**Nonlinear Differential Equations
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Precise homogenization rates for the Fučík spectrum

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Abstract. Given a bounded domain Ω in $\mathbb{R}^N$, $N \geq 1$ we study the homogenization of the weighted Fučík spectrum with Dirichlet boundary conditions. In the case of periodic weight functions, precise asymptotic rates of the curves are obtained.

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1. Introduction

Given a bounded domain Ω in $\mathbb{R}^N$, $N \geq 1$ we study the asymptotic behavior as $\varepsilon \to 0$ of the spectrum of the following asymmetric elliptic problem

$$\begin{cases} -\Delta_p u_\varepsilon = \alpha_\varepsilon m_\varepsilon (u_\varepsilon^+)^{p-1} - \beta_\varepsilon n_\varepsilon (u_\varepsilon^-)^{p-1} & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

Here, $\Delta_p u$ denotes the p-Laplace operator with $2 \leq p < \infty$ and, as usual, $u^\pm := \max\{\pm u, 0\}$. The parameters α_ε and β_ε are real numbers depending on $\varepsilon > 0$. The functions m_ε and n_ε are assumed to be positive and uniformly bounded away from zero and infinity, that is, there are constants θ_-, θ_+ such that

$$0 < \theta_- \leq m_\varepsilon(x), n_\varepsilon(x) \leq \theta_+ < +\infty. \quad (1.2)$$

Also, we assume the existence of functions $m_0(x)$ and $n_0(x)$ satisfying (1.2), such that, as $\varepsilon \to 0$,

$$m_\varepsilon \rightharpoonup m_0 \quad \text{weakly* in } L^\infty(\Omega), \quad n_\varepsilon \rightharpoonup n_0 \quad \text{weakly* in } L^\infty(\Omega). \quad (1.3)$$

Under these considerations, as $\varepsilon \to 0$ the following natural limit problem for (1.1) is obtained

$$\begin{cases} -\Delta_p u_0 = \alpha_0 m_0 (u_0^+)^{p-1} - \beta_0 n_0 (u_0^-)^{p-1} & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.4)$$

For a fixed value of ε, problem (1.1) was introduced by Dancer [7] and Fučík [13] in the '70s in connection with jumping nonlinearities. In these works they considered the constant weights case in one-dimensional settings. Later on, the extension to higher dimensions and the case with weights, were widely studied. We refer to [2,10,19,20] for positive weights, and [1,14] for indefinite weights.

For a fixed $\varepsilon > 0$, the Fučík spectrum of (1.1) is defined as the set

$$\Sigma_\varepsilon = \Sigma_\varepsilon(m_\varepsilon, n_\varepsilon) := \{(\alpha_\varepsilon, \beta_\varepsilon) \in \mathbb{R}^2 \colon (1.1) \text{ has a nontrivial solution}\}.$$

We say that a nontrivial function $u_\varepsilon \in W_0^{1,p}(\Omega)$ is an eigenfunction of (1.1) associated to $(\alpha_\varepsilon, \beta_\varepsilon) \in \mathbb{R}^+ \times \mathbb{R}^+$ if it satisfies the weak formulation

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx = \int_\Omega (\alpha_\varepsilon m_\varepsilon(x)(u_\varepsilon^+)^{p-1}v - \beta_\varepsilon n(x)(u_\varepsilon^-)^{p-1}v) \, dx \quad (1.5)$$

for all $v \in W_0^{1,p}(\Omega)$.

Observe that when both weights are the same, let us say, r_ε, and both parameters are equal, let us say, λ_ε, Eq. (1.1) becomes the weighted p-Laplacian eigenvalue problem with Dirichlet boundary conditions, i.e.,

$$\begin{cases} -\Delta_p u_\varepsilon = \lambda_\varepsilon r_\varepsilon |u_\varepsilon|^{p-2}u_\varepsilon & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.6)$$

One immediately observe that Σ_ε contains the trivial lines $\lambda_1(m_\varepsilon) \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1(n_\varepsilon)$, being $\lambda_1(r_\varepsilon)$ the first eigenvalue of (1.6). In contrast with the one-dimensional case, where a full description of the spectrum is obtained, when $N > 1$ it is only known the existence of a curve $\mathcal{C}_\varepsilon$ beyond the trivial lines, see [8]. Such curve can be written by considering its intersection with the line of slope $s \in \mathbb{R}^+$ passing through the origin in $\mathbb{R}^2$ as

$$\mathcal{C}_\varepsilon = \mathcal{C}_\varepsilon(m_\varepsilon, n_\varepsilon) := \{(\alpha_\varepsilon(s), \beta_\varepsilon(s)), s \in \mathbb{R}^+\}. \quad (1.7)$$

The authors in [3] deal with a variational characterization for $\alpha(s)$ and $\beta(s)$.

When $\varepsilon \to 0$ the limit problem (1.4) has a corresponding first nontrivial curve defined as

$$\mathcal{C}_0 = \mathcal{C}_0(m_0, n_0) := \{(\alpha_0(s), \beta_0(s)), s \in \mathbb{R}^+\}.$$

In this context, the natural question that arises is whether the eigencurve $\mathcal{C}_\varepsilon$ converges to the limit curve $\mathcal{C}_0$, and, under which considerations the rate of the convergence of this curve can be estimated.

These inquiries were answered in [21]: $\mathcal{C}_\varepsilon$ converges to $\mathcal{C}_0$ in the sense that

$$\alpha_\varepsilon(s) \to \alpha_0(s) \quad \text{and} \quad \beta_\varepsilon(s) \to \beta_0(s)$$

as $\varepsilon \to 0$, for each fixed $s \in \mathbb{R}^+$.

In the case of periodic homogenization, i.e., when the family of functions m_ε and n_ε are given in terms of $Q-$periodic functions in the form $m_\varepsilon(x) = m(x/\varepsilon)$ and $n_\varepsilon(x) = n(x/\varepsilon)$, Q being the unit cube in $\mathbb{R}^N$ and m and n

satisfying (1.2), more information is known about the behavior of the curves. In this case it is well-known that as $\varepsilon \to 0$,

$$m_\varepsilon \rightharpoonup m_0 := \bar{m} = \fint_Q m(x)\, dx, n_\varepsilon \rightharpoonup n_0 := \bar{n}$$

$$= \fint_Q n(x)\, dx \text{ weakly* in } L^\infty(\Omega). \qquad (1.8)$$

By using the variational characterization the curves provided by [3], it was established the convergence rate of the curve:

Theorem 1.1. (Theorem 4.2, [21]) *Let $s \in \mathbb{R}^+$ be a fixed value, and m and n two $Q-$periodic functions satisfying* (1.2). *Given $\varepsilon > 0$ let us consider $m_\varepsilon := m(\frac{x}{\varepsilon})$ and $n_\varepsilon := n(\frac{x}{\varepsilon})$. Then, the following estimates hold for $(\alpha_\varepsilon(s), \beta_\varepsilon(s)) \in \mathcal{C}_\varepsilon$ and $(\alpha_0(s), \beta_0(s)) \in \mathcal{C}_0$*

$$|\alpha_\varepsilon(s) - \alpha_0(s)| \leq \begin{cases} c\varepsilon s & s \geq 1 \\ c\varepsilon s^{-2} & s < 1, \end{cases} \quad |\beta_\varepsilon(s) - \beta_0(s)| \leq \begin{cases} c\varepsilon s^2 & s \geq 1 \\ c\varepsilon s^{-1} & s < 1 \end{cases}$$
$$(1.9)$$

where c is a computable constant independent on ε and s.

