Proceedings of the Royal Society of Edinburgh: page 1 of 18
DOI:10.1017/S0308210517000361

Fractional eigenvalue problems that approximate Steklov eigenvalue problems

Leandro M. Del Pezzo*, **Julio D. Rossi** and **Ariel M. Salort**
Departamento de Matemática and
Instituto de Investigaciones Matemáticas 'Luis A. Santaló',
Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires,
Ciudad Universitaria, Pabellón I (1428), Av. Cantilo s/n. Buenos Aires,
Argentina (ldpezzo@dm.uba.ar; jrossi@dm.uba.ar;
asalort@dm.uba.ar)

(MS received 20 January 2016; accepted 25 July 2016)

In this paper we analyse possible extensions of the classical Steklov eigenvalue problem to the fractional setting. In particular, we find a non-local eigenvalue problem of fractional type that approximates, when taking a suitable limit, the classical Steklov eigenvalue problem.

Keywords: Steklov eigenvalues; nonlocal problems; Sobolev inequalities

2010 *Mathematics subject classification:* Primary 35P30; 35J61; 45C05

1. Introduction

Of crucial importance in the study of boundary-value problems for differential operators are the Sobolev spaces and inequalities. Hence, the Sobolev inequalities and their optimal constants is a subject of interest in the analysis of partial differential equations (PDEs) and related topics. They have been widely studied in the past by many authors and are still an area of intensive research – see the book [1] and the survey [10] for an introduction to this field.

When analysing elliptic or parabolic problems with nonlinear boundary conditions it turns out that among the Sobolev embeddings a fundamental role is played by the Sobolev trace theorem. The study of the best constant in the Sobolev trace theorem leads naturally to eigenvalue problems known in the literature as Steklov eigenvalue problems.

Our main goal in this paper is to analyse a fractional approximation for Steklov eigenvalues. Given a bounded domain $\Omega \subset \mathbb{R}^n$, $s \in (0,1)$ and $p \in (1,\infty)$, we aim to study the non-local problem

$$\left.\begin{aligned} \mathcal{K}_{n,p}(1-s)(-\Delta)^s_p u + |u|^{p-2}u &= \frac{\lambda}{\varepsilon}\chi_{\Omega_\varepsilon}|u|^{p-2}u && \text{in } \Omega, \\ \mathcal{N}_{s,p}u &= 0 && \text{in } \Omega^{\mathrm{c}} = \mathbb{R}^n \setminus \overline{\Omega}, \end{aligned}\right\} \tag{1.1}$$

*Present address: CONICET and Departamento de Matemática y Estadística, Universidad Torcuato Di Tella, Av. Figueroa Alcorta 7350 (C1428BCW), C. A. de Buenos Aires, Argentina.

where s and ε are real numbers belonging to $(0,1)$ and $\Omega_\varepsilon \coloneqq \{x \in \Omega \colon d(x,\Omega) \leqslant \varepsilon\}$. The fractional p-Laplacian is defined as

$$(-\Delta)_p^s u(x) = 2\,\mathrm{p.\,v.}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}\,\mathrm{d}y,$$

where 'p.v.' indicates the principal value, and $\mathcal{N}_{s,p}$ is the associated non-local derivative defined in [9] by

$$\mathcal{N}_{s,p}u(x) \coloneqq 2\int_{\Omega} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}\,\mathrm{d}y, \quad x \in \mathbb{R}^n \setminus \overline{\Omega}. \tag{1.2}$$

The constant $\mathcal{K}_{n,p}$ is the normalization constant computed in [4]. In fact, although the fractional seminorm satisfies $[u]_{s,p} \to \infty$ as $s \to 1^-$, Bourgain *et al.* [4] proved that for any smooth bounded domain $\Omega \subset \mathbb{R}^n$ and $u \in W^{1,p}(\Omega)$ with $p \in (1,\infty)$ there exists a constant $\mathcal{K}_{n,p}$ such that

$$\lim_{s\to 1^-} \mathcal{K}_{n,p}(1-s)\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega |\nabla u|^p\,\mathrm{d}x. \tag{1.3}$$

The constant can be explicitly computed and is given by

$$\mathcal{K}_{n,p} = \frac{p\Gamma(\frac{1}{2}(n+p))}{2\pi^{(n-1)/2}\Gamma(\frac{1}{2}(p+1))}.$$

As the authors of [9] pointed out, one of the main advantages in using this form of non-local derivative arises in the following non-local divergence theorem: for any bounded smooth enough functions u and v it holds that

$$\int_\Omega (-\Delta)_p^s u(x)\,\mathrm{d}x = -\int_{\Omega^c} \mathcal{N}_{s,p}u(x)\,\mathrm{d}x. \tag{1.4}$$

Moreover, the integration by parts formula

$$\mathcal{H}_{s,p}(u,v) = \int_\Omega v(x)(-\Delta)_p^s u(x)\,\mathrm{d}x + \int_{\Omega^c} v(x)\mathcal{N}_{s,p}u(x)\,\mathrm{d}x \tag{1.5}$$

is true, where

$$\mathcal{H}_{s,p}(u,v) \coloneqq \iint_{\mathbb{R}^{2n}\setminus(\Omega^c)^2} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y.$$

Multiplying (1.1) by a bounded smooth enough function v, integrating in Ω and using (1.5), we obtain the following weak formulation for (1.1):

$$\mathcal{K}_{n,p}(1-s)\mathcal{H}_{s,p}(u,v) + \int_\Omega |u|^{p-2}uv\,\mathrm{d}x = \frac{\lambda}{\varepsilon}\int_{\Omega_\varepsilon} |u|^{p-2}uv\,\mathrm{d}x. \tag{1.6}$$

We now introduce some notation that we will use in the paper. Given a measurable function $u\colon \mathbb{R}^n \to \mathbb{R}$ we set

$$\|u\|_{s,p} \coloneqq (\|u\|^p_{L^p(\Omega)} + [u]^p_{s,p})^{1/p}, \quad \text{where } [u]_{s,p} \coloneqq (\mathcal{H}_{s,p}(u,u))^{1/p}.$$

The natural space associated with this norm is

$$\mathcal{W}^{s,p}(\Omega) := \{u\colon \mathbb{R}^n \to \mathbb{R} \text{ measurable}\colon \|u\|_{s,p} < \infty\}.$$

For a fixed value $\varepsilon > 0$, we say that the value $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1.1) if there is $u \in \mathcal{W}^{s,p}(\Omega)$ such that (1.6) holds for any $v \in \mathcal{W}^{s,p}(\Omega)$. Note that if $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1.1) and u is an associated eigenfunction, then $\lambda > 0$ and $u \not\equiv 0$ in Ω_ε. Thus the first eigenvalue of (1.1) is given by

$$\lambda_{1,\varepsilon}(s,p) = \inf_{\substack{u\in\mathcal{W}^{s,p}(\Omega),\\ \|u\|^p_{L^p(\Omega_\varepsilon)}\neq 0}} \frac{\mathcal{K}_{n,p}(1-s)[u]^p_{s,p} + \|u\|^p_{L^p(\Omega)}}{\|u\|^p_{L^p(\Omega_\varepsilon)}/\varepsilon}. \tag{1.7}$$

Recall that it is well known that the first eigenvalue of the Steklov problem

$$\left.\begin{aligned} -\Delta_p u + |u|^{p-2}u &= 0 && \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \lambda |u|^{p-2}u && \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.8}$$

is given by

$$\lambda_1(p) = \inf_{\substack{u\in W^{1,p}(\Omega),\\ u\neq 0}} \frac{\|\nabla u\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)}}{\|u\|^p_{L^p(\partial\Omega)}}. \tag{1.9}$$

Here the p-Laplacian is defined as $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ for $p \in (1,\infty)$.

