)^s u_A^s = 1$ in A, the last inequality reads as

$$\mathcal{E}(u_A^s, v) \le \int_\Omega v\,dx.$$

Since $v \in H_0^s(\Omega)$ is nonnegative but otherwise arbitrary, we get that $(-\Delta)^s u_A^s \le 1$ in Ω. Finally, if $w \le 0$ in $\mathbb{R}^n \setminus A$ and $(-\Delta)^s w \le 1$ in Ω, then

$$(-\Delta)^s w \le (-\Delta)^s u_A^s \quad \text{in } A \qquad \text{and} \qquad w \le u_A^s \quad \text{in } \mathbb{R}^n \setminus A.$$

Hence, by comparison, $w \le u_A^s$ in $\mathbb{R}^n$. □

The set $\mathcal{K}_s$ satisfies the following properties:

Proposition 3.3. *The set $\mathcal{K}_s$ is a convex, closed and bounded subset of $H_0^s(\Omega)$.*

Proof. Clearly, $\mathcal{K}_s$ is a convex set. It is also bounded. Indeed, given $u \in \mathcal{K}_s$, by Hölder's and Poincaré's inequalities we get

$$c(n,s)[u]_s^2 \le \int_\Omega u\,dx \le |\Omega|^{\frac{1}{2}} \|u\|_{L^2(\Omega)} \le C|\Omega|^{\frac{1}{2}}[u]_s.$$

In order to see that $\mathcal{K}_s$ is closed, let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathcal{K}_s$ such that $u_k \to u$ in $H_0^s(\Omega)$. For any $k \in \mathbb{N}$ and any $v \in H_0^s(\Omega)$, $v \geq 0$, it holds that

$$\mathcal{E}(u_k, v) \leq \int_\Omega v\,dx.$$

Since $\mathcal{E}(\,\cdot\,, v)$ is continuous in $H_0^s(\Omega)$ (in fact is weakly continuous), taking the limit as $k \to \infty$ we obtain that $\mathcal{E}(u, v) \leq \int_\Omega v\,dx$, but, since $v \in H_0^s(\Omega)$ is nonnegative but otherwise arbitrary, we obtain that $(-\Delta)^s u \leq 1$ in Ω and then $u \in \mathcal{K}_s$. □

Remark 3.4. Observe that optimal constant in Poincaré's inequality

$$\|u\|^2_{L^2(\Omega)} \leq C(\Omega, s)[u]_s^2,$$

has a dependence on s of the form

$$C(\Omega, s) \leq (1-s)C(\Omega).$$

See [5]. Therefore, the proof of Proposition 3.3 gives that if $u \in \mathcal{K}_s$, then

$$(1-s)[u]_s^2 \leq C, \tag{3.6}$$

where C depends on Ω but is independent on $0 < s < 1$.

Now, in order to prove the existence of solution to (2.3) we define a functional G_s on $\mathcal{K}_s$ satisfying conditions (G_1)–(G_3). We will use the notation, for $0 < s \leq 1$,

$$\mathcal{A}_s^c(\Omega) := \{A \in \mathcal{A}_s(\Omega) : |A| \leq c\}. \tag{3.7}$$

For any $0 < s \leq 1$, given $w \in \mathcal{K}_s$ we define

$$J_s(w) = \inf\{F_s(A) : A \in \mathcal{A}_s^c(\Omega),\ u_A^s \leq w\}.$$

This functional J_s is not lower semicontinuous in general. So we define G_s to be the lower semicontinuous envelope of J_s in $\mathcal{K}_s$ with respect to the strong topology in $L^2(\Omega)$, i.e.,

$$G_s(w) = \inf\Big\{\liminf_{k\to\infty} J_s(w_k)\Big\}, \tag{3.8}$$

where the infimum is taken over all sequences $\{w_k\}_{k\in\mathbb{N}}$ in $\mathcal{K}_s$ such that $w_k \to w$ in $L^2(\Omega)$.

Observe that G_s automatically verifies (G_2).

Proposition 3.5. *Let $0 < s \leq 1$. The functional G_s satisfies conditions* (G_1).

Proof. The case $s = 1$ is considered in [9], so we take $0 < s < 1$.