Nevertheless, since estimates (1.9) do not depend on p, we suspect that Theorem 1.1 does not turn out to be enough accurate. Moreover, the one-dimensional problem, where the spectrum is known to be a family of infinite curves, is not treated in [21] due to the lack of a variational characterization of the curves.

Our first aim in this paper is to refine (1.9) by using an alternative characterization of the curves. By following the arguments of [6] it is possible to define $\mathcal{C}_\varepsilon$ by minimizing the first eigenvalue of weighted p-Laplacian problems over all possible partition of the kind

$$\mathcal{P} = \{\{\omega_+, \omega_-\} \subset \Omega : \omega_\pm \text{ is open and connected}, \omega_+ \cap \omega_- = \emptyset\},$$

see Theorem 2.1 in Sect. 2 for the precise statement. Such optimal partition characterization reduces our analysis to study the homogenization rates of the first eigenvalue of the weighted p-Laplacian and it leads to a precise estimation.

Our first result reads as follows.

Theorem 1.2. *Under the same hypothesis of Thorem 1.1, given $\varepsilon > 0$ and $s \in \mathbb{R}^+$, let $(\alpha_\varepsilon(s), \beta_\varepsilon(s)) \in \mathcal{C}_\varepsilon$ and $(\alpha_0(s), \beta_0(s)) \in \mathcal{C}_0$. Then the following estimates hold*

$$|\alpha_\varepsilon(s) - \alpha_0(s)| \leq \begin{cases} C\varepsilon s^{\frac{1}{p}} & s \geq 1 \\ C\varepsilon s^{-1-\frac{1}{p}} & s < 1 \end{cases} \quad |\beta_\varepsilon(s) - \beta_0(s)| \leq \begin{cases} C\varepsilon s^{1+\frac{1}{p}} & s \geq 1 \\ C\varepsilon s^{-\frac{1}{p}} & s < 1 \end{cases}$$

where C is a constant independent on ε and s.

Remark 1.3. A careful computation allow us to compute explicitly the constant in Theorem 1.2 as

$$C = \left(\frac{\theta_+}{\theta_-}\right)^{1+\frac{1}{p}} \mu_2(\Omega)^{1+\frac{1}{p}} \max\{C_m, C_n\} \qquad (1.10)$$

where μ_2 is the second eigenvalue of the Dirichlet p-Laplacian in Ω and

$$C_r = p\frac{\sqrt{N}}{2}\|r - \bar{r}\|_{L^\infty(\mathbb{R}^N)}\theta_+(\theta_-)^{-\frac{1}{p}-2}. \tag{1.11}$$

In the second part of the work we deal with the homogenization of the one-dimensional version of (1.1), i.e.,

$$\begin{cases} -\Delta_p u_\varepsilon = \alpha_\varepsilon m_\varepsilon(u_\varepsilon^+)^{p-1} - \beta_\varepsilon n_\varepsilon(u_\varepsilon^-)^{p-1} & \text{in } (a,b) \subset \mathbb{R} \\ u_\varepsilon(a) = u_\varepsilon(b) = 0. \end{cases} \tag{1.12}$$

In contrast with the general case, for a fixed value of $\varepsilon > 0$, Dancer and Fučík showed that the nontrivial solutions of (1.12) consist in a family of infinite hyperbolic-like curves (see [7,13]).

The existence of similar curves in the spectrum was proved later for non-constant weights by Rynne in [20], together with several properties about simplicity of zeros. The asymptotic behavior of the curves was studied in [16]. For sign-changing weights, similar results were obtained by Alif and Gossez, see [1].

The main advantage with regard to the higher dimensional case is the fact of knowing the structure of the whole spectrum and precise information about the curves. By means of shooting arguments Rynne proved that the spectrum of (1.12) can be described as an union of curves

$$\Sigma_\varepsilon(m_\varepsilon, n_\varepsilon) := \bigcup_{k\in\mathbb{N}_0} \mathcal{C}_{k,\varepsilon}.$$

Here, $\mathcal{C}_{0,\varepsilon} = \mathcal{C}_{0,\varepsilon}^+ \cup \mathcal{C}_{0,\varepsilon}^-$ are the trivial lines, which are given by $\lambda_1(m_\varepsilon) \times \mathbb{R}$, and $\mathbb{R} \times \lambda_1(n_\varepsilon)$, respectively, being $\lambda_1(r_\varepsilon)$ the first eigenvalue of the Dirichlet p-Laplacian with weight r_ε. These curve are characterized for having eigenfunctions which do not change signs. The remaining curves are made as the union $\mathcal{C}_{k,\varepsilon} = \mathcal{C}_{k,\varepsilon}^+ \cup \mathcal{C}_{k,\varepsilon}^-$, where $\mathcal{C}_{k,\varepsilon}^+$ (resp. $\mathcal{C}_{k,\varepsilon}^-$) it is composed of pairs whose corresponding eigenfunctions have k internal zeros, and positive (resp. negative) slope at $x = a$.

As $\varepsilon \to 0$, the following natural limit problem for (1.12) is obtained

$$\begin{cases} -\Delta_p u = \alpha_0 m_0(u^+)^{p-1} - \beta_0 n_0(u^-)^{p-1} & \text{in } (a,b) \\ u(a) = u(b) = 0, \end{cases} \tag{1.13}$$

where m_0 and n_0 are given in (1.8). Similarly, its corresponding spectrum is composed as the union

$$\Sigma_0(m_0, n_0) := \bigcup_{k\in\mathbb{N}_0} \mathcal{C}_{k,0},$$

with curves $\mathcal{C}_{k,0}$ satisfying analogous properties to $\mathcal{C}_{k,\varepsilon}$.

In order to describe the curves in the spectrum of Σ_ε and Σ_0 we will denote $(\alpha_{k,\varepsilon}(s), \beta_{k,\varepsilon}(s))$ and $(\alpha_{k,0}(s), \beta_{k,0}(s))$ the intersection of the curves $\mathcal{C}_{k,\varepsilon}$ and $\mathcal{C}_{k,0}$ with the line of slope s passing through the origin, respectively. In the paper [15] it is shown that such intersections consist in a divergent sequence of pairs as $k \to \infty$.