Taking $\varepsilon = 1-s$, we are interested in studying the behaviour of $\lambda_{1,1-s}(s,p)$ as $s \to 1^-$. Intuitively, a connection between the limit of such an eigenvalue and $\lambda_1(p)$, the first eigenvalue of the Steklov p-Laplacian in Ω, is expected to be found. Indeed, note that from (1.3) one has that, for a fixed u,

$$\lim_{s\to 1^-} \mathcal{K}_{n,p}(1-s)[u]^p_{s,p\Omega} + \|u\|^p_{L^p(\Omega)} = \|\nabla u\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)};$$

moreover, since $\Omega_\varepsilon := \{x \in \Omega\colon d(x,\Omega) \leqslant \varepsilon\}$ is a strip around the boundary $\partial\Omega$ of size $|\Omega_\varepsilon| \sim \varepsilon \times |\partial\Omega|$, one expects that

$$\lim_{s\to 1^-} \frac{1}{1-s}\int_{\Omega_{1-s}} |u|^p \,\mathrm{d}x = \int_{\partial\Omega} |u|^p \,\mathrm{d}\sigma.$$

Note that the precise choice of $\varepsilon = 1-s$ is made in order for this limit to hold.

Our main results can be summarized as follows.

Theorem 1.1. *There exists a sequence of eigenvalues of (1.1) $\lambda_{k,\varepsilon}(s,p)$ such that $\lambda_{k,\varepsilon}(s,p) \to +\infty$ as $k \to +\infty$. Every eigenfunction of (1.1) is in $L^\infty(\Omega)$.*

The first eigenvalue $\lambda_{1,\varepsilon}(s,p)$ of (1.1) is isolated and simple and has eigenfunctions that do not change sign.

Moreover, choosing $\varepsilon = 1-s$, we have the convergence of the first eigenvalue to the first Steklov eigenvalue as $s \to 1^-$, that is,

$$\lim_{s\to 1^-} \lambda_{1,1-s}(s,p) = \lambda_1(p).$$

https://doi.org/10.1017/S0308210517000361

Remark 1.2. It seems natural to consider

$$\left.\begin{aligned} \mathcal{K}_{n,p}(1-s)(-\Delta)_p^s u + |u|^{p-2}u = 0 \quad & \text{in } \Omega, \\ \mathcal{N}_{s,p}u = \lambda|u|^{p-2}u \quad & \text{in } \Omega^{\mathrm{c}}. \end{aligned}\right\} \tag{1.10}$$

Associated with the first eigenvalue in this problem is the following minimization problem:

$$\tilde{\lambda}_1(s,p) = \inf_{\substack{u\in\mathcal{W}^{s,p}(\Omega),\\ \|u\|^p_{L^p(\Omega^{\mathrm{c}})}\neq 0}} \frac{\mathcal{K}_{n,p}(1-s)[u]^p_{s,p\Omega} + \|u\|^p_{L^p(\Omega)}}{\|u\|^p_{L^p(\Omega^{\mathrm{c}})}}. \tag{1.11}$$

However, this gives

$$\tilde{\lambda}_1(s,p) = 0,$$

as can be easily obtained just by considering as a minimizing sequence $u_k(x) = \phi(x+ke_1)$ with ϕ a C^∞ compactly supported profile.

Remark 1.3. When a trace embedding theorem holds (that is, when $ps > 1$) we can consider the best fractional Sobolev trace constant, which is given by

$$\Lambda_1(s,p) = \inf_{\substack{u\in W^{s,p}(\Omega),\\ u|_{\partial\Omega}\not\equiv 0}} \frac{\mathcal{K}_{n,p}(1-s)\iint_{\Omega\times\Omega}(|u(x)-u(y)|^p/|x-y|^{n+sp})\,\mathrm{d}x\,\mathrm{d}y + \|u\|^p_{L^p(\Omega)}}{\int_{\partial\Omega}|u|^p\,\mathrm{d}\sigma}. \tag{1.12}$$

Thanks to the compactness of the embedding $W^{s,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$, this infimum is attained and the minimizers are solutions to

$$\mathcal{K}_{n,p}(1-s)\iint_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega |u|^{p-2}uv\,\mathrm{d}x = \Lambda_1(s,p)\int_{\partial\Omega}|u|^{p-2}uv\,\mathrm{d}x$$

for every $v \in W^{s,p}(\Omega)$. Note that with this formulation it is not clear how to identify the 'boundary condition' satisfied by a minimizer u; the equation inside the domain reads as

$$\mathcal{K}_{n,p}(1-s)\int_\Omega \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}\,\mathrm{d}y + |u|^{p-2}u(x) = 0$$

for $x \in \Omega$. This is why we choose to analyse (1.7) (which has (1.1) as an associated PDE problem) instead of (1.12).

With the same ideas used in the study of the limit as $s \to 1^-$ in theorem 1.1 (see § 4), one can show that

$$\lim_{s\to 1^-}\Lambda_1(s,p) = \lambda_1(p).$$

We leave the details to the reader.

https://doi.org/10.1017/S0308210517000361

The paper is organized as follows: in § 2 we gather some preliminary results and, in particular, we show a minimum principle for our problem; in § 3 we deal with the eigenvalue problem (1.1) and prove the first part of theorem 1.1; finally, in § 4 we analyse the limit as $s \to 1^-$.

2. Preliminaries

We denote the usual fractional Sobolev spaces by $W^{s,p}(\Omega)$ for $p \in [1, \infty)$ and $s \in (0, 1)$ endowed with the norm

$$\|u\|^p_{W^{s,p}(\Omega)} := \|u\|^p_{L^p(\Omega)} + \iint_{\Omega^2} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}}\, \mathrm{d}x\, \mathrm{d}y.$$

In the following, $|u|_{W^{s,p}(\Omega)}$ denotes the usual Gagliardo seminorm defined as

$$|u|_{W^{s,p}(\Omega)} := \left(\iint_{\Omega^2} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}}\, \mathrm{d}x\, \mathrm{d}y \right)^{1/p}$$

for $1 \leqslant p < \infty$. It is easy to check that $\mathcal{W}^{s,p}(\Omega)$ is a subset of $W^{s,p}(\Omega)$ for all $s \in (0, 1)$.

It will be quite useful here to establish the fractional compact embeddings. For the proof see [7].

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $s \in (0, 1)$ and $p \in (1, \infty)$. Then we have the following compact embeddings:*

$$\begin{aligned} &W^{s,p}(\Omega) \hookrightarrow L^q(\Omega) && \text{for all } q \in [1, p^\star_s) && \text{if } sp \leqslant n,\\ &W^{s,p}(\Omega) \hookrightarrow C^{0,\lambda}_b(\Omega) && \text{for all } \lambda < s - n/p && \text{if } sp > n. \end{aligned}$$

Here $p^\star_s$ is the fractional critical Sobolev exponent, that is,

$$p^\star_s := \begin{cases} \dfrac{np}{n - sp} & \text{if } sp < n,\\ \infty & \text{if } sp \geqslant n. \end{cases}$$

2.1. A minimum principle

Here, we follow the ideas in [5].

Given $s, \varepsilon \in (0, 1)$ and $p \in (1, \infty)$, we say that $u \in \mathcal{W}^{s,p}(\Omega)$ is a weak supersolution of

$$\left.\begin{aligned} \mathcal{K}_{n,p}(1 - s)(-\Delta)^s_p u + |u|^{p-2}u = 0 \quad &\text{in } \Omega,\\ \mathcal{N}_{s,p}u = 0 \quad &\text{in } \Omega^c, \end{aligned}\right\} \tag{2.1}$$

if and only if

$$\mathcal{K}_{n,p}(1 - s)\mathcal{H}_{s,p}(u, v) + \int_\Omega |u|^{p-2}uv\, \mathrm{d}x \geqslant 0 \tag{2.2}$$

for every $v \in \mathcal{W}^{s,p}(\Omega)$, $v \geqslant 0$.