Let us begin by noticing that if $u, v \in \mathcal{K}_s$, then $\max\{u, v\} \in \mathcal{K}_s$. In fact, let us denote $w = \max\{u, v\}$ and consider the convex set

$$E = \{z \in H_0^s(\Omega) : z \leq w \text{ in } \Omega\}.$$

By Stampacchia's theorem there exists a unique function $z_0 \in E$ such that

$$I_s(z_0) = \min_E I_s,$$

where I_s is defined in (2.2). In addition, z_0 satisfies that

$$\mathcal{E}(z_0, z - z_0) \geq \int_\Omega (z - z_0)\,dx \quad \text{for all } z \in E, \tag{3.9}$$

where $\mathcal{E}$ is defined in (3.5).

Let us see that $(-\Delta)^s z_0 \leq 1$. Given $\varphi \in H_0^s(\Omega)$ such that $\varphi \leq 0$, we define the functional

$$i(t) = I_s(z_0 + t\varphi) \quad \text{for all } t \geq 0.$$

Observe that $i'(0) \geq 0$. In consequence, for any nonpositive $\varphi \in H_0^s(\Omega)$ it holds that $\mathcal{E}(z_0, \varphi) \geq \int_\Omega \varphi\,dx$, and then $\mathcal{E}(z, \varphi) \leq \int_\Omega \varphi\,dx$ for any $\varphi \in H_0^s(\Omega)$, $\varphi \geq 0$ and the claim follows.

Now, we will prove that $z_0 \geq u$ (and for symmetry reasons that $z_0 \geq v$), from where it will follow that $z_0 \geq w$. Since $z_0 \in E$, the reverse inequality holds and we can conclude that $z_0 = w \in \mathcal{K}_s$.

Let $\eta = \max\{z_0, u\}$ and let us see that $z_0 = \eta$. Observe that $\eta \in E$ and thus it can be consider as a test function in (3.9). Thus,

$$\mathcal{E}(z_0, \eta - z_0) \geq \int_\Omega (\eta - z_0)\,dx.$$

On the other hand, since $\eta - z_0 \geq 0$ and $(-\Delta)^s u \leq 1$ in Ω, it follows that

$$\mathcal{E}(u, \eta - z_0) \leq \int_\Omega (\eta - z_0)\,dx.$$

From both inequalities it is straightforward to see that

$$0 \leq \mathcal{E}(z_0 - u, \eta - z_0) \leq -c(n, s)[(u - z_0)^+]_s^2$$

and then $(u - z_0)^+ = 0$ in $\mathbb{R}^n$, which implies that $z_0 \geq u$ in $\mathbb{R}^n$, as we required.

Now we proceed with the proof. Let $u, v \in \mathcal{K}_s$ be such that $u \leq v$ and let $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{K}_s$ be such that $u_k \to u$ in $L^2(\Omega)$ and $J_s(u_k) \to G_s(u)$. By our previous claim, we have $v_k = \max\{v, u_k\} \in \mathcal{K}_s$ for each $k \in \mathbb{N}$ and $v_k \to v = \max\{v, u\}$ in $L^2(\Omega)$. Consequently, since J_s is nonincreasing and $v_k \geq u_k$ for any $k \in \mathbb{N}$, we get

$$G_s(v) \leq \liminf_{k\to\infty} J_s(v_k) \leq \lim_{k\to\infty} J_s(u_k) = G_s(u),$$

as we wanted to show. □

We will need the following lemma in order to prove condition (G$_3$). We omit the proof since it is completely analogous to that of [9, Lemmas 3.2 and 3.3] where the case $s = 1$ was considered.

Lemma 3.6. *Let $0 < s < 1$. Let $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{A}_s(\Omega)$ be a sequence such that $u^s_{A_k} \to u$ in $L^2(\Omega)$, with $u \leq u^s_A$. We define $A^\varepsilon = \{u^s_A > \varepsilon\}$. Then, if $u^s_{A_k\cup A^\varepsilon} \to u^\varepsilon$ in $L^2(\Omega)$, it holds that $u^\varepsilon \leq u^s_A$.*

With the help of Lemma 3.6 we are able to show that G_s satisfies condition (G$_3$).

Proposition 3.7. *Let $0 < s \leq 1$. Then the functional G_s satisfies* (G$_3$).