Under these considerations, in [12] it was studied the behavior of the eigencurves of (1.12) as ε approaches zero. It was proved that for each $k \in \mathbb{N}_0$, the curve $\mathcal{C}_{k,\varepsilon}$ converges to $\mathcal{C}_{k,0}$ in the sense that

$$\alpha_{k,\varepsilon}(s) \to \alpha_{k,0}(s) \quad \text{and} \quad \beta_{k,\varepsilon}(s) \to \beta_{k,0}(s) \tag{1.14}$$

as $\varepsilon \to 0$, for each fixed $s \in \mathbb{R}^+$. Moreover, in [12], Theorem 1.3, it is proved that given $\varepsilon, s > 0$, the following convergence rates hold for the curves in (1.13) for the case $p = 2$

$$|\alpha_\varepsilon(s) - \alpha(s)| \le Ck^3 \max\{t^{-\frac{3}{2}}, t^{\frac{1}{2}}\}, \quad |\beta_\varepsilon(s) - \beta(s)| \le Ck^3 \max\{t^{-\frac{1}{2}}, t^{\frac{3}{2}}\} \tag{1.15}$$

for some universal constant $C > 0$.

Again, following [6] it is possible to obtain a representation of the curves $\mathcal{C}_{k,\varepsilon}$ and $\mathcal{C}_{k,0}$ by minimizing the first eigenvalue of weighted p-Laplacian problems over all possible partition of the kind

$$\mathcal{P}_{k+1} : \{a = t_0 < t_1 < \cdots < t_{k+1} = b\},$$

see Theorem 4.1 in Sect. 4 for the precise statement. When the weights m_ε and n_ε are considered to be periodic functions, such characterization allows us to prove the following result concerning to the convergence rates of (1.14) for the periodic weight case.

Theorem 1.4. *Under the same hypothesis of Thorem 1.1, given $\varepsilon > 0$ and $s \in \mathbb{R}^+$, let $(\alpha_\varepsilon(s), \beta_\varepsilon(s)) \in \mathcal{C}_{k,\varepsilon}$ and $(\alpha_0(s), \beta_0(s)) \in \mathcal{C}_{k,0}$. Then the following estimates hold*

$$|\alpha_\varepsilon(s) - \alpha_0(s)| \le \begin{cases} C\varepsilon k^{p+1} s^{\frac{1}{p}} & s \ge 1 \\ C\varepsilon k^{p+1} s^{-1-\frac{1}{p}} & s < 1, \end{cases} \quad |\beta_\varepsilon(s) - \beta_0(s)|$$

$$\le \begin{cases} C\varepsilon k^{p+1} s^{1+\frac{1}{p}} & s \ge 1 \\ C\varepsilon k^{p+1} s^{-\frac{1}{p}} & s < 1 \end{cases}$$

where $(\alpha_0(s), \beta_0(s)) \in \mathcal{C}_{k,0}$ and

$$C = \left(\frac{\theta_+}{\theta_-}\right)^{1+\frac{1}{p}} \left(\frac{\pi_p}{b-a}\right)^{1+p} \max\{C_m, C_n\}, \tag{1.16}$$

being C_m and C_n given in (1.11).

We want to highlight that the convergence rates obtained in Theorem 1.4 are the natural generalization to (1.15) for $p \ge 2$. Moreover, here, in contradistinction to (1.15) the constant C can be explicitly computed.

Observe that when we specialize Theorem 1.4 with both weight functions being the same $1-$periodic function $r(\frac{x}{\varepsilon})$, and both parameters being the same, that is, $\lambda_\varepsilon = \alpha_\varepsilon = \beta_\varepsilon$, it follows that $s = 1$, and we recover the homogenization rates for the eigenvalue convergence of

$$-\Delta_p u_\varepsilon = \lambda_\varepsilon r_\varepsilon |u_\varepsilon|^{p-2} u_\varepsilon \quad \text{in } (a,b), \qquad u_\varepsilon(a) = u_\varepsilon(b) = 0, \tag{1.17}$$

to the limit problem

$$-\Delta_p u = \lambda_0 \bar{r} |u|^{p-2} u \quad \text{in } (a,b), \qquad u(a) = u(b) = 0 \tag{1.18}$$

as $\varepsilon \to 0$, which has been widely studied, see for instance [11,17,18]. More precisely, in particular Theorem 1.4 states that

$$|\lambda_k(r_\varepsilon) - \lambda_k(\bar{r})| \leq ck^{p+1}\varepsilon$$

where $\lambda_k(r_\varepsilon)$ is $k-$th eigenvalue of (1.17), $\lambda_k(\bar{r})$ is the $k-$th eigenvalue of (1.18), and c is a constant independent on k and ε.

The paper is organized as follows: in Sect. 2 we state some properties concerning to the first nontrivial curve in the Fučík spectrum for $N \geq 1$; in Sect. 3 we deal with the proof of Theorem 1.2; in Sect. 4 we study the one-dimensional Fučík eigencurves; in Sect. 5 we provide a proof for Theorem 1.4. Finally, in Sect. 6 we made some numerical computations.

2. The Fučík spectrum in $\mathbb{R}^N$

As we pointed in the introduction, given $\Omega \subset \mathbb{R}^N$, $N \geq 1$, and functions m and n satisfying (1.2), the structure of the spectrum $\Sigma(m,n)$ of the following asymmetric equation

$$\begin{cases} -\Delta_p u = \alpha m(u^+)^{p-1} - \beta n(u^-)^{p-1} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

is not completely understood, even in the constant weight case. Immediately one can check that $\Sigma(m,n)$ contains the lines $\lambda_1(m) \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1(n)$. Here, given a function r satisfying (1.2), $\lambda_1(r)$ denotes the first eigenvalue of

$$\begin{cases} -\Delta_p u = \lambda r|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

The first eigenvalue of (2.2) can be written variationally by minimizing the following quotient over all the non-zero functions belonging to $W_0^{1,p}(\Omega)$

$$\lambda_1(r) = \inf \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega r(x)|u|^p \, dx}. \tag{2.3}$$

When $r \equiv 1$ we just write μ_1 to denote (2.3). When it is precise to emphasize the dependence on the domain we will write $\lambda_k(r, \Omega)$ and $\mu_k(\Omega)$ to denote the $k-$th variational eigenvalue of (2.2).

In [2,3] it was shown the existence of a first variational nontrivial curve $\mathcal{C}_1(m,n)$ given by minimizing the Rayleigh quotient associated to (2.1) along a family of sign-changing paths. More precisely, the authors in [8] proved that

$$C_1(m,n) = \{(\alpha(s), \beta(s)), s \in \mathbb{R}^+\}$$

where $(\alpha(s), \beta(s))$ is the intersection between $\Sigma(m,n)$ and the line of slope s passing through the origin in $\mathbb{R}^2$. Each component is given by

$$\alpha(s) = c(m, sn), \qquad \beta(s) = s\alpha(s)$$

where

$$c(m,n) = \inf_{\gamma \in \Gamma} \max_{u \in \gamma[-1,1]} \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega (m(u^+)^p + n(u^-)^p) \, dx}$$

and $\Gamma := \{\gamma \in C([-1,1]) \colon \gamma(-1) \geq 0 \text{ and } \gamma(1) \leq 0\}$.