First we need a subtle adaptation of [8, lemma 1.3].

https://doi.org/10.1017/S0308210517000361

LEMMA 2.2. *Let $s, \varepsilon \in (0,1)$ and $p \in (1,\infty)$. Suppose that u is a weak supersolution of (2.1), and $u \geqslant 0$ in $\mathbb{R}^n$. If $B_R(x_0) \subset \mathbb{R}^n \setminus \partial\Omega$, then, for any $B_r = B_r(x_0) \subset B_{R/2}(x_0)$ and $0 < \delta < 1$,*

$$\iint_{B_r \times A} \frac{1}{|x-y|^{n+sp}} \left| \log\left(\frac{u(x)+\delta}{u(y)+\delta}\right)\right|^p \mathrm{d}x\,\mathrm{d}y \leqslant C r^{n-sp}(1+r^{sp}),$$

where

$$A = \begin{cases} B_r & \text{if } B_R \subset \Omega, \\ \Omega & \text{if } B_R \subset \mathbb{R}^n \setminus \bar{\Omega}, \end{cases}$$

and C is a constant independent of δ.

Proof. Let $0 < r < R/2$, $0 < \delta$ and $\phi \in C_0^\infty(B_{3r/2})$ be such that

$$0 \leqslant \phi \leqslant 1, \quad \phi \equiv 1 \text{ in } B_r \quad \text{and} \quad |D\phi| < Cr^{-1} \text{ in } B_{3r/2} \subset B_R.$$

Taking $v = (u+\delta)^{1-p}\phi^p$ as a test function in (2.2), we have that

$$0 \leqslant \mathcal{K}_{n,p}(1-s)\mathcal{H}_{s,p}(u,(u+\delta)^{1-p}\phi^p) + \int_{B_{3r/2}\cap\Omega} \frac{u^{p-1}}{(u+\delta)^{p-1}}\phi^p\,\mathrm{d}x. \tag{2.3}$$

On the other hand, in the proof of lemma 1.3 in [8], it was shown that

$$\begin{aligned} &\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}(v(x)-v(y)) \\ &\qquad \leqslant -\frac{1}{C}\frac{1}{|x-y|^{n+sp}}\left|\log\left(\frac{u(x)+\delta}{u(y)+\delta}\right)\right|^p \phi(y)^p + C\frac{|\phi(x)-\phi(y)|^p}{|x-y|^{n+sp}} \end{aligned} \tag{2.4}$$

for a constant $C \equiv C(p)$. Moreover, in the $B_R \subset \Omega$ case, it was shown that

$$\mathcal{H}_{s,p}(u,(u+\delta)^{1-p}\phi^p) \leqslant Cr^{n-sp} - \iint_{B_r\times B_r} \frac{1}{|x-y|^{n+sp}}\left|\log\left(\frac{u(x)+\delta}{u(y)+\delta}\right)\right|^p \mathrm{d}x\,\mathrm{d}y,$$

where C is independent of δ. Then, by (2.3) and using that $0 \leqslant u^{p-1}(u+\delta)^{1-p}\phi^p \leqslant 1$ in $B_{3r/2} \cap \Omega = B_{3r/2}$, the lemma holds.

We proceed now to consider the case in which $B_R \subset \mathbb{R}^n \setminus \bar{\Omega}$. Since $B_{3r/2} \cap \Omega = \emptyset$, by (2.3) and (2.4),

$$\begin{aligned} \iint_{B_r\times\Omega} \frac{1}{|x-y|^{n+sp}}\left|\log\left(\frac{u(x)+\delta}{u(y)+\delta}\right)\right|^p \mathrm{d}x\,\mathrm{d}y &\leqslant C\iint_{B_{3r/2}\times\Omega} \frac{|\phi(x)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant C\frac{r^n}{\mathrm{dist}(B_R,\Omega)^{sp}} \end{aligned}$$

for $C = C(n,s,p)$ □

Proceeding as in the proof of theorem A.1 in [5] and using the previous lemma, we get the following minimum principle.

THEOREM 2.3 (minimum principle). *Let $s, \varepsilon \in (0,1)$ and $p \in (1,\infty)$. If u is a weak supersolution of (2.1) such that $u \geqslant 0$ in $\mathbb{R}^n$ and $u \not\equiv 0$ in all connected components of $\mathbb{R}^n \setminus \partial\Omega$, then $u > 0$ almost everywhere (a.e.) in Ω.*

https://doi.org/10.1017/S0308210517000361

Proof. We argue by contradiction and we assume that $Z = \{x\colon u(x) = 0\}$ has positive measure. Since $u \not\equiv 0$ in all connected components of $\mathbb{R}^n \setminus \Omega$, there are a ball $B_R(x_0) \subset \mathbb{R}^n \setminus \partial\Omega$ and $r \in (0, 2R)$ such that $|B_r(x_0) \cap Z| > 0$ and $u \not\equiv 0$ in $B_r(x_0)$.

For any $\delta > 0$ and $x \in \mathbb{R}^n$, we define

$$F_\delta(x) \coloneqq \log\left(1 + \frac{u(x)}{\delta}\right).$$

Observe that if $y \in B_r(x_0) \cap Z$, then

$$|F_\delta(x)|^p = |F_\delta(x) - F_\delta(y)|^p \leqslant \frac{(2r)^{n+sp}}{|x-y|^{n+sp}}\left|\log\left(\frac{u(x)+\delta}{u(y)+\delta}\right)\right|^p \quad \forall x \in \mathbb{R}^n.$$

Then

$$|F_\delta(x)|^p \leqslant \frac{(2r)^{n+sp}}{|Z \cap B_r(x_0)|}\int_{B_r(x_0)} \frac{1}{|x-y|^{n+sp}}\left|\log\left(\frac{u(x)+\delta}{u(y)+\delta}\right)\right|^p \mathrm{d}y \quad \forall x \in \mathbb{R}^n.$$

Therefore,

$$\int_A |F_\delta(x)|^p \,\mathrm{d}x \leqslant \frac{(2r)^{n+sp}}{|Z \cap B_r(x_0)|}\iint_{B_r(x_0)\times A} \frac{1}{|x-y|^{n+sp}}\left|\log\left(\frac{u(x)+\delta}{u(y)+\delta}\right)\right|^p \mathrm{d}x\,\mathrm{d}y,$$

where

$$A = \begin{cases} B_r & \text{if } B_R \subset \Omega, \\ \Omega & \text{if } B_R \subset \mathbb{R}^n \setminus \bar{\Omega}. \end{cases}$$

By lemma 2.2, there is a constant C independent of δ such that

$$\int_A |F_\delta(x)|^p \,\mathrm{d}x \leqslant C\frac{r^{2n}(1 + r^{sp})}{|Z \cap B_r(x_0)|}.$$

Taking $\delta \to 0$ in the above inequality, we obtain

$$u \equiv 0 \quad \text{in } A,$$

which is a contradiction since $u \not\equiv 0$ in all connected components of $\mathbb{R}^n \setminus \partial\Omega$. Thus, $u > 0$ in $\mathbb{R}^n$. □

3. The eigenvalue problem

In this section we prove that $\lambda_{1,\varepsilon}(s,p)$ is the first non-zero eigenvalue of (1.1), that there is a sequence of eigenvalues, and that the eigenfunctions are bounded. Additionally, we show that $\lambda_{1,\varepsilon}(s,p)$ is simple and isolated using variational methods for non-local operators of elliptic type. For more details about the construction of the eigenvalues in non-local settings, see, for instance, [16, appendix A] and [15].

THEOREM 3.1. *The first non-zero eigenvalue of (1.1) is* $\lambda_{1,\varepsilon}(s,p)$.

Proof. Take a minimizing sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{W}^{s,p}(\Omega)$ of $\lambda_{1,\varepsilon}(s,p)$ and normalize it according to $\|u_k\|_{L^p(\Omega_\varepsilon)} = \varepsilon$. Then there is a constant C such that

$$\|u_k\|_{s,p} \leqslant C.$$

https://doi.org/10.1017/S0308210517000361

Thus, by theorem 2.1, up to a subsequence,

$$\left.\begin{aligned} u_k &\rightharpoonup u \quad \text{weakly in } \mathcal{W}^{s,p}(\Omega),\\ u_k &\to u \quad \text{strongly in } L^p(\Omega). \end{aligned}\right\} \tag{3.1}$$

In particular, $u_k \to u$ strongly in $L^p(\Omega_\varepsilon)$ and therefore $\|u\|_{L^p(\Omega_\varepsilon)} = \varepsilon$.