Proof. We only need to consider $0 < s < 1$. Let us fix $A \in \mathcal{A}^c_s(\Omega)$. From (3.8) it follows that $G_s(u^s_A) \leq F_s(A)$. To prove the reverse inequality, it suffices to see that

$$F_s(A) \leq \liminf_{k\to\infty} J_s(w_k)$$

for any sequence $\{w_k\}_{k\in\mathbb{N}} \subset \mathcal{K}_s$ such that $w_k \to u^s_A$ in $L^2(\Omega)$. By the definition of J_s, there exists $A_k \in \mathcal{A}^c_s(\Omega)$ such that

$$F_s(A_k) \leq J_s(w_k) + \frac{1}{k} \quad \text{and} \quad u^s_{A_k} \leq w_k.$$

Observe that $u^s_{A_k} \in \mathcal{K}_s$ for each $k \in \mathbb{N}$ and by Proposition 3.3, $\{u^s_{A_k}\}_{k\in\mathbb{N}}$ is bounded in $H^s_0(\Omega)$. Then, up to a subsequence, there exists $u \in \mathcal{K}_s$ such that $u^s_{A_k} \to u$ in $L^2(\Omega)$. Since $w_k \to u^s_A$ in $L^2(\Omega)$, from $u^s_{A_k} \leq w_k$ we get $u \leq u^s_A$.

Let us consider the set $A^\varepsilon = \{u^s_A > \varepsilon\}$ and observe that $u^s_{A_k\cup A^\varepsilon} \in \mathcal{K}_s$. Again by Proposition 3.3, it follows that and $u^s_{A_k\cup A^\varepsilon} \to u^\varepsilon$ in $L^2(\Omega)$ for some $u^\varepsilon \in \mathcal{K}_s$. By Lemma 3.6, the inequality $u^\varepsilon \leq u^s_A$ follows.

We claim that $(u^s_A - \varepsilon)^+ \leq u^s_{A^\varepsilon}$. Indeed,

$$(u^s_A - \varepsilon)^+(x) - (u^s_A - \varepsilon)^+(y) = \begin{cases} u^s_A(x) - u^s_A(y) & \text{if } x, y \in A^\varepsilon, \\ u^s_A(x) - \varepsilon & \text{if } x \in A^\varepsilon \text{ and } y \notin A^\varepsilon, \\ -u^s_A(y) + \varepsilon & \text{if } x \notin A^\varepsilon \text{ and } y \in A^\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for any $v \in H_0^s(A^\varepsilon)$ such that $v \geq 0$, we get

$$\begin{aligned}
&\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{((u_A^s(x)-\varepsilon)^+ - ((u_A^s(y)-\varepsilon)^+)(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy\\
&\quad= \iint_{A^\varepsilon\times A^\varepsilon} \frac{(u_A^s(x)-u_A^s(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy + 2\iint_{A^\varepsilon\times(A^\varepsilon)^c} \frac{(u_A^s(x)-\varepsilon)v(x)}{|x-y|^{n+2s}}\,dy\,dx\\
&\quad= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u_A^s(x)-u_A^s(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy + 2\iint_{A^\varepsilon\times(A^\varepsilon)^c} \frac{(u_A^s(y)-\varepsilon)v(x)}{|x-y|^{n+2s}}\,dy\,dx\\
&\quad\leq \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u_A^s(x)-u_A^s(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy.
\end{aligned}$$

That is, $(-\Delta)^s(u_A^s-\varepsilon)^+ \leq (-\Delta)^s u_A^s = 1 = (-\Delta^s)u_{A^\varepsilon}^s$ in A^ε. Moreover, since $0 = (u_A^s-\varepsilon)^+ = u_{A^\varepsilon}^s$ in $\mathbb{R}^n \setminus A^\varepsilon$, from the comparison principle it follows that $(u_A^s-\varepsilon)^+ \leq u_{A^\varepsilon}^s$ in $\mathbb{R}^n$.

We have obtained the following chain of inequalities:

$$(u_A^s-\varepsilon)^+ \leq u_{A^\varepsilon}^s \leq u_{A_k\cup A^\varepsilon}^s.$$

Taking limit as $k\to\infty$, we conclude that

$$(u_A^s-\varepsilon)^+ \leq u^\varepsilon \leq u_A^s,$$

since $u^\varepsilon \leq u_A^s$ and $u_{A_k\cup A^\varepsilon}^s \to u^\varepsilon$. Since $u^\varepsilon \in \mathcal{K}_s$, by (3.6), $\{u^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $H_0^s(\Omega)$. Consequently, up to a subsequence, $u^\varepsilon \to u_A^s \in L^2(\Omega)$. By a standard diagonal argument, there exists a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that $u_{A_k\cup A^{\varepsilon_k}}^s \to u_A^s$ in $L^2(\Omega)$.