Later on, problem (2.1) was considered for the case $p = 2$ and constant weights in [6]. In that paper the authors, among other things, obtain an alternative representation of $\mathcal{C}_1$ in the framework of optimal partitions, see Theorem 1.2 in [6]. However, as the they notice (in Remark 2.2, [6]) the procedure leading to the proof of such result can be trivially adapted for any $p \geq 2$ and by considering weights. Therefore, the result corresponding to (2.1) can be stated as follows.

Theorem 2.1. *The first nontrivial curve $\mathcal{C}_1(m, n)$ in the spectrum of (2.1) can be written as*

$$\mathcal{C}(m, n) = \{(\alpha(s), \beta(s)), s \in \mathbb{R}^+\}, \qquad with \quad \alpha(s) = s^{-1} c(s), \quad \beta(s) = c(s)$$

where

$$c(s) := \inf_{(\omega_i) \in \mathcal{P}_2} \max\{s \lambda_1(m, \omega_+), \lambda_1(n, \omega_-)\} \tag{2.4}$$

and

$$\mathcal{P}_2 = \{(\omega_+, \omega_-) \subset \Omega : \omega_i \text{ is open and connected}, \omega_+ \cap \omega_- = \emptyset\}.$$

Moreover, for every $s > 0$ there exists $u \in W_0^{1,p}(\Omega)$ such that $(\{u^+ > 0\}, \{u^- > 0\})$ achieves $c(s)$.

In order to prove our main result we state some properties concerning the curve $\mathcal{C}_1$. First, we establish bounds for points belonging to $\mathcal{C}_1$ in terms of the parameter s and the auxiliary function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ defined as

$$\gamma(s) = \begin{cases} 1 & \text{if } s \geq 1 \\ s^{-1} & \text{if } s < 1. \end{cases} \tag{2.5}$$

Lemma 2.2. (Lemma 3.1, [21])   *Given $s \in \mathbb{R}^+$, let $(\alpha(s), \beta(s)) \in \mathcal{C}_1(m, n)$. Then*

$$\alpha(s) \leq \theta_-^{-1} \mu_2(\Omega) \gamma(s), \qquad \beta(s) \leq \theta_-^{-1} \mu_2(\Omega) s \gamma(s)$$

where γ is defined in (2.5) and μ_2 denotes the second eigenvalue of the p-Laplacian in Ω with Dirichlet boundary conditions.

In the following lemma we consider the first eigenvalue of the p-Laplacian on nodal domains of eigenfunctions corresponding to points belonging to $\mathcal{C}_1$.

Lemma 2.3. *Given $s \in \mathbb{R}^+$, let $(\alpha(s), \beta(s)) \in \mathcal{C}_1(m, n)$ and let u be a corresponding eigenfunction. If we denote $\omega_\pm = supp(u^\pm)$, then*

$$\mu_1(\omega_+) \leq C\gamma(s), \qquad \mu_1(\omega_-) \leq Cs\gamma(s)$$

where $C = \frac{\theta_+}{\theta_-} \mu_2(\Omega)$ and $\gamma(s)$ is given in (2.5).

Proof. By taking $v = u^+$ in the weak formulation of (2.1) we obtain that

$$\int_{\omega_+} |\nabla u^+|^p \, dx = \int_\Omega |\nabla u^+|^p \, dx = \alpha \int_\Omega m |u^+|^p \, dx = \alpha \int_{\omega_+} m |u^+|^p \, dx, \tag{2.6}$$

from where it follows that $\alpha = \lambda_1(m, \omega_+)$ and $u|_{\omega_+} \in W_0^{1,p}(\omega_+)$ is an eigenfunction associated to $\lambda_1(m, \omega_+)$. Since

$$\frac{1}{\theta_+} \frac{\int_{\omega_+} |\nabla v|^p}{\int_{\omega_+} |v|^p} \leq \frac{\int_{\omega_+} |\nabla v|^p}{\int_{\omega_+} m|v|^p} \leq \frac{1}{\theta_-} \frac{\int_{\omega_+} |\nabla v|^p}{\int_{\omega_+} |v|^p}$$

for all $v \in W_0^{1,p}(\omega_+)$, from (2.3) it follows that

$$\tfrac{1}{\theta_+} \mu_1(\omega_+) \leq \lambda_1(m, \omega_+) \leq \tfrac{1}{\theta_-} \mu_1(\omega_+),$$

and the desired inequality follows by using Lemma 2.2. Analogously, by using u^- as a test function in the weak formulation of (2.1) the another inequality is obtained.

3. Proof of the results for $N \geq 1$

Before proving our main result, we state an auxiliary results concerning the homogenization of eigenvalues of the weighted p-Laplacian.

Given a bounded domain $\Omega \subset \mathbb{R}^N$ and a function r satisfying (1.2), we denote $\lambda_1(r_\varepsilon)$ the first eigenvalue of

$$\begin{cases} -\Delta_p u_\varepsilon = \lambda_\varepsilon r_\varepsilon(x)|u_\varepsilon|^{p-2}u_\varepsilon & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

When an explicit emphasis on the domain is required, we denote $\lambda_1(r_\varepsilon, \Omega)$ the first eigenvalue of (3.1); additionally, when $r_\varepsilon \equiv 1$ we write $\mu_1(\Omega)$ instead of $\lambda_1(1, \Omega)$.

As we pointed out in the introduction, convergence rates in the homogenization of the Fučík spectrum are closely related with the convergence rates in the homogenization of eigenvalues of the p-Laplacian. Given a $Q-$periodic function r satisfying (1.2), Q being the unit cube in $\mathbb{R}^N$, we denote $r_\varepsilon(x) = r(\frac{x}{\varepsilon})$. As $\varepsilon \to 0$ the following limit problem for (3.1) is obtained

$$\begin{cases} -\Delta_p u_0 = \lambda_0 \bar{r}|u_0|^{p-2}u_0 & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where $\bar{r}$ is the average of r over Q. The eigenvalue $\lambda_1(r_\varepsilon)$ converges to the first eigenvalue of (3.2). Furthermore, the rate of the convergence of $\lambda_1(r_\varepsilon)$ is stated in the following result.

Theorem 3.1. (Theorem 2.2, [21]) *Given a $Q-$periodic function r satisfying (1.2), let us denote $\lambda_1(r_\varepsilon)$ and $\lambda_1(\bar{r})$ the first eigenvalue of equations (3.1) and (3.2), respectively. Then*

$$|\lambda_1(r_\varepsilon, \Omega) - \lambda_1(\bar{r}, \Omega)| \leq C_r \mu_1(\Omega)^{\frac{1}{p}+1}\varepsilon$$

with C_r given by

$$C_r = p\frac{\sqrt{N}}{2}\|r - \bar{r}\|_{L^\infty(\mathbb{R}^N)}\theta_+(\theta_-)^{-\frac{1}{p}-2}.$$

We are ready to prove our main result in this section.