Since (3.1) holds,

$$\begin{aligned} \mathcal{K}_{n,p}(1-s)[u]_{s,p}^p + \|u\|_{L^p(\Omega)}^p &\leqslant \liminf_{k\to\infty} \mathcal{K}_{n,p}(1-s)[u_k]_{s,p}^p + \|u_k\|_{L^p(\Omega)}^2 \\ &= \lim_{k\to\infty} \mathcal{K}_{n,p}(1-s)[u_k]_{s,p}^p + \|u_k\|_{L^p(\Omega)}^p \\ &= \lambda_{1,\varepsilon}(s,p). \end{aligned}$$

Then, by (1.7), we have that

$$\mathcal{K}_{n,p}(1-s)[u]_{s,p}^p + \|u\|_{L^p(\Omega)}^p = \lambda_{1,\varepsilon}(s,p).$$

The fact that a minimizer verifies (1.6) is standard but we include a short proof here for the sake of completeness. Let u be a non-trivial minimizer of (1.7). Then, using Lagrange's multipliers, we get the existence of a value $\lambda \in \mathbb{R}$ such that

$$\mathcal{K}_{n,p}(1-s)\mathcal{H}_{s,p}(u,v) + \int_\Omega |u_p|^{p-2}uv\,\mathrm{d}x = \frac{\lambda}{\varepsilon}\int_{\Omega_\varepsilon} |u|^{p-2}uv\,\mathrm{d}x \tag{3.2}$$

for all $v \in \mathcal{W}^{s,p}(\Omega)$ with $\|v\|_{L^p(\Omega_\varepsilon)} = \varepsilon$. Therefore, (3.2) also holds for all $v \in \mathcal{W}^{s,p}(\Omega)$. Finally, taking $v = u$ we get that $\lambda = \lambda_{1,\varepsilon}(s,p)$. □

Using a topological tool (the genus), we can construct an unbounded sequence of eigenvalues.

THEOREM 3.2. *There is a sequence of eigenvalues $\lambda_{k,\varepsilon}(s,p)$ such that $\lambda_{k,\varepsilon}(s,p) \to \infty$ as $k \to \infty$.*

Proof. We follow ideas from [13], and hence we omit the details. Let us consider

$$M_\alpha = \{u \in \mathcal{W}^{s,p}(\Omega) \colon \|u\|_{s,p} = p\alpha\}$$

and

$$\varphi(u) = \frac{1}{p}\int_{\Omega_\varepsilon} |u(x)|^p\,\mathrm{d}x.$$

We are looking for critical points of φ restricted to the manifold M_α using a minimax technique. We consider the class

$$\Sigma = \{A \subset \mathcal{W}^{s,p}(\Omega) \setminus \{0\} \colon A \text{ is closed}, A = -A\}.$$

Over this class we define the genus, $\gamma \colon \Sigma \to \mathbb{N} \cup \{\infty\}$, as

$$\gamma(A) = \min\{k \in \mathbb{N} \colon \text{there exists } \phi \in C(A, \mathbb{R}^k - \{0\}),\ \phi(x) = -\phi(-x)\}.$$

Now, we let $C_k = \{C \subset M_\alpha \colon C \text{ is compact, symmetric and } \gamma(C) \leqslant k\}$ and let

$$\beta_k = \sup_{C\in C_k} \min_{u\in C} \varphi(u). \tag{3.3}$$

https://doi.org/10.1017/S0308210517000361

Then $\beta_k > 0$ and there exists $u_k \in M_\alpha$ such that $\varphi(u_k) = \beta_k$ and u_k is a weak eigenfunction with $\lambda_k = \alpha/\beta_k$. □

Our next aim is to prove that the eigenfunctions are bounded. We follow ideas from [12].

LEMMA 3.3. *Let $s, \varepsilon \in (0,1)$, let $p \in (1,\infty)$ and let λ be an eigenvalue of (1.1). If u is an eigenfunction associated with λ, then $u \in L^\infty(\Omega)$.*

Proof. If $ps > n$, then the assertion holds by theorem 2.1. Then let us suppose that $sp \leqslant n$. We will show that if $\|u_+\|_{L^p(\Omega)} \leqslant \delta$, then u_+ is bounded, where $\delta > 0$ must be determined.

For $k \in \mathbb{N}_0$ we define the function u_k by

$$u_k \coloneqq (u(x) - 1 + 2^{-k})_+.$$

Observe that $u_0 = u_+$ and for any $k \in \mathbb{N}_0$ we have that $u_k \in \mathcal{W}^{s,p}(\Omega)$,

$$\left.\begin{aligned} u_{k+1} \leqslant u_k \quad & \text{a.e. } \mathbb{R}^n, \\ u < (2^{k+1}-1)u_k \quad & \text{in } \{u_{k+1} > 0\}, \\ \{u_{k+1} > 0\} \subset \{u_k > 2^{-(k+1)}\}&. \end{aligned}\right\} \tag{3.4}$$

Now, since

$$|v_+(x) - v_+(y)|^p \leqslant |v(x) - v(y)|^{p-2}(v(x) - v(y))(v_+(x) - v_+(y)) \quad \forall x, y \in \mathbb{R}^n,$$

for any function $v\colon \mathbb{R}^n \to \mathbb{R}$, by taking $v = u - 1 + 2^{-k}$ we have that

$$\begin{aligned} \mathcal{K}_{n,p}(1-s)[u_{k+1}]^p_{s,p} + \|u_{k+1}\|^p_{L^p(\Omega)} \\ &\leqslant \mathcal{K}_{n,p}(1-s)\mathcal{H}_{s,p}(u, u_{k+1}) + \int_\Omega |u|^{p-2} u u_{k+1}\,\mathrm{d}x \\ &= \frac{\lambda}{\varepsilon}\int_{\Omega_\varepsilon} |u|^{p-2} u u_{k+1}\,\mathrm{d}x \end{aligned}$$

for all $k \in \mathbb{N}_0$. Then, by (3.4), we have that

$$\begin{aligned} \mathcal{K}_{n,p}(1-s)[u_{k+1}]^p_{s,p} + \|u_{k+1}\|^p_{L^p(\Omega)} &\leqslant \frac{\lambda}{\varepsilon}\int_{\Omega_\varepsilon} u^{p-1} u_{k+1}\,\mathrm{d}x \\ &\leqslant \frac{\lambda}{\varepsilon}(2^{k+1}-1)^{p-1}\|u_k\|^p_{L^p(\Omega)} \end{aligned} \tag{3.5}$$

for all $k \in \mathbb{N}_0$.

On the other hand, in the $sp < n$ case, using Hölder's inequality, fractional Sobolev embeddings and Chebyshev's inequality, for any $k \in \mathbb{N}_0$ we have that

$$\begin{aligned} \|u_{k+1}\|^p_{L^p(\Omega)} &\leqslant \|u_{k+1}\|^p_{L^{p^*_s}(\Omega)} |\{u_{k+1} > 0\}|^{sp/n} \\ &\leqslant C\|u_{k+1}\|^p_{s,p} |\{u_{k+1} > 0\}|^{sp/n} \\ &\leqslant C\|u_{k+1}\|^p_{s,p} |\{u_k > 2^{-(k+1)}\}|^{sp/n} \\ &\leqslant C\|u_{k+1}\|^p_{s,p} (2^{(k+1)p}\|u_k\|^p_{L^p(\Omega)})^{sp/n}. \end{aligned} \tag{3.6}$$

https://doi.org/10.1017/S0308210517000361

Similarly, in the $sp = n$ case, taking $r > p$ and proceeding as in the previous case in which $sp < n$ (with r in place of p_s^*), we have that (3.6) holds with $1 - p/r > 0$ in place of sp/n.