In conclusion, we have proved that $(A_k \cup A^{\varepsilon_k})$ γ_s-converges to A. Therefore

$$F_s(A) \leq \liminf_{k\to\infty} F_s(A_k\cup A^{\varepsilon_k}) \leq \liminf_{k\to\infty} F_s(A_k) \leq \liminf_{k\to\infty} J_s(w_k).$$

This fact concludes the proof of the proposition. □

Having proved these preliminary results, the proof of Theorem 2.7 follows in the same way of that of [9, Theorem 2.5]. We include the details for the reader convenience.

Proof of Theorem 2.7. First, we solve (3.3). Take $\{w_k\}_{k\in\mathbb{N}} \subset \mathcal{K}_s$ such that $|\{w_k>0\}| \leq c$ and

$$\lim_{k\to\infty} G_s(w_k) = \inf\{G_s(w) : w \in \mathcal{K}_s,\ |\{w>0\}| \leq c\} =: m_{G_s}.$$

By Proposition 3.3, there exists $w_0 \in \mathcal{K}_s$ such that $w_k \to w_0$ strongly in $L^2(\Omega)$, up to a subsequence. Thus, $|\{w_0>0\}| \leq c$. Then, by (G_2),

$$m_{G_s} \leq G_s(w_0) \leq \liminf_{k\to\infty} G_s(w_k) = m_{G_s}.$$

So, w_0 is a solution to (3.3).

Now, consider $A_0 := \{w_0>0\}$. Then $A_0 \in \mathcal{A}_s^c(\Omega)$. By Lemma 3.2, $w_0 \leq u_{A_0}^s$. For every $A \in \mathcal{A}_s^c(\Omega)$, we know that $u_A^s \in \mathcal{K}_s$, $|\{u_A^s>0\}| \leq c$. Then, by (G_3), (G_1) and the fact that w_0 is the solution to (3.3), we have

$$F_s(A_0) = G_s(u_{A_0}^s) \leq G_s(w_0) \leq G_s(u_A^s) = F_s(A).$$

Therefore, A_0 is a solution to (2.3). □

4 Proof of Theorem 2.10

In this section we prove Theorem 2.10 following the same spirit of [9]; however, nontrivial changes must be performed due to the nonlocal settings.

Our first goal is to show that a sequence $\{u_k\}_{k\in\mathbb{N}} \subset L^2(\Omega)$ such that $u_k \in \mathcal{K}_{s_k}$ is precompact and that every accumulation point belongs to $\mathcal{K}_1$. This is the content of the next lemma.

Lemma 4.1. *Let $0 < s_k \uparrow 1$ and let $u_k \in \mathcal{K}_{s_k}$. Then there exist $u \in H_0^1(\Omega)$ and a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}} \subset \{u_k\}_{k\in\mathbb{N}}$ such that $u_{k_j} \to u$ strongly in $L^2(\Omega)$. Moreover, if $u_k \in \mathcal{K}_{s_k}$ is such that $u_k \to u$ strongly in $L^2(\Omega)$, then $u \in \mathcal{K}_1$.*

Proof. From Remark 3.4, there exists a constant $C > 0$ such that

$$\sup_{k\in\mathbb{N}}(1-s_k)[u_k]_{s_k}^2 \leq C.$$

Now the first claim follows from [3, Theorem 4].

Assume that $u_k \to u$ in $L^2(\Omega)$. It is clear that $u \geq 0$. Since $(-\Delta)^{s_k}u_k \leq 1$ in Ω, for every nonnegative $\varphi \in C_c^\infty(\Omega)$ we have that

$$\int_\Omega (-\Delta)^{s_k}\varphi u_k\,dx = \langle(-\Delta)^{s_k}u_k, \varphi\rangle \leq \int_\Omega \varphi\,dx.$$

By the convergence assumption on u_k and the fact that the convergence (2.1) is also strong in $L^2(\Omega)$, we can take limit as $k \to \infty$ in the previous inequality to obtain that

$$\int_\Omega -\Delta\varphi u\,dx = \langle-\Delta u, \varphi\rangle \leq \int_\Omega \varphi\,dx,$$

and conclude that $-\Delta u \leq 1$ in Ω. Consequently, $u \in \mathcal{K}_1$ as required. □

Analogously as in the previous section, we define the following functionals for the limit problem:

$$J_1(w) := \inf\{F_1(A) : A \in \mathcal{A}_1^c(\Omega),\ u_A \leq w\},$$

where $\mathcal{A}_1^c(\Omega)$ is defined in (3.7) for $s = 1$ and we define G_1 to be the lower semicontinuous envelope of J_1 in $\mathcal{K}_1$.