Proof of Theorem 1.2. According to Theorem 2.1 the curve $\mathcal{C}_\varepsilon$ associated to (1.1) is given by

$$\mathcal{C}_\varepsilon := \{(\alpha_\varepsilon(s), \beta_\varepsilon(s)),\ s \in \mathbb{R}^+\} = \{(s^{-1}c_\varepsilon(s), c_\varepsilon(s)),\ s \in \mathbb{R}^+\}$$

where

$$c_\varepsilon(s) := \inf_{(\xi_+,\xi_-)\in\mathcal{P}_2} \max\{s\lambda_1(m_\varepsilon, \xi_+), \lambda_1(n_\varepsilon, \xi_-)\}. \tag{3.3}$$

In a similar way the limit curve $\mathcal{C}_0$ associated to (1.4) is given by

$$\mathcal{C}_0 := \{(\alpha_0(s), \beta_0(s)),\ s \in \mathbb{R}^+\} = \{(s^{-1}c_0(s), c_0(s)),\ s \in \mathbb{R}^+\}$$

where

$$c_0(s) := \inf_{(\xi_+,\xi_-)\in\mathcal{P}_2} \max\{s\lambda_1(\bar{m}, \xi_+), \lambda_1(\bar{n}, \xi_-)\}. \tag{3.4}$$

Let $(\omega_+, \omega_-) \in \mathcal{P}_2$ be a partition such that

$$c_0(s) = \max\{s\lambda_1(\bar{m}, \omega_+), \lambda_1(\bar{n}, \omega_-)\}.$$

By putting (ω_+, ω_-) in (3.3) it follows that

$$c_\varepsilon(s) \le \max\{s\lambda_1(m_\varepsilon, \omega_+), \lambda_1(n_\varepsilon, \omega_-)\}. \tag{3.5}$$

Now, Theorem 3.1 allows as to bound $\lambda_1(m_\varepsilon, \omega_+)$ and $\lambda_1(n_\varepsilon, \omega_-)$ in terms of $\lambda_1(\bar{m}, \omega_+)$ and $\lambda_1(\bar{n}, \omega_-)$, from where we bound (3.5) as

$$\max\{s\big(\lambda_1(\bar{m}, \omega_+) + C_m\mu_1(\omega_+)^{\frac{1}{p}+1}\varepsilon\big), \lambda_1(\bar{n}, \omega_-) + C_n\mu_1(\omega_-)^{\frac{1}{p}+1}\varepsilon\}$$

$$\le \max\{s\lambda_1(\bar{m}, \omega_+), \lambda_1(\bar{n}, \omega_-)\} + C_1\varepsilon \max\{s\mu_1(\omega_+)^{\frac{1}{p}+1}, \mu_1(\omega_-)^{\frac{1}{p}+1}\}$$

$$= c_0(s) + C_1\varepsilon \max\{s\mu_1(\omega_+)^{\frac{1}{p}+1}, \mu_1(\omega_-)^{\frac{1}{p}+1}\} \tag{3.6}$$

where $C_1 = \max\{C_m, C_n\}$.

In the another hand, by using Lemma 2.3 we obtain that

$$\max\{s\mu_1(\omega_+)^{\frac{1}{p}+1}, \mu_1(\omega_-)^{\frac{1}{p}+1}\}$$

$$\le \left(\frac{\theta_+}{\theta_-}\right)^{1+\frac{1}{p}} \max\{s(\mu_2(\omega_+)\gamma(s))^{\frac{1}{p}+1}, (s\mu_2(\omega_-)\gamma(s))^{\frac{1}{p}+1}\}$$

$$\le C_2 \max\{s\gamma(s)^{\frac{1}{p}+1}, (s\gamma(s))^{\frac{1}{p}+1}\}$$

$$\le C_2\gamma(s)^{1+\frac{1}{p}}s\max\{1, s^{\frac{1}{p}}\}, \tag{3.7}$$

where $C_2 = \left(\frac{\theta_+}{\theta_-}\mu_2(\Omega)\right)^{1+\frac{1}{p}}$.

Collecting (3.5)–(3.7) we obtain that

$$c_\varepsilon(s) \le c_0(s) + C\varepsilon\gamma(s)^{1+\frac{1}{p}}s\max\{1, s^{\frac{1}{p}}\}, \tag{3.8}$$

where $C = C_1C_2$. Interchanging the roles of $c_\varepsilon(s)$ and $c_0(s)$ we similarly obtain that

$$c_0(s) \le c_\varepsilon(s) + C\varepsilon\gamma(s)^{1+\frac{1}{p}}s\max\{1, s^{\frac{1}{p}}\}, \tag{3.9}$$

where C is the same constant that in (3.8).

Mixing up (3.8) and (3.9) it follows that

$$|c_\varepsilon(s) - c_0(s)| \le C\varepsilon\gamma(s)^{1+\frac{1}{p}}s\max\{1, s^{\frac{1}{p}}\}.$$

Now, from Theorem 2.1 we get

$$|\beta_\varepsilon(s) - \beta_0(s)| = |c_\varepsilon(s) - c_0(s)| \leq C\varepsilon\gamma(s)^{1+\frac{1}{p}} s \max\{1, s^{\frac{1}{p}}\},$$

$$|\alpha_\varepsilon(s) - \alpha_0(s)| = s^{-1}|c_\varepsilon(s) - c_0(s)| \leq C\varepsilon\gamma(s)^{1+\frac{1}{p}} \max\{1, s^{\frac{1}{p}}\}$$

as it was required.

4. The one-dimensional Fučík problem

In this section we state some properties related with the following one-dimensional asymmetric equation in $\Omega = (a, b)$

$$\begin{cases} -\Delta_p u = \alpha m(u^+)^{p-1} - \beta n(u^-)^{p-1} & \text{in } \Omega \\ u(a) = u(b) = 0. \end{cases} \tag{4.1}$$

As it was pointed out in the introduction, Rynne [20] showed that its spectrum is given by

$$\Sigma(m, n) := \bigcup_{k \in \mathbb{N}} \mathcal{C}_k,$$

where the curves $\mathcal{C}_k = \mathcal{C}_k^+ \cup \mathcal{C}_k^-$, $k \in \mathbb{N}_0$ are composed of pairs $(\alpha, \beta) \in \mathbb{R}^2$ whose corresponding eigenfunctions have k internal zeros and positive (resp. negative) slope at $x = a$. In particular, $\mathcal{C}_0^+ = \lambda_1(m) \times \mathbb{R}$ and $\mathcal{C}_0^- = \mathbb{R} \times \lambda_1(n)$ have eigenfunctions which do not change signs in Ω, being $\lambda_1(r)$ the first eigenvalue of

$$\begin{cases} -\Delta_p u = \lambda r|u|^{p-2}u & \text{in } \Omega \\ u(a) = u(b) = 0. \end{cases} \tag{4.2}$$

For simplicity, when $r = 1$ we denote μ_k the $k-$th eigenvalue of (4.2).