Then, by (3.5) and (3.6), there exist a constant $C > 1$ and $\alpha > 0$ both independent of k such that

$$\|u_{k+1}\|_{L^p(\Omega)}^p \leqslant C^k (\|u_k\|_{L^p(\Omega)}^p)^{1+\alpha}.$$

Therefore, if $\|u_+\|_{L^p(\Omega)}^p = \|u_0\|_{L^p(\Omega)}^p \leqslant C^{-1/\alpha^2} = \delta^p$, then

$$\lim_{k\to+\infty} \|u_k\|_{L^p(\Omega)} = 0.$$

On the other hand, as $u_k \to (u-1)_+$ a.e in $\mathbb{R}^n$, we obtain that $(u-1)_+ \equiv 0$ in $\mathbb{R}^n$. Therefore, u_+ is bounded.

Finally, taking $-u$ in place of u we have that u_- is bounded if $\|u_-\|_{L^p(\Omega)} < \delta$. Therefore, u is bounded. □

Now, using theorem 2.3, we show that a non-negative eigenfunction is positive.

Lemma 3.4. *Let $s, \varepsilon \in (0,1)$, let $p \in (1,\infty)$ and let λ be an eigenvalue of (1.1). If u is a non-negative eigenfunction associated with λ, then $u > 0$ in $\mathbb{R}^n$.*

Proof. By theorem 2.3, we only need to show that $u \not\equiv 0$ in all connected components of $\mathbb{R}^n \setminus \partial\Omega$. Suppose, by contradiction, that there is a connected component Z of $\mathbb{R}^n \setminus \partial\Omega$ such that $u \equiv 0$ in Z. Taking $\phi \in C_0^\infty(Z)$ as a test function in (1.6), we get

$$\mathcal{H}_{s,p}(u, \phi) = 0.$$

Therefore,

$$\int_\Omega (u(x))^{p-1} \int_Z \frac{\phi(y)}{|x-y|^{n+sp}} \, \mathrm{d}y \, \mathrm{d}x = 0 \quad \forall \phi \in C_0^\infty(Z).$$

Then $u = 0$ in Ω. Thus, since u s a non-negative eigenfunction associated with λ, we obtain that

$$[u]_{s,p} = \mathcal{H}_{s,p}(u,u) = \frac{1}{\mathcal{K}_{n,p}(1-s)} \left(\frac{\lambda}{\varepsilon} \int_{\Omega_\varepsilon} |u|^p \, \mathrm{d}x - \int_\Omega |u|^p \, \mathrm{d}x \right) = 0.$$

Hence, $u \equiv 0$ in $\mathbb{R}^n$, which is a contradiction since $u \not\equiv 0$ in $\mathbb{R}^n$. □

Note that if u is an eigenfunction associated with $\lambda_{1,\varepsilon}(s,p)$, then

$$u_+(x) = \max\{u(x), 0\} \not\equiv 0 \quad \text{or} \quad u_-(x) = \max\{-u(x), 0\} \not\equiv 0$$

in Ω_ε. If $u_+(x) \not\equiv 0$ in Ω_ε, then

$$\begin{aligned} \mathcal{K}_{n,p}(1-s)[u_+]_{s,p}^p + \|u_+\|_{L^p(\Omega)}^p &\leqslant \mathcal{K}_{n,p}(1-s)\mathcal{H}_{s,p}(u, u_+) + \int_\Omega |u|^{p-2} u u_+ \, \mathrm{d}x \\ &= \frac{\lambda_{1,\varepsilon}(s,p)}{\varepsilon} \int_{\Omega_\varepsilon} |u|^{p-2} u u_+ \, \mathrm{d}x \\ &= \frac{\lambda_{1,\varepsilon}(s,p)}{\varepsilon} \|u_+\|_{L^p(\Omega_\varepsilon)}^p, \end{aligned}$$

https://doi.org/10.1017/S0308210517000361

that is, u_+ is a minimizer of (1.7). Therefore, u_+ is a non-negative eigenfunction associated with $\lambda_{1,\varepsilon}(s,p)$. Then, by lemma 3.4, $u_+ > 0$ in Ω.

In the same manner we can see that if $u_-(x) \not\equiv 0$ in Ω_ε, then $u_- > 0$ in Ω. Thus the next theorem is proved.

THEOREM 3.5. *Any eigenfunction associated with $\lambda_{1,\varepsilon}(s,p)$ has constant sign.*

A key ingredient in the following sections is the simplicity of the first eigenvalue $\lambda_{1,\varepsilon}(s,p)$. In order to prove this result we need the following Picone-type identity (see [2, lemma 6.2]).

LEMMA 3.6. *Let $p \in (1,\infty)$. For $u, v\colon \mathbb{R}^n \to \mathbb{R}$ such that $u \geqslant 0$ and $v > 0$, we have*

$$L(u,v) \geqslant 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}^n,$$

where

$$L(u,v)(x,y) = |u(x)-u(y)|^p - |v(x)-v(y)|^{p-2}(v(x)-v(y))\left(\frac{u^p(x)}{v^{p-1}(x)} - \frac{u^p(y)}{v^{p-1}(y)}\right).$$

The equality holds if and only if $u = kv$ a.e. in $\mathbb{R}^n$ for some constant k.

THEOREM 3.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open connected set with Lipschitz boundary. Assume that u is a positive eigenfunction corresponding to $\lambda_{1,\varepsilon}(s,p)$. Then if $\lambda > 0$ is such that there exists a non-negative eigenfunction v of (1.1) with eigenvalue λ, then $\lambda = \lambda_{1,\varepsilon}(s,p)$ and there exists $c \in \mathbb{R}$ such that $v = cu$ a.e. in $\mathbb{R}^n$.*

Proof. Since $\lambda_{1,\varepsilon}(s,p)$ is the first eigenvalue, we have that $\lambda_{1,\varepsilon}(s,p) \leqslant \lambda$. On the other hand, by lemma 3.4, $v > 0$ in $\mathbb{R}^n$.

For $k \in \mathbb{N}$ take $v_k \coloneqq v + 1/k$. We begin by proving that $w_k \coloneqq u^p/v_k^{p-1} \in \mathcal{W}^{s,p}(\Omega)$. First observe that $w_k \in L^p(\Omega)$, due to $u \in L^\infty(\Omega)$ (see lemma 3.3). Now, for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ we have

$$\begin{aligned}
|w_k(x)-w_k(y)| &= \left|\frac{u^p(x)-u^p(y)}{v_k^{p-1}(x)} - \frac{u^p(y)(v_k^{p-1}(x)-v_k^{p-1}(y))}{v_k^{p-1}(x)v_k^{p-1}(y)}\right| \\
&\leqslant k^{p-1}|u^p(x)-u^p(y)| + \|u\|_\infty^p \frac{|v_k^{p-1}(x)-v_k^{p-1}(y)|}{v_k^{p-1}(x)v_k^{p-1}(y)} \\
&\leqslant pk^{p-1}(u^{p-1}(x)+u^{p-1}(y))|u(x)-u(y)| \\
&\qquad + (p-1)\|u\|_\infty^p \frac{v_k^{p-2}(x)+v_k^{p-2}(y)}{v_k^{p-1}(x)v_k^{p-1}(y)}|v_k(x)-v_k(y)| \\
&\leqslant 2pk^{p-1}\|u\|_\infty^p|u(x)-u(y)| \\
&\qquad + (p-1)\|u\|_\infty^p\left(\frac{1}{v_k(x)v_k^{p-1}(y)} + \frac{1}{v_k^{p-1}(x)v_k(y)}\right)|v(x)-v(y)| \\
&\leqslant C(k,\|u\|_\infty,p)(|u(x)-u(y)| + |v(x)-v(y)|).
\end{aligned}$$

As $u, v \in \mathcal{W}^{s,p}(\Omega)$, we deduce that $w_k \in W(\Omega)$ for all $k \in \mathbb{N}$.