The next lemma gives the continuity of u_A^s when $s \uparrow 1$.

Lemma 4.2. *For every $A \in \mathcal{A}_1(\Omega)$, $u_A^s \to u_A$ strongly in $L^2(\Omega)$, when $s \uparrow 1$.*

Proof. Let us remind that, from Lemma 3.1, u_A^s is also the solution to the minimization problem

$$I_s(u_A^s) = \min\{I_s(w) : w \in L^2(\Omega)\},$$

where

$$I_s(w) = \begin{cases} \dfrac{c(n,s)}{2}[w]_s^2 - \displaystyle\int_\Omega w\,dx & \text{if } w \in H_0^s(A), \\ \infty & \text{otherwise.} \end{cases}$$

Notice that, by [28], we have $\frac{c(n,s)}{2}[w]_s^2 \xrightarrow{\Gamma} \frac{1}{2}\|\nabla w\|_2^2$. Since the Γ-convergence is stable under continuous perturbations, we have $I_s \xrightarrow{\Gamma} I_1$ in $L^2(\Omega)$, where

$$I_1(w) = \begin{cases} \dfrac{1}{2}\|\nabla w\|_2^2 - \displaystyle\int_\Omega w\,dx & \text{if } w \in H_0^1(A), \\ \infty & \text{otherwise.} \end{cases}$$

Thus, the minimizer of I_s converges to the minimizer of I_1. That is $u_A^s \to u_A$ strongly in $L^2(\Omega)$. □

Now we address the more difficult problem of understanding the limit behavior of u_A^s when the domains also are varying with s.

This first lemma is key in understanding this limit behavior and the ideas are taken from [9].

Lemma 4.3. *Let $0 < s_k \uparrow 1$ and for every $k \in \mathbb{N}$ let $A_k \in \mathcal{A}_{s_k}(\Omega)$ be such that $u_{A_k}^{s_k} \to u$ strongly in $L^2(\Omega)$. Let $\{w_k\}_{k\in\mathbb{N}} \subset L^2(\Omega)$ be such that $w_k \in H_0^{s_k}(A_k)$ for every $k \in \mathbb{N}$ and $\sup_{k\in\mathbb{N}}(1-s_k)[w_k]_{s_k}^2 < \infty$. Assume, moreover, that $w_k \to w$ strongly in $L^2(\Omega)$. Then $w \in H_0^1(\{u > 0\})$.*

Proof. We need to show that $w = 0$ in $\mathbb{R}^n \setminus \{u > 0\}$, i.e., $w = 0$ in $\{u = 0\}$.

Let us define the functional

$$\Phi_k(v) = \begin{cases} \dfrac{c(n,s_k)}{2}[v]_{s_k}^2 & \text{if } v \in H_0^{s_k}(A_k), \\ \infty & \text{otherwise,} \end{cases}$$

defined in $L^2(\Omega)$. By the compactness of Γ-convergence, there exists a subsequence still denote by Φ_k such that

$$\Phi_k \xrightarrow{\Gamma} \Phi \quad \text{in } L^2(\Omega).$$

From [12, Theorem 11.10], Φ is a quadratic form in $L^2(\Omega)$ with domain $D(\Phi) \subset L^2(\Omega)$. Observe that $w \in D(\Phi)$, since

$$\Phi(w) \le \liminf_{k\to+\infty} \Phi_k(w_k) \le \sup_{k\in\mathbb{N}} \frac{c(n,s_k)}{2}[w_k]^2_{s_k} \le C \sup_{k\in\mathbb{N}}(1-s_k)[w_k]^2_{s_k} < \infty.$$

Let $B : D(\Phi) \times D(\Phi) \to \mathbb{R}$ be the bilinear form associated to Φ, which is defined by

$$B(v,\eta) = \frac{1}{4}(\Phi(v+\eta) - \Phi(v-\eta)).$$

Let us denote by V the closure of $D(\Phi)$ in $L^2(\Omega)$ and consider the linear operator $T : D(T) \subset L^2(\Omega) \to L^2(\Omega)$ defined as $Tv = f$, where

$$D(T) = \Big\{ v \in D(\Phi) : \text{there exists } f \in V \text{ such that } B(v,\eta) = \int_\Omega f\eta \, dx \text{ for all } \eta \in D(\Phi) \Big\}.$$