Sometimes, in order to empathize the dependence on the domain we write $\lambda_k(r, \Omega)$ and $\mu_k(\Omega)$ to denote the $k-$th eigenvalues of (4.2).

Observe that, in contrast with the higher dimensional case, eigenvalues of the Dirichlet p-Laplacian can be explicitly computed as

$$\mu_k(I) = \pi_p^p k^p |\Omega|^{-p}$$

where $\pi_p = 2(p-1)^{1/p} \int_0^1 (1 - s^p)^{-1/p}\, ds$, see [9].

Moreover, the sequence of variational eigenvalues of (4.2) can be described as

$$\lambda_k = \inf_{U \in T_k} \sup_{u \in U} \frac{\int_a^b |u'|^p\, dx}{\int_a^b r(x)|u|^p\, dx} \tag{4.3}$$

where

$$T_k = \{U \subset W_0^{1,p}(\Omega) \ : \ U \text{ is compact}, \ U = -U, \ \gamma(U) \geq k\},$$

and γ is the Krasnoselskii genus, see [4] for details.

From (4.3) it follows that

$$\theta_+^{-1}\mu_k \leq \lambda_k \leq \theta_-^{-1}\mu_k \tag{4.4}$$

for any $k \geq 1$.

The paper [6] characterizes the curves of $\Sigma(m, n)$ in terms of the first eigenvalue of weighted p-Laplacian problems (see Theorem 1.3 and Remark 2.2). The description of the curves is made as follows. A couple (α, β) belonging to $\mathcal{C}_1$ has eigenfunctions with an internal zero, i.e., it has two nodal domains. Such couple can be written as $(s^{-1}c_2(s), c_2(s))$, where

$$c_2(s) = \inf \max\{s\lambda_1(m, I_1), \lambda_2(n, I_2)\}$$

and s is the slope of the line ℓ_s passing through the origin such that $(\alpha, \beta) = \mathcal{C}_1 \cap \ell_s$. The infumum is taken over all the partitions $\mathcal{P}_2$ of Ω such that $a = t_0 < t_1 < t_2 = b$, and $I_1 = t_1 - t_0$, $I_2 = t_2 - t_1$.

Now, a couple belonging to $\mathcal{C}_2$ has associated eigenfunctions with three nodal domains. Such pair can be characterized as $(s^{-1}c_3(s), c_3(s))$, where

$$c_3(s) = \inf \max\{s\lambda_1(m, I_1), \lambda_2(n, I_2), s\lambda_1(m, I_3)\}$$

and s is the slope of the line ℓ_s passing through the origin such that $(\alpha, \beta) = \mathcal{C}_2 \cap \ell_s$. Here the infumum is taken over all the partition $\mathcal{P}_3$ of Ω such that $a = t_0 < t_1 < t_2 < t_3 = b$, with $I_1 = t_1 - t_0$, $I_2 = t_2 - t_1$ and $I_3 = t_3 - t_2$.

In order to state the general case we introduce the following notation: for $k \geq 0$ we denote

$$\mathcal{P}_{k+1} : \{a = t_0 < t_1 < \cdots < t_{k+1} = b\}$$

a partition of $\Omega = (a, b)$, and we write $I_{i+1} = t_{i+1} - t_i$ for $0 \leq i \leq k$.

Although the result in [6] was proved for the case $p = 2$ of (4.1) and with constant weights, as the authors comment, by mixing Theorem 1.3 and Remark 2.2 from [6] it is straightforward to obtain the following result concerning to the spectrum of the weighted Eq. (4.1) for any $p > 2$.

Theorem 4.1. *Given $k \geq 1$ let us define*

$$\begin{aligned}
c_{k+1}^+(s) &= \inf_{\mathcal{P}_{k+1}} \max_{0 \leq i \leq k} \{s\lambda_1(m, I_{2i+1}), \lambda_1(n, I_{2i+2})\}, \\
c_{k+1}^-(s) &= \inf_{\mathcal{P}_{k+1}} \max_{0 \leq i \leq k} \{s\lambda_1(m, I_{2i+2}), \lambda_1(n, I_{2i+1})\}
\end{aligned} \tag{4.5}$$

for all $s > 0$. Then the pair $(s^{-1}c_{k+1}^\pm(s), c_{k+1}^\pm, (s))$ belongs to a curve $\mathcal{C}_k^\pm$.

Moreover, the infima above are attained for suitable optimal partitions $P^\pm \in \mathcal{P}_{k+1}$. Furthermore, there are eigenfunctions $u^\pm \in W_0^{1,p}(\Omega)$ of (4.1) associated to $(\alpha = s^{-1}c_{k+1}^\pm(s), \beta = c_{k+1}^\pm, (s))$ whose nodal domains are given by $P^\pm$.

From the definition of $c_{k+1}(s)$ it is easy to check that $s_2 > s_1$ implies $c_{k+1}(s_2) > c_{k+1}(s_1)$. Moreover, it can be proved that $s_2^{-1}c_{k+1}(s_2) < s_1^{-1}c_{k+1}(s_1)$, from where the monotonicity of $\mathcal{C}_{k+1}$ follows:

Lemma 4.2. *(Theorem 21, [20]) The curve $\mathcal{C}_{k+1}$ is decreasing in the sense that if the points $(\alpha(s_1), \beta(s_1))$ and $(\alpha(s_2), \beta(s_2))$ belong to $\mathcal{C}_{k+1}$ then*

$$\alpha(s_1) > \alpha(s_2) \qquad and \qquad \beta(s_2) > \beta(s_1)$$

whenever $s_2 > s_1$.

The following inequality relates $c_k^{\pm}(1)$ with the $k-$th eigenvalue of the Dirichlet p-Laplacian.

Lemma 4.3. *Let $k \geq 1$ and $c_{k+1}^{\pm}(\cdot)$ given in (4.5). It holds that*

$$c_{k+1}^{\pm}(1) \leq \theta_-^{-1} \mu_{k+1}(\Omega).$$

Proof. By using (1.2) and (4.4) we have that

$$\begin{aligned}
c_{k+1}^{+}(1) &\leq \inf_{\mathcal{P}_{k+1}} \max_i \{\lambda_1(\theta_-, I_{2i+1}), \lambda_1(\theta_-, I_{2i+2})\} \\
&\leq \theta_-^{-1} \inf_{\mathcal{P}_{k+1}} \max_i \{\mu_1(I_i)\}
\end{aligned} \tag{4.6}$$

In particular, if we take an uniform partition of Ω, i.e., $|I_i| = |\Omega|/(k+1)$, it follows that $\mu_1(I_i) = \pi_p^p |I_i|^{-p} = \mu_{k+1}(\Omega)$ for each $0 \leq i \leq k$ and the result follows. $\square$

As a consequence of Lemma 4.3, we obtain upper bounds for $\alpha(s)$ and $\beta(s)$. The following result is a one-dimensional version of Lemma 2.2 for every curve in the spectrum of (4.1).