Recall that $u, v \in \mathcal{W}^{s,p}(\Omega)$ are two eigenfunctions of problem (1.1) with eigenvalues $\lambda_1(s,p)$ and λ, respectively. Then, by using the previous lemma, we deduce

https://doi.org/10.1017/S0308210517000361

that

$$\begin{aligned}0 &\leqslant \mathcal{K}_{n,p}(1-s)\iint_{\mathbb{R}^{2n}\setminus(\Omega^c)^2} \frac{L(u,v_k)(x,y)}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \mathcal{K}_{n,p}(1-s)\iint_{\mathbb{R}^{2n}\setminus(\Omega^c)^2} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad - \mathcal{K}_{n,p}(1-s)\iint_{\mathbb{R}^{2n}\setminus(\Omega^c)^2} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))}{|x-y|^{n+sp}} \\ &\qquad\qquad \times\left(\frac{u^p(x)}{v_k^{p-1}(x)} - \frac{u^p(y)}{v_k^{p-1}(y)}\right)\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \mathcal{K}_{n,p}(1-s)\iint_{\mathbb{R}^{2n}\setminus(\Omega^c)^2} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y - \frac{\lambda}{\varepsilon}\int_{\Omega_\varepsilon} v^{p-1}\frac{u^p}{v_k^{p-1}}\,\mathrm{d}x \\ &\quad + \int_\Omega v^{p-1}\frac{u^p}{v_k^{p-1}}\,\mathrm{d}x \\ &\leqslant \frac{\lambda_{1,\varepsilon}(s,p)}{\varepsilon}\int_{\Omega_\varepsilon} u^p\,\mathrm{d}x - \int_\Omega |u|^p\,\mathrm{d}x - \frac{\lambda}{\varepsilon}\int_{\Omega_\varepsilon} v^{p-1}\frac{u^p}{v_k^{p-1}}\,\mathrm{d}x + \int_\Omega v^{p-1}\frac{u^p}{v_k^{p-1}}\,\mathrm{d}x.\end{aligned}$$

Taking $k \to \infty$ and using Fatou's lemma and the dominated convergence theorem, we infer that

$$\iint_{\mathbb{R}^{2n}\setminus(\Omega^c)^2} \frac{L(u,v)(x,y)}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y = 0$$

(recall that $\lambda_{1,\varepsilon}(s,p) \leqslant \lambda$). Therefore, by the previous lemma, $L(u,v)(x,y) = 0$ a.e. in $\mathbb{R}^{2n} \setminus (\Omega^c)^2$ and $u = cv$ for some constant $c > 0$. □

We will need the following lemma.

Lemma 3.8. *Let $\varepsilon > 0$. If u is an eigenfunction associated with $\lambda > \lambda_{1,\varepsilon}(s,p)$, there exist $C > 0$ and $\alpha > 0$ independent of λ, u and ε such that*

$$\left(\frac{C\varepsilon}{\lambda}\right)^\alpha \leqslant |\Omega^\pm|.$$

Here $\Omega^+ = \{x \in \Omega\colon u(x) > 0\}$ and $\Omega^- = \{x \in \Omega\colon u(x) < 0\}$.

Proof. Let $u^+(x) = \max\{0, u(x)\}$. Since u is an eigenfunction associated with $\lambda > \lambda_{1,\varepsilon}(s,p)$, we have that u changes sign, and then $u^+ \not\equiv 0$. In addition,

$$\begin{aligned}\min\{\mathcal{K}_{n,p}(1-s),1\}\|u^+\|_{s,p}^p &\leqslant \mathcal{K}_{n,p}(1-s)[u^+]_{s,p}^p + \|u^+\|_{L^p(\Omega)}^p \\ &\leqslant \mathcal{K}_{n,p}(1-s)\mathcal{H}_{s,p}(u,u^+) + \int_\Omega |u|^{p-2}uu^+\,\mathrm{d}x \\ &= \frac{\lambda}{\varepsilon}\int_{\Omega_\varepsilon} |u|^{p-2}uu^+\,\mathrm{d}x \\ &= \frac{\lambda}{\varepsilon}\|u^+\|_{L^p(\Omega_\varepsilon)}^p. \qquad (3.7)\end{aligned}$$

https://doi.org/10.1017/S0308210517000361

On the other hand, by the Sobolev embedding theorem, there exists a constant C independent of λ, u and ε such that

$$\|u^+\|_{L^q(\Omega)} \leqslant C\|u^+\|_{s,p},$$

where $1 < q < p_s^\star$. Then, by (3.7) and Hölder's inequality, there exists a constant C independent of λ, u and ε such that

$$\|u^+\|^p_{L^q(\Omega)} \leqslant C\frac{\lambda}{\varepsilon}\|u^+\|^p_{L^q(\Omega)}|\Omega^+|^{(q-p)/q} \quad \forall p < q < p_s^\star.$$

Fix any $p < q < p_s^\star$ and take $\alpha = q/(q-p)$. Then

$$\left(\frac{\varepsilon}{C\lambda}\right)^\alpha \leqslant |\Omega^+|.$$

In order to prove the second inequality, it will suffice to proceed as above, using the function $u^-(x) = \max\{0, -u(x)\}$ instead of u^+. □

Theorem 3.9. *For each fixed value $\varepsilon > 0$, $\lambda_{1,\varepsilon}(s,p)$ is isolated.*

Proof. From its definition, we have that $\lambda_{1,\varepsilon}(s,p)$ is left-isolated.

To prove that $\lambda_{1,\varepsilon}(s,p)$ is right-isolated, we argue by contradiction. We assume that there exists a sequence of eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ such that $\lambda_k > \lambda_{1,\varepsilon}(s,p)$ and $\lambda_k \searrow \lambda_{1,\varepsilon}(s,p)$ as $k \to +\infty$. Let u_k be an eigenfunction associated with λ_k; we can assume that

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} |u_k(x)|^p \,\mathrm{d}x = 1.$$

Then $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $\mathcal{W}^{s,p}(\Omega)$, and therefore we can extract a subsequence (that we still denote by $\{u_k\}_{k\in\mathbb{N}}$) such that

$$\begin{aligned} u_k &\rightharpoonup u \quad \text{weakly in } \mathcal{W}^{s,p}(\Omega), \\ u_k &\to u \quad \text{strongly in } L^p(\Omega). \end{aligned}$$

Then

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} |u(x)|^p \,\mathrm{d}x = 1$$

and

$$\begin{aligned} \mathcal{K}_{n,p}(1-s)[u]^p_{s,p} + \|u\|^p_{L^p(\Omega)} &\leqslant \mathcal{K}_{n,p}(1-s)\liminf_{k\to+\infty}[u_k]^p_{s,p} + \|u\|^p_{L^p(\Omega)} \\ &= \lim_{k\to+\infty}\lambda_k \\ &= \lambda_{1,\varepsilon}(s,p). \end{aligned}$$

Hence, u is an eigenfunction associated with $\lambda_{1,\varepsilon}(s,p)$. By theorem 3.5, we can assume that $u > 0$.

On the other hand, by the Egorov's theorem, for any $\delta > 0$ there exists a subset A_δ of Ω such that $|A_\delta| < \delta$ and $u_k \to u > 0$ uniformly in $\Omega \setminus A_\delta$. This contradicts the fact that, by lemma 3.8,

$$\left(\frac{C\varepsilon}{\lambda_k}\right)^\alpha \leqslant |\{x \in \Omega\colon u_k(x) < 0\}|.$$

This proves the theorem. □

4. The limit of $\lambda_{1,1-s}(s,p)$ as $s \to 1^-$

Throughout this section, we assume that Ω is a smooth bounded domain and take $\varepsilon = 1-s$.

Here we analyse the behaviour of $\lambda_{1,1-s}(s,p)$ as $s \to 1^-$. For simplicity, we omit the subscript $1-s$ and we just write $\lambda_1(s,p)$.

First we show that

$$\limsup_{s\to 1^-} \lambda_1(s,p) \leqslant \lambda_1(p).$$

For this purpose, we state some convergence results. We start with the following lemma.