By [12, Proposition 12.17], $D(T)$ is dense in $D(\Phi)$ with respect to the norm

$$\|v\|_\Phi = (\|v\|_{L^2(\Omega)} + \Phi(v))^{\frac{1}{2}}.$$

Moreover, the following relation holds:

$$\sqrt{2}\|\cdot\|_\Phi \ge \|\cdot\|_{H^1_0(\Omega)}. \tag{4.1}$$

Indeed, if $z \in D(\Phi)$, as $\Phi_k \xrightarrow{\Gamma} \Phi$ in $L^2(\Omega)$, there exists $\{z_k\}_{k\in\mathbb{N}}$ such that $z_k \to z$ in $L^2(\Omega)$ and

$$\infty > \Phi(z) = \lim_{k\to\infty} \Phi_k(z_k) = \begin{cases} \lim\limits_{k\to\infty} \dfrac{c(n,s_k)}{2}[z_k]^2_{s_k} & \text{if } z_k \in H^{s_k}_0(A_k), \\ \infty & \text{otherwise.} \end{cases}$$

Thus, $z_k \in H^{s_k}_0(A_k)$ and then

$$\|z\|^2_{H^1_0(\Omega)} \le \liminf_{k\to\infty} c(n,s_k)[z_k]^2_{s_k} = 2\lim_{k\to\infty} \Phi_k(z_k) = 2\Phi(z) \le 2\|z\|^2_\Phi.$$

Since (4.1) holds, $D(T)$ is dense in $D(\Phi)$ with respect to the strong topology of $H^1_0(\Omega)$. Now to achieve the proof it is enough to prove that $v = 0$ in $\{u = 0\}$ for all $v \in D(T)$.

Let $v \in D(T)$ and let $f \in Tv$; then v is a minimum point of the functional

$$\Psi(\eta) = \frac{1}{2}\Phi(\eta) - \int_\Omega f\eta \, dx$$

(see [12, Proposition 12.12]). Let v_k be the minimum point of functional

$$\Psi_k(\eta) := \frac{1}{2}\Phi_k(\eta) - \int_\Omega f\eta \, dx;$$

then v_k is the solution of the problem

$$(-\Delta)^{s_k} v_k = f, \qquad v \in H^{s_k}_0(A_k).$$

Since $\Phi_k \xrightarrow{\Gamma} \Phi$, it follows that $\Psi_k \xrightarrow{\Gamma} \Psi$ and so we have that $v_k \to v$ strongly in $L^2(\Omega)$.

For $\varepsilon > 0$ we consider f^ε to be a bounded function with compact support such that $\|f^\varepsilon - f\|_2 < \varepsilon$ and v^ε_k is a solution of

$$(-\Delta)^{s_k} v^\varepsilon_k = f^\varepsilon \quad \text{in } A_k, \qquad v^\varepsilon_k \in H^{s_k}_0(A_k).$$

By using the linearity of the operator together with Hölder's and Poincaré's inequalities, we get

$$\frac{c(n,s_k)}{2}[v^\varepsilon_k - v_k]^2_{s_k} = \int_\Omega (f^\varepsilon - f)(v^\varepsilon_k - v_k)\, dx \le \|f_\varepsilon - f\|_2 \|v^\varepsilon_k - v_k\|_2.$$

From Poincaré's inequality we obtain that

$$(1-s_k)[v_k^\varepsilon - v_k]_{s_k}^2 \le C\varepsilon^2,$$

where C is independent on k. Then from [3, Theorem 4], up to a subsequence, $v_k^\varepsilon \to v^\varepsilon$ strongly in $L^2(\Omega)$ and $\|v^\varepsilon - v\|_{H_0^1(\Omega)} \le C\varepsilon$. At this point it is enough to prove that $v^\varepsilon = 0$ in $\{u = 0\}$ for all $\varepsilon > 0$.

Since $f^\varepsilon \le c^\varepsilon := \|f^\varepsilon\|_\infty$ and

$$(-\Delta)^{s_k} v_k^\varepsilon = f^\varepsilon \le c^\varepsilon = (-\Delta)^{s_k}(c^\varepsilon u_{A_k}^{s_k}) \quad \text{in } A_k, \qquad v_k^\varepsilon = c^\varepsilon u_{A_k}^{s_k} = 0 \quad \text{in } \mathbb{R}^n \setminus A_k,$$

the comparison principle gives that $v_k^\varepsilon \le c^\varepsilon u_{A_k}^{s_k}$. Analogously, $-v_k^\varepsilon \le c^\varepsilon u_{A_k}^{s_k}$.

