EIGENVALUE HOMOGENIZATION FOR QUASILINEAR ELLIPTIC EQUATIONS WITH DIFFERENT BOUNDARY CONDITIONS

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Abstract. We study the rate of convergence for (variational) eigenvalues of several non-linear problems involving oscillating weights and subject to different kinds of boundary conditions in bounded domains.

1. Introduction

In this work we study the asymptotic behavior as $\varepsilon \to 0$ of the (variational) eigenvalues of

$$-div(a(x,\nabla u^\varepsilon)) + V(\tfrac{x}{\varepsilon})|u^\varepsilon|^{p-2}u^\varepsilon = \lambda^\varepsilon \rho(\tfrac{x}{\varepsilon})|u^\varepsilon|^{p-2}u^\varepsilon \quad \text{in } \Omega \tag{1.1}$$

with different boundary conditions (Dirichlet, Neumann, etc.), where $\Omega \subset \mathbb{R}^N$ is a bounded domain, ε is a positive real number, and λ^ε is the eigenvalue parameter. We also consider eigenvalue dependent boundary conditions,

$$\begin{cases} -div(a(x,\nabla u^\varepsilon)) + V(\frac{x}{\varepsilon})|u^\varepsilon|^{p-2}u^\varepsilon = \lambda^\varepsilon \rho(\frac{x}{\varepsilon})|u^\varepsilon|^{p-2}u^\varepsilon & \text{in } \Omega, \\ a(x,\nabla u^\varepsilon)\nu = \lambda^\varepsilon |u^\varepsilon|^{p-2}u^\varepsilon & \text{in } \partial\Omega. \end{cases}$$

and the Steklov problem

$$\begin{cases} -div(a(x,\nabla u^\varepsilon)) + V(\frac{x}{\varepsilon})|u^\varepsilon|^{p-2}u^\varepsilon = 0 & \text{in } \Omega, \\ a(x,\nabla u^\varepsilon)\nu = \lambda^\varepsilon |u^\varepsilon|^{p-2}u^\varepsilon & \text{in } \partial\Omega. \end{cases}$$

The weight function $\rho(x)$ is assumed to be bounded away from zero and infinity, the potential function $V(x)$ is bounded and the operator $a(x,\xi)$ has precise hypotheses that are stated below, but the prototypical example is

$$\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(\left| \sum_{k,m=1}^N a_{km}(x)\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_m}\right|^{\frac{p-2}{2}} a_{ij}(x)\frac{\partial u}{\partial x_j}\right), \tag{1.2}$$

where $A = (a_{i,j})_{i,j=1}^N$ is a $N \times N$ symmetric matrix with measurable coefficients which satisfies a uniform ellipticity condition

$$\alpha'|\xi|^2 \le A(x)\xi\cdot\xi, \quad |A(x)\xi| \le \beta'|\xi| \quad \forall \xi \in \mathbb{R}^N, \text{ a.e. } x \in \Omega.$$

for some positive constants α' and β', and $1 < p < \infty$.

2010 *Mathematics Subject Classification.* 35B27, 35P15, 35P30.

Key words and phrases. Eigenvalue homogenization, nonlinear eigenvalues, order of convergence.

The problem of finding the asymptotic behavior of the eigenvalues of (1.1) has relevance in different fields of applications, and it is an important part of *Homogenization Theory*.

We will consider the important case of periodic homogenization, i.e., the functions $\rho(x)$ and $V(x)$ are assume to be $Q-$periodic functions, Q being the unit cube in $\mathbb{R}^N$.

The natural limit problem of (1.1) as $\varepsilon \to 0$ is given by

$$-div(a(x, \nabla u)) + \bar{V}|u|^{p-2}u = \lambda\bar{\rho}|u|^{p-2}u \quad \text{in } \Omega \tag{1.3}$$

with the corresponding boundary condition, where $\bar{\rho}$ and $\bar{V}$ are the averages of ρ and V in Q, respectively.

In this work, we focus our attention on the order of convergence of the eigenvalues, that is, an estimate of $|\lambda_k^\varepsilon - \lambda_k|$ in terms of k and ε, where λ_k^ε and λ_k are the k–th variational eigenvalues of problems (1.1) and (1.3) respectively.

The homogenization problem for eigenvalues has deserved a great deal of attention in the past, specially in the linear case, that is, problem (1.1) with $a(x, \xi)$ given by (1.2) and $p = 2$.