Lemma 4.1. *Let Ω be a domain in $\mathbb{R}^n$ with Lipschitz boundary and let $p \in (1,\infty)$. If $u \in W^{1,p}(\Omega)$, then*

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} |u|^p \, \mathrm{d}x = \int_{\partial\Omega} |u|^p \, \mathrm{d}S.$$

In order to deal with the integrals on Ω_ε we will state the following lemma, which is an immediate consequence of the coarea formula. See [11, section 3.4.4] for details.

Lemma 4.2. *Given $g\colon \mathbb{R}^n \to \mathbb{R}$ an integrable function, and $f\colon \mathbb{R}^n \to \mathbb{R}$ a Lipschitz function such that* ess inf $|Df| > 0$, *it follows that*

$$\int_{\{0<f<t\}} g \, \mathrm{d}x = \int_0^t \left(\int_{\{f=r\}} \frac{g}{|Df|} \, \mathrm{d}S \right) \mathrm{d}r. \tag{4.1}$$

Now we are ready to proceed with the proof of lemma 4.1.

Proof of lemma 4.1. We consider the $(n-1)$-dimensional hyper-surface in $\mathbb{R}^n$ given by $\omega_r = \{x \in \mathbb{R}^n \colon d(x,\Omega^{\mathrm{c}}) = r\}$, where $d(x,\Omega) = \inf_{y\in\Omega} |x-y|$. Observe that $\Omega_\varepsilon = \{x \in \mathbb{R}^n \colon x \in \omega_r \text{ for } r \in [0,\varepsilon]\}$ and $\omega_0 = \partial\Omega$. By applying lemma 4.2 with $g = |u|^p$ and $f(x) = d(x,\Omega^{\mathrm{c}})$, we get

$$\int_{\Omega_\varepsilon} |u|^p \, \mathrm{d}x = \int_0^\varepsilon \left(\int_{\omega_r} |u|^p \, \mathrm{d}S \right) \mathrm{d}r$$

since $|Df| = 1$. The mean-value theorem for integrals asserts that there exists $r_0 \in [0,\varepsilon]$ such that

$$\int_0^\varepsilon \left(\int_{\omega_r} |u|^p \, \mathrm{d}S \right) \mathrm{d}r = \varepsilon \int_{\omega_{r_0}} |u|^p \, \mathrm{d}S.$$

Since r_0 tends to 0 as $\varepsilon \to 0^+$, we get that ω_{r_0} tends to $\partial\Omega$ as $\varepsilon \to 0^+$; the result follows by using that the trace operator for a fixed function in $W^{1,p}(\Omega)$ depends continuously on the hyper-surface (see [3]), and hence

$$\int_{\omega_{r_0}} |u|^p \, \mathrm{d}S \to \int_{\partial\Omega} |u|^p \, \mathrm{d}S$$

as $r_0 \to 0^+$. □

https://doi.org/10.1017/S0308210517000361

If Ω is a smooth bounded domain in $\mathbb{R}^n$, then, by [14, theorem 7.25], for any open ball $B_R \supset\supset \Omega$ there is a bounded linear extension operator E from $W^{1,p}(\Omega)$ into $W_0^{1,p}(B_R)$ such that $Eu = u$ in Ω. Our next goal is to prove that

$$\mathcal{K}_{n,p}(1-s)[Eu]_{s,p} \to \|\nabla u\|^p_{L^p(\Omega)} \tag{4.2}$$

as $s \to 1^-$. To this end, we need the following result. For the proof we refer the reader to [4, corollary 2].

THEOREM 4.3. *Let Ω be a smooth bounded domain and let $p \in (1, \infty)$. Assume that $u \in L^p(\Omega)$. Then*

$$\lim_{s\to 1^-} \mathcal{K}_{n,p}(1-s)|u|^p_{W^{s,p}(\Omega)} = |u|^p_{W^{1,p}(\Omega)}$$

with

$$|u|^p_{W^{1,p}(\Omega)} = \begin{cases} \|\nabla u\|^p_{L^p(\Omega)} & \text{if } u \in W^{1,p}(\Omega), \\ \infty & \text{otherwise.} \end{cases}$$

We now show (4.2), which will be key in the proof of the next results.

LEMMA 4.4. *If $u \in W^{1,p}(\Omega)$, then*

$$\lim_{s\to 1^-} \mathcal{K}_{n,p}(1-s)[Eu]^p_{s,p} = \|\nabla u\|^p_{L^p(\Omega)}.$$

Proof. Observe that

$$\begin{aligned}[Eu]^p_{s,p} = |u|^p_{W^{s,p}(\Omega)} &+ 2\iint_{\Omega\times(\Omega^c\cap B_R)} \frac{|Eu(x)-Eu(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x \\ &+ 2\iint_{\Omega\times B_R^c} \frac{|Eu(x)-Eu(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Then, by theorem 4.3, we need to show that

$$(1-s)\iint_{\Omega\times(\Omega^c\cap B_R)} \frac{|Eu(x)-Eu(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x \to 0,$$
$$(1-s)\iint_{\Omega\times B_R^c} \frac{|Eu(x)-Eu(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x \to 0$$

as $s \to 1^-$.

By theorem 4.3, we have that

$$\mathcal{K}_{n,p}(1-s)|Eu|^p_{W^{s,p}(B_R)} \to \|\nabla Eu\|^p_{L^p(B_R)},$$
$$\mathcal{K}_{n,p}(1-s)|Eu|^p_{W^{s,p}(\Omega)} \to \|\nabla Eu\|^p_{L^p(\Omega)},$$
$$\mathcal{K}_{n,p}(1-s)|Eu|^p_{W^{s,p}(\Omega\cap B_R^c)} \to \|\nabla Eu\|^p_{L^p(\Omega^c\cap B_R)}$$

https://doi.org/10.1017/S0308210517000361

as $s \to 1^-$. Therefore,

$$\begin{aligned}(1-s)\iint_{\Omega\times(\Omega^c\cap B_R)} &\frac{|Eu(x)-Eu(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x\\ &= \tfrac{1}{2}(1-s)(|Eu|^p_{W^{s,p}(B_R)} - |Eu|^p_{W^{s,p}(\Omega)} - |Eu|^p_{W^{s,p}(\Omega\cap B_R^c)})\\ &\to 0 \quad \text{as } s\to 1^-.\end{aligned}$$

On the other hand,

$$(1-s)\iint_{\Omega\times B_R^c} \frac{|Eu(x)-Eu(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x \leqslant C_n \frac{1-s}{sp}\frac{1}{d(\Omega,B_R^c)^{sp}} \to 0$$

as $s \to 1^-$. □

From lemmas 4.1 and 4.4, we get the following corollary.

Corollary 4.5. *Let Ω be a smooth bounded domain and let $p \in (1,\infty)$. For a fixed $u \in W^{1,p}(\Omega)\setminus W_0^{1,p}(\Omega)$, it holds that*

$$\lim_{s\to 1^-} \frac{\mathcal{K}_{n,p}(1-s)[Eu]^p_{s,p} + \|Eu\|^p_{L^p(\Omega)}}{\|Eu\|^p_{L^p(\Omega_{1-s})}/(1-s)} = \frac{\|\nabla u\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)}}{\|u\|^p_{L^p(\partial\Omega)}}.$$

From this result the following corollary is straightforward.

Corollary 4.6. *Let Ω be a smooth bounded domain and let $p \in (1,\infty)$. Then*

$$\limsup_{s\to 1^-} \lambda_1(s,p) \leqslant \lambda_1(p).$$

With this result in mind, to prove the last part of theorem 1.1 we need to show that

$$\lambda_1(p) \leqslant \liminf_{s\to 1^-} \lambda_1(s,p). \tag{4.3}$$

Before proving this, we need to state some auxiliary results.

The next theorem was established in [4, corollary 7].