Lemma 4.4. *Let $(\alpha(s), \beta(s)) \in \mathcal{C}_k(m, n)$. For each $s > 0$ it holds that*

$$\alpha(s) \leq \theta_-^{-1} \mu_{k+1}(\Omega)\gamma(s), \qquad \beta(s) \leq \theta_-^{-1} \mu_{k+1}(\Omega)s\gamma(s)$$

with γ defined by

$$\gamma(s) = \begin{cases} 1 & if \ s \geq 1 \\ s^{-1} & if \ s \leq 1. \end{cases} \tag{4.7}$$

Proof. Let $s > 0$ and $(\alpha(s), \beta(s)) \in \mathcal{C}_k$. From Theorem 4.1 we can write $\alpha(s) = s^{-1}c_{k+1}(s)$ and $\beta(s) = c_{k+1}(s)$ (here c_k denotes any of $c_k^{\pm}$). We emphasize that $\mathcal{C}_k(m, n)$ is an decreasing curve.

When $s \geq 1$, by using Lemma 4.3 we can bound

$$\alpha(s) \leq \alpha(1) = c_{k+1}(1) \leq \theta_-^{-p} \mu_{k+1}(\Omega). \tag{4.8}$$

When $s \leq 1$ we have that $\beta(s) \leq \beta(1)$, from where $s^{-1}\beta(s) \leq s^{-1}\beta(1)$. Since $\beta(s) = s\alpha(s)$, we conclude that

$$\alpha(s) = s^{-1}\beta(s) \leq s^{-1}\alpha(1) = s^{-1}c_{k+1}(1) \leq s^{-1}\theta_-^{-p} \mu_{k+1}(\Omega). \tag{4.9}$$

By using (4.8) and (4.9) together with the relation $\beta = s\alpha$ the conclusion of the lemma follows. $\square$

Finally, the following lemma allows us to estimate eigenvalues of the p-Laplacian on nodal domains corresponding to eigenfunctions of (4.1).

Lemma 4.5. *Let $(\alpha(s), \beta(s)) \in \mathcal{C}_k(m, n)$ with associated eigenfunction u. Let I_+ (resp. I_-) be a nodal domain of u in which $u > 0$ (resp. $u < 0$). Then*

$$\mu_1(I_+) \leq C\gamma(s), \qquad \mu_1(I_-) \leq Cs\gamma(s)$$

where $C = \frac{\theta_+}{\theta_-}\mu_{k+1}(\Omega)$ and $\gamma(s)$ is given in (4.7).

Proof. By arguing in the same way than in the proof of Lemma 2.3 it is obtained that

$$\mu_1(I_+) \le \theta_+\alpha(s), \qquad \mu_1(I_-) \le \theta_+\beta(s).$$

The result now follows by applying Lemma 4.4.   □

5. Proof of the result in the one-dimensional case

Aimed at proving our main result for one-dimensional Fučík spectrum, first we introduce the following notation we will use along this section. As we have pointed out in the introduction, given the bounded interval $\Omega = (a,b) \subset \mathbb{R}$ and a $(b-a)$−periodic function r satisfying (1.2), we denote $r_\varepsilon(x) = r(\frac{x}{\varepsilon})$, and $\lambda_k(r_\varepsilon, \Omega)$ the k−th eigenvalue of

$$\begin{cases} -\Delta_p u_\varepsilon = \lambda_\varepsilon r_\varepsilon |u_\varepsilon|^{p-2} u_\varepsilon & \text{in } \Omega \\ u_\varepsilon(a) = u_\varepsilon(b) = 0. \end{cases} \tag{5.1}$$

In the case in which $r \equiv 1$ we just put $\mu_k(\Omega)$.

Observe that, since $\mathcal{C}_{k,\varepsilon} \to \mathcal{C}_{k,0}$ (see Theorem 1, [12]) in the sense that

$$\alpha_{k,\varepsilon}(s) \to \alpha_{k,0}(s) \quad \text{and} \quad \beta_{k,\varepsilon}(s) \to \beta_{k,0}(s) \tag{5.2}$$

where, for $s \in \mathbb{R}^+$,

$$(\alpha_\varepsilon(s), \beta_\varepsilon(s)) \in \mathcal{C}_{k\,\varepsilon}(m_\varepsilon, n_\varepsilon) \quad \text{and} \quad (\alpha_0(s), \beta_0(s)) \in \mathcal{C}_{k,0}(\bar{m}, \bar{n})$$

eigenfunctions corresponding to $(\alpha_0(s), \beta_0(s))$ have exactly k nodal domains on Ω.

With the previous remarks and lemmas stated in Sect. 4 we are ready to prove the rates of the convergences (5.2).

Proof of Theorem 1.4. We consider the curve $\mathcal{C}_{k,\varepsilon}^+$. An eigenfunction corresponding to a pair over this curve has positive slope at $x = a$, therefore it is positive over odd nodal domains and negative over even nodal domains. The treatment for $\mathcal{C}_{k,\varepsilon}^-$ is analogous.

Let $s > 0$. According to Theorem 4.1 a pair $(\alpha_\varepsilon(s), \beta_\varepsilon(s)) \in \mathcal{C}_{k,\varepsilon}^+$ can be written as

$$(\alpha_\varepsilon(s), \beta_\varepsilon(s)) = (s^{-1} c_{k+1,\varepsilon}(s), c_{k+1,\varepsilon}(s))$$

where

$$c_{k+1,\varepsilon}(s) = \inf_{\mathcal{P}_{k+1}} \max_i \{s\lambda_{1,\varepsilon}(m_\varepsilon, I_{2i+1}), \lambda_{1,\varepsilon}(n_\varepsilon, I_{2i+2})\} \tag{5.3}$$

and in a similar way, the limit pair $(\alpha_0(s), \beta_0(s))$ belonging to the limit curve $\mathcal{C}_{k,0}^+$ can be written as

$$(\alpha_0(s), \beta_0(s)) = (s^{-1} c_{k+1,0}(s), c_{k+1,0}(s))$$

where

$$c_{k+1,0}(s) = \inf_{\mathcal{P}_{k+1}} \max_i \{s\lambda_1(\bar{m}, I_{2i+1}), \lambda_1(\bar{n}, I_{2i+2})\}. \tag{5.4}$$

Let $P_{k+1} \in \mathcal{P}_{k+1}$ a partition where the infimum is attained in (5.4). By considering P_{k+1} in the expression (5.3) we get

$$c_{k+1,\varepsilon}(s) \leq \max_i \{ s\lambda_1(m_\varepsilon, I_{2i+1}), \lambda_1(n_\varepsilon, I_{2i+2}) \}. \tag{5.5}$$