As $k \to \infty$, we obtain that $|v^\varepsilon| \le c^\varepsilon u$, which implies that $v^\varepsilon = 0$ in $\{u = 0\}$ for any $\varepsilon > 0$ and that completes the proof. □

The next lemma is the counterpart of Lemma 3.6 of the previous section. We include here the details for completeness. The main modifications with respect to the previous proof (which was analogous to that of [9, Lemmas 3.2 and 3.3]) were carried out in the previous lemmas of this section.

Lemma 4.4. *Let $0 < s_k \uparrow 1$ and for every $k \in \mathbb{N}$, let $A_k \in \mathcal{A}_{s_k}(\Omega)$, $A \in \mathcal{A}_1(\Omega)$. Assume that $u_{A_k}^{s_k} \to u$ in $L^2(\Omega)$ and that $u \le u_A$. Then, if $u_{A_k \cup A^\varepsilon}^{s_k} \to u^\varepsilon$ strongly in $L^2(\Omega)$, where $A^\varepsilon := \{u_A > \varepsilon\}$, it holds that $u^\varepsilon \le u_A$.*

Proof. By Lemma 3.2 with $s = 1$, the inequality $u^\varepsilon \le u_A$ will follow if we prove that $u^\varepsilon \in H_0^1(\Omega)$, $u^\varepsilon \le 0$ in $\mathbb{R}^n \setminus A$ and $-\Delta u^\varepsilon \le 1$ in Ω.

Observe that by Lemma 4.1 we have that $u, u^\varepsilon \in H_0^1(\Omega)$. Let us define

$$v^\varepsilon := 1 - \frac{1}{\varepsilon}\min\{u_A, \varepsilon\} = \frac{1}{\varepsilon}(\varepsilon - u_A)^+$$

and observe that $0 \le v^\varepsilon \le 1$ and $v^\varepsilon = 0$ in A^ε since $0 \le \min\{u_A, \varepsilon\} \le \varepsilon$ and $\frac{1}{\varepsilon}\min\{u_A, \varepsilon\} = 1$ in A^ε. If we define

$$u_{k,\varepsilon} := u_{A_k \cup A^\varepsilon}^{s_k}, \quad w_{k,\varepsilon} := \min\{v^\varepsilon, u_{k,\varepsilon}\},$$

it holds that $w_{k,\varepsilon} \ge 0$ since the comparison principle gives $u_{k,\varepsilon} \ge 0$, and also $v^\varepsilon \ge 0$. Since $v^\varepsilon = 0$ in A^ε, it follows that $w_{k,\varepsilon} = 0$ in A^ε. Moreover, since $u_{k,\varepsilon} = 0$ in $\mathbb{R}^n \setminus (A_k \cup A^\varepsilon)$, it holds that $w_{k,\varepsilon} = 0$ in $\mathbb{R}^n \setminus (A_k \cup A^\varepsilon)$, and consequently, $w_{k,\varepsilon} \in H_0^{s_k}(A_k)$. Notice that $w_{k,\varepsilon} \to w_\varepsilon := \min\{v^\varepsilon, u^\varepsilon\}$ strongly in $L^2(\Omega)$, and then, applying Lemma 4.3, we get $w_\varepsilon \in H_0^1(\{u > 0\})$, from where $w_\varepsilon = 0$ in $\{u = 0\}$. The relation $0 \le u \le u_A$ implies the inclusion $\{u_A = 0\} \subset \{u = 0\}$, from where $w_\varepsilon \in H_0^1(\{u_A > 0\})$. Moreover, since $\{u_A > 0\} \subset A$, we have that $w_\varepsilon = 0$ in $\mathbb{R}^n \setminus A$. Now, being $v^\varepsilon = 1$ in $\mathbb{R}^n \setminus A$, we get $u^\varepsilon = 0$ in $\mathbb{R}^n \setminus A$, and in particular, $u^\varepsilon \le 0$ in $\mathbb{R}^n \setminus A$.

Finally, it remains to see that $-\Delta u^\varepsilon \le 1$ in Ω. Observe that $u_{k,\varepsilon} \in \mathcal{K}_{s_k}$ and $u_{k,\varepsilon} \to u^\varepsilon$ strongly in $L^2(\Omega)$. Then $u^\varepsilon \in \mathcal{K}_1$ by Lemma 4.1. Thus $-\Delta u^\varepsilon \le 1$ in Ω and the proof is complete. □

With the help of these lemmas, we are now in a position to prove the main tool needed in the proof of Theorem 2.10.