Theorem 4.7. *Let Ω be a smooth bounded domain, let $p \in (1,\infty)$ and let $u_s \in W^{s,p}(\Omega)$. Assume that*

$$\|u_s\|_{L^p(\Omega)} \leqslant C \quad \textit{and} \quad (1-s)|u_s|_{W^{s,p}(\Omega)} < C \quad \forall s > 0.$$

Then, up to a subsequence, $\{u_s\}$ converges in $L^p(\Omega)$ (and, in fact, in $W^{s_0,p}(\Omega)$ for all $s_0 \in (0,1)$) to some $u \in W^{1,p}(\Omega)$.

The proof of the following proposition can be found in [6, proposition 3.10].

Proposition 4.8. *Let Ω be a smooth bounded domain and let $p \in (1,\infty)$. Given $\{s_k\} \subset (0,1)$ an increasing sequence converging to 1 and $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega)$ converging to u in $L^p(\Omega)$, we have that*

$$\|\nabla u\|^p_{L^p(\Omega)} \leqslant \lim_{k\to\infty} \mathcal{K}_{n,p}(1-s_k)|u_{s_k}|^p_{W^{s_k,p}(\Omega)}$$

Lemma 4.9. *Let Ω be a smooth bounded domain, let $p \in (1,\infty)$ and let $\{u_s\}_{s\in(0,1)}$ be such that $u_s \to u$ strongly in $W^{t,p}(\Omega)$ for some $t \in (1/p, 1)$. Then*

$$\frac{1}{1-s}\int_{\Omega_{1-s}} |u_s|^p \,\mathrm{d}x \to \int_{\partial\Omega} |u|^p \,\mathrm{d}S$$

as $s \to 1^-$.

Proof. We start by observing that, since $\partial\Omega \in C^2$ and $t > 1/p$, the trace constant in the embedding $W^{t,p}(\Omega) \hookrightarrow L^p(\partial\Omega_\varepsilon)$ for all $\varepsilon \in (0,\varepsilon_0)$ is bounded uniformly (independently of ε). Then there is a constant C independent of s such that

$$\|u_s - u\|_{L^p(\partial\Omega_\varepsilon)} \leqslant C\|u_s - u\|_{W^{t,p}(\Omega)}.$$

Therefore,

$$\frac{1}{1-s}\int_{\Omega_{1-s}} |u_s(x)|^p \,\mathrm{d}x = \frac{1}{1-s}\int_0^{1-s}\left(\int_{\partial\Omega_r} |u_s|^p \,\mathrm{d}S\right)\mathrm{d}r \to \int_{\partial\Omega} |u|^p \,\mathrm{d}S$$

as $s \to 1^-$. □

Now we are ready to prove (4.3).

Corollary 4.10. *Let Ω be a smooth bounded domain and let $p \in (1,\infty)$. Then*

$$\lambda_1(p) \leqslant \liminf_{s\to 1^-} \lambda_1(s,p).$$

Proof. Let $\{s_k\}_{k\in\mathbb{N}}$ be a sequence in $(0,1)$ such that $s_k \to 1^-$ as $k \to \infty$ and

$$\lim_{k\to\infty} \lambda_1(s_k,p) = \liminf_{s\to 1^-} \lambda_1(s,p).$$

For $k \in \mathbb{N}$, let u_k be the eigenfunctions of problem (1.1) with $s = s_k$ and $\lambda = \lambda_1(s_k,p)$ normalized such that

$$\frac{1}{1-s_k}\int_{\Omega_{1-s_k}} |u_k|^p \,\mathrm{d}x = 1.$$

Moreover, by corollary 4.6, there is a positive constant C such that

$$\|u_k\|_{L^p(\Omega)} \leqslant C \quad \text{and} \quad (1-s_k)|u_k|_{W^{s,p}(\Omega)} < C \quad \forall k \in \mathbb{N}.$$

Then, by theorem 4.7, up to a subsequence, $\{u_k\}$ converges in $L^p(\Omega)$ (and, in fact, in $W^{s_0,p}(\Omega)$ for all $s_0 \in (0,1)$) to some $u \in W^{1,p}(\Omega)$. Thus, by proposition 4.8 and lemma 4.9, we get

$$\|\nabla u\|^p_{L^p(\Omega)} \leqslant \lim_{k\to\infty} \mathcal{K}_{n,p}(1-s_k)|u_{s_k}|^p_{W^{s_k,p}(\Omega)}$$

and

$$\lim_{k\to\infty}\frac{1}{1-s_k}\int_{\Omega_{1-s_k}} |u_k(x)|^p \,\mathrm{d}x = \int_{\partial\Omega} |u|^p \,\mathrm{d}S.$$

https://doi.org/10.1017/S0308210517000361

Then $\|u\|^p_{L^p(\partial\Omega)} = 1$ and

$$\begin{aligned}\|\nabla u\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)} &\leqslant \lim_{k\to\infty} \mathcal{K}_{n,p}(1-s_k)|u_{s_k}|^p_{W^{s_k,p}(\Omega)} + \|u_k\|^p_{L^p(\Omega)} \\ &\leqslant \lim_{k\to\infty} \lambda_1(s_k,p) = \liminf_{s\to 1^-} \lambda_1(s,p).\end{aligned}$$

Therefore,

$$\lambda_1(p) \leqslant \liminf_{s\to 1^-} \lambda_1(s,p).$$

□

Acknowledgements

We want to thank Ricardo Duran for several interesting discussions.

References

1 R. A. Adams. *Sobolev spaces*, Pure and Applied Mathematics, vol. 65 (Academic Press, 1975).

2 S. Amghibech. On the discrete version of Picone's identity. *Discrete Appl. Math.* **156** (2008), 1–10.

3 J. M. Arrieta, A. Rodriguez-Bernal and J. D. Rossi. The best Sobolev trace constant as limit of the usual Sobolev constant for small strips near the boundary. *Proc. R. Soc. Edinb.* A **138** (2008), 1–15.

4 J. Bourgain, H. Brezis and P. Mironescu. Another look at Sobolev spaces. In *Optimal control and partial differential equations*, pp. 439–455 (IOS Press, 2001).

5 L. Brasco and F. Giovanni. Convexity properties of Dirichlet integrals and Picone-type inequalities. *Kōdai Math. J.* **37** (2014), 769–799.

6 L. Brasco, E. Parini and M. Squassina. Stability of variational eigenvalues for the fractional p-Laplacian. *Discrete Contin. Dynam. Syst.* **36** (2016), 1813–1845. (English summary.)

7 F. Demengel and G. Demengel. *Functional spaces for the theory of elliptic partial differential equations*, Universitext (Springer, 2012).

8 A. Di Castro, T. Kuusi and G. Palatucci. Local behavior of fractional p-minimizers. *Annales Inst. H. Poincaré Analyse Non Linéaire* **33** (2016), 1279–1299.

9 S. Dipierro, X. Ros-Oton and E. Valdinoci. Nonlocal problems with Neumann boundary conditions. *Rev. Mat. Ibero.* **33** (2017), 377–416.

10 O. Druet and E. Hebey. *The AB program in geometric analysis: sharp Sobolev inequalities and related problems*, Memoirs of the American Mathematical Society, vol. 160, no. 761 (Providence, RI: American Mathematical Society, 2002).

11 L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions* (CRC press, 2015).

12 G. Franzina and G. Palatucci. Fractional p-eigenvalues. *Riv. Mat. Univ. Parma Ser. 5* **5** (2014), 373–386.

13 J. Garcia-Azorero and I. Peral. Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term. *Trans. Am. Math. Soc.* **323** (1991), 877–895.

14 D. Gilbarg and N. Trudinger. *Elliptic partial differential equations of second order*, edn 2, Classics in Mathematics (Springer, 2001).

15 E. Lindgren and P. Lindqvist. Fractional eigenvalues. *Calc. Var. PDEs* **49** (2014), 795–826.

16 R. Servadei and E. Valdinoci. Variational methods for non-local operators of elliptic type. *Discrete Contin. Dynam. Syst.* **33** (2013), 2105–2137.