Now, using Theorem 3.1 we can bound $\lambda_1(m_\varepsilon, I_{2i+1})$ and $\lambda_1(n_\varepsilon, I_{2i+2})$ in term of $\lambda_1(\bar{m}, I_{2i+1})$ and $\lambda_1(\bar{n}, I_{2i+2})$, from where we find an upper bound of (5.5) as:

$$\max_i \{ s \big(\lambda_1(\bar{m}, I_{2i+1}) + C_m \mu_1(I_{2i+1})^{\frac{1}{p}+1} \varepsilon \big), \lambda_1(\bar{n}, I_{2i+2}) + C_n \mu_1(I_{2i+2})^{\frac{1}{p}+1} \varepsilon \}$$

$$\leq \max \{ s\lambda_1(\bar{m}, I_{2i+1}), \lambda_1(\bar{n}, I_{2i+2}) \} + C\varepsilon \max \{ s\mu_1(I_{2i+1})^{\frac{1}{p}+1}, \mu_1(I_{2i+2})^{\frac{1}{p}+1} \}$$

$$= c_{k+1,0}(s) + C_1 \varepsilon \max \{ s\mu_1(I_{2i+1})^{\frac{1}{p}+1}, \mu_1(I_{2i+2})^{\frac{1}{p}+1} \} \tag{5.6}$$

where $C_1 = \max\{C_m, C_n\}$.

On the other hand, by using Lemma 4.5 we obtain that

$$\max \{ s\mu_1(I_{2i+1})^{\frac{1}{p}+1}, \mu_1(I_{2i+2})^{\frac{1}{p}+1} \}$$

$$\leq C_2 \max \{ s(\mu_{k+1}(I)\gamma(s))^{\frac{1}{p}+1}, (s\mu_{k+1}(I)\gamma(s))^{\frac{1}{p}+1} \} \tag{5.7}$$

$$\leq C_2 \mu_{k+1}(I)^{1+\frac{1}{p}} \gamma(s)^{1+\frac{1}{p}} s \max\{1, s^{\frac{1}{p}}\}.$$

where $C_2 = \left(\frac{\theta_+}{\theta_-} \right)^{\frac{1}{p}+1}$.

Collecting (5.5)–(5.7) we obtain that

$$c_{k+1,\varepsilon}(s) \leq c_{k+1,0}(s) + C_1 C_2 \varepsilon \mu_{k+1}(I)^{1+\frac{1}{p}} \gamma(s)^{1+\frac{1}{p}} s \max\{1, s^{\frac{1}{p}}\}. \tag{5.8}$$

Interchanging the roles of $c_{k+1,\varepsilon}(s)$ and $c_{k+1,0}(s)$ we similarly obtain that

$$c_{k+1,0}(s) \leq c_{k+1,\varepsilon}(s) + C_1 C_2 \varepsilon \mu_{k+1}(I)^{1+\frac{1}{p}} \gamma(s)^{1+\frac{1}{p}} s \max\{1, s^{\frac{1}{p}}\}. \tag{5.9}$$

Mixing up (5.8) and (5.9) it follows that

$$|c_\varepsilon(s) - c_0(s)| \leq C_1 C_2 \varepsilon \mu_{k+1}(I)^{1+\frac{1}{p}} \gamma(s)^{1+\frac{1}{p}} s \max\{1, s^{\frac{1}{p}}\}.$$

Finally, since $\mu_k(I) = k^p \pi_p^p |I|^{-p}$, we get

$$|\beta_\varepsilon(s) - \beta_0(s)| = |c_{k+1,\varepsilon}(s) - c_{k+1,0}(s)| \leq C\varepsilon(k+1)^{p+1} \gamma(s)^{1+\frac{1}{p}} s \max\{1, s^{\frac{1}{p}}\},$$

$$|\alpha_\varepsilon(s) - \alpha_0(s)| = s^{-1}|c_{k+1,\varepsilon}(s) - c_{k+1,0}(s)| \leq C\varepsilon(k+1)^{p+1} \gamma(s)^{1+\frac{1}{p}} \max\{1, s^{\frac{1}{p}}\}$$

where $C = C_1 C_2 \left(\frac{\pi_p}{b-a} \right)^{1+p}$, and the result is proved. $\square$

6. Some numerical experiments

In this section we exhibit some numerical computation made on problem (1.1). Let us consider (1.1) in $\mathbb{R}$, with $p = 2$, $m(x) = n(x) = \sin(\pi x) + 2$ and $\Omega = (0, 1)$. As $\varepsilon \to 0$, according to (1.3), we have that $m_0 = n_0 = 2$. Observe that the value of C given in (1.16) is independent on k and ε, in this case is given by $C = 30.2088$.

By using a numerical method based on the Prüfer transform (see, for instance [5] for further information) it is possible to compute the Fučík curves

TABLE 1. Comparison for diferent values of ε when $k = 1$ and $s = 1$ (left), $k = 3$ and $s = 1$ (right)

ε	$|\alpha_\varepsilon(1) - \alpha_0(1)|$	$C\varepsilon$	$|\alpha_\varepsilon(1) - \alpha_0(1)|$	$C\varepsilon k^3$
0.01	0.000068811	0.302088	0.00504723	8.15636
0.005	0.000018349	0.1510440	0.00123687	4.07818
0.001	0.000013698	0.0302088	0.00017597	0.815636
0.0005	0.000037889	0.0151044	0.000200674	0.407818

TABLE 2. Comparison for diferent values of ε when $k = 2$ and $s = 0.7$ (left), $k = 2$ and $s = 1.7$ (right)

ε	$|\beta_\varepsilon(0.7) - \beta_0(0.7)|$	$C\varepsilon k^3 s^{-1/2}$	$|\beta_\varepsilon(1.7) - \beta_0(1.7)|$	$C\varepsilon k^3 s^{3/2}$
0.01	0.00126839	2.88851	0.0273438	5.35669
0.005	0.00030517	1.44425	0.0219727	2.67834
0.001	0.00015258	0.28885	0.0195313	0.53566
0.0005	0.00004577	0.14442	0.0164270	0.26783

and compare the difference $|\alpha_\varepsilon(s) - \alpha_0(s)|$ with the theoretical bound given in Theorem 1.4. Here, given fixed values of ε, k and s, we consider the couples $(\alpha_\varepsilon(s), \beta_\varepsilon(s)) \in \mathcal{C}_{k,\varepsilon}$ and $(\alpha_0(s), \beta_0(s)) \in \mathcal{C}_{k,0}$.

In Table 1 we consider the intersection of the first nontrivial curve with the line of slope 1 passing through the origin, and the same intersection for the third curve.

Table 2 contains computations made for the intersection of the lines of slopes 0.7 and 1.7 with the Fučík spectrum. In this case, a second nontrivial curve is considered.

Acknowledgements

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