Proposition 4.5. *Let $0 < s_k \uparrow 1$, and let $A_k \in \mathcal{A}_{s_k}^c(\Omega)$ be such that $u_{A_k}^{s_k} \to u$ strongly in $L^2(\Omega)$. Then there exist $\tilde{A}_k \in \mathcal{A}_{s_k}(\Omega)$ such that $A_k \subset \tilde{A}_k$ and $\tilde{A}_k$ γ-converges to $A := \{u > 0\}$.*

Proof. Since $u_{A_k}^{s_k} \in \mathcal{K}_{s_k}$ and $u_{A_k}^{s_k} \to u$, by Lemma 4.1, $u \in \mathcal{K}_1$. Then, by Lemma 3.2, $u \le u_A$. As in the previous proof, consider $A^\varepsilon := \{u_A > \varepsilon\}$ and observe that

$$u_{A^\varepsilon}^{s_k} \le u_{A_k \cup A^\varepsilon}^{s_k}$$

Since $u_{A_k \cup A^\varepsilon}^{s_k} \in \mathcal{K}_{s_k}$, by Lemma 4.1, there exists $u^\varepsilon \in H_0^1(\Omega)$ such that $u_{A_k \cup A^\varepsilon}^{s_k} \to u^\varepsilon$ strongly in $L^2(\Omega)$, up to a subsequence. Also, by Lemma 4.2, $u_{A^\varepsilon}^{s_k} \to u_{A^\varepsilon}$ strongly in $L^2(\Omega)$. Then we can pass to the limit as $k \to \infty$ in the previous inequality to conclude that

$$u_{A^\varepsilon} \le u^\varepsilon.$$

It can be easily checked that $u_{A^\varepsilon} = (u_A - \varepsilon)_+$. Moreover, from Lemma 4.4,

$$(u_A - \varepsilon)_+ \le u^\varepsilon \le u_A.$$

Thus, there exists a sequence $0 < \varepsilon_k \downarrow 0$ such that

$$u^{s_k}_{A_k \cup A^{\varepsilon_k}} \to u_A \quad \text{strongly in } L^2(\Omega).$$

That is, $A_k \cup A^{\varepsilon_k} =: \tilde{A}_k$ γ-converges to A. □

Now we are ready to prove the main result.

Proof of Theorem 2.10. By Theorem 2.7, there exists $A_k \in \mathcal{A}^c_{s_k}(\Omega)$ such that

$$F_{s_k}(A_k) = \min\{F_{s_k}(A) : A \in \mathcal{A}^c_{s_k}(\Omega)\}.$$

Then, if $A \in \mathcal{A}^c_1(\Omega)$, by condition (H_1) we know that

$$\limsup_{k\to\infty} F_{s_k}(A_k) \le \lim_{k\to\infty} F_{s_k}(A) = F_1(A),$$

from where it follows that

$$\limsup_{k\to\infty} \min\{F_{s_k}(A) : A \in \mathcal{A}^c_{s_k}(\Omega)\} \le \min\{F_1(A) : A \in \mathcal{A}^c_1(\Omega)\}. \tag{4.2}$$

Let us see the reverse inequality. By simplicity, let us denote $u_k := u^{s_k}_{A_k} \in \mathcal{K}_{s_k}$. By Lemma 4.1, there is $u \in H^1_0(\Omega)$ such that, up to a subsequence, $u_k \to u$ strongly in $L^2(\Omega)$. Moreover, by Proposition 4.5, there exists $\tilde{A}_k \in \mathcal{A}_{s_k}(\Omega)$ such that $A_k \subset \tilde{A}_k$ and $\tilde{A}_k$ γ-converges to $A := \{u > 0\}$. Since $u_k \to u$ in $L^2(\Omega)$, we have $|A| \le c$. Finally, from conditions (H_2) and (H^s_2) we conclude that

$$F_1(A) \le \liminf_{k\to\infty} F_{s_k}(\tilde{A}_k) \le \liminf_{k\to\infty} F_{s_k}(A_k),$$

from where it follows that

$$\min\{F_1(A) : A \in \mathcal{A}^c_1(\Omega)\} \le \liminf_{k\to\infty} \min\{F_{s_k}(A) : A \in \mathcal{A}^c_{s_k}(\Omega)\}. \tag{4.3}$$

Putting together (4.2) and (4.3) the result follows. □

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