In particular, the first result on the order of convergence for the linear problem complemented with homogeneous Dirichlet boundary conditions can be found in [17] where it is proved that

$$|\lambda_k^\varepsilon - \lambda_k| \le Ck^{\frac{6}{N}}\varepsilon^{\frac{1}{2}}.$$

where C is a positive constant independent of k and ε

Later on, again in the linear case and with homogeneous Dirichlet boundary conditions, Santonsa and Vogelius [21] proved, by using asymptotic expansions, that

$$|\lambda_k^\varepsilon - \lambda_k| \le C\varepsilon$$

where C depends on k.

We want to remark that in the above mentioned works, the authors allowed for an ε dependance on the diffusion matrix in (1.2).

More recently, the linear problem in dimension $N = 1$, and for the operator $a(x, \xi) = \xi$, was studied by Castro and Zuazua in [5, 6]. In those articles the authors, using the so-called WKB method which relays on asymptotic expansions of the solutions of the problem, and the explicit knowledge of the eigenfunctions and eigenvalues of the constant coefficient limit problem, proved the bound

$$|\lambda_k^\varepsilon - \lambda_k| \le Ck^3\varepsilon.$$

Let us mention that their method needs higher regularity on the weight ρ, which must belong at least to C^2.

Recently, Kenig, Lin and Shen [14] studied the linear problem in any dimension (allowing an ε dependance in the diffusion matrix of the elliptic operator) and proved that for Lipschitz domains Ω one has

$$|\lambda_k^\varepsilon - \lambda_k| \le C\varepsilon|\log\varepsilon|^{\frac{1}{2}+\sigma}$$

for any $\sigma > 0$, C depending on k and σ.

Also, the authors show that if the domain Ω is more regular ($C^{1,1}$ is enough) they can get rid of the logarithmic term in the above estimate. However, no explicit dependance of C on k is obtained in that work.

In the non-linear case without dependence on ε in $a(x,\xi)$, and $N \geq 1$, we proved in [8] (by means of a precise order of convergence for oscillating integrals), that

$$|\lambda_k^\varepsilon - \lambda_k| \leq C k^{\frac{p+1}{N}} \varepsilon$$

with C independent of k and ε.

Moreover, in [8], for the one dimensional problem, we show that the constant entering in the above estimate can be found explicitly and, moreover, a dependence on ε on the operator $a(x,\xi)$ was treated.

Let us stress the fact that all the above mentioned works deal with the homogenous Dirichlet boundary condition case, with the exception of the aforementioned paper [14] where also it is considered Neumann and Steklov boundary conditions and similar results as in the Dirichlet boundary condition case were found.

Some results are known in the linear case when different boundary conditions are considered. In the one-dimensional case with ε dependence in the operator $a(x,\xi)$ Moskov and Vogelius [16] by using the Osborn's eigenvalues estimate proved that

$$|\lambda_k^\varepsilon - \lambda_k| \leq C\varepsilon$$

where C depends on k.

Recently, in [20] by similar methods to those in [8] the rate of convergence of the first non-trivial curve in a weighted Fucik problem with Neumann boundary conditions was found. As corollary it follows that

$$|\lambda_1^\varepsilon - \lambda_1| \leq C\varepsilon, \qquad |\lambda_2^\varepsilon - \lambda_2| \leq C\varepsilon$$

where λ_1^ε and λ_2^ε are the first and second eigenvalues of the nonlinear equation (1.1) with Neumann boundary conditions and C does not depends on ε.

There is a large class of usually studied eigenvalue problems related to problem (1.1) with different boundary conditions. In this paper we analyze the most common ones.

Let us consider ρ, V two $Q-$periodic functions, being Q the unit cube in $\mathbb{R}^N$, which satisfy

$$0 < \rho^- \leq \rho(x) \leq \rho^+ < +\infty \quad \text{a.e. } \Omega \quad \text{and} \quad V \in L^\infty(\Omega). \tag{1.4}$$

for certain constants $\rho^- < \rho^+$. We denote $\rho_\varepsilon(x) := \rho(\frac{x}{\varepsilon})$ and $V_\varepsilon(x) := V(\frac{x}{\varepsilon})$.

For each $\varepsilon > 0$ fixed, we define the following eigenvalue problems:

- Dirichlet problem:

$$D_\varepsilon(\Omega) : \begin{cases} -div(a(x,\nabla u^\varepsilon)) + V_\varepsilon |u^\varepsilon|^{p-2}u^\varepsilon = \lambda^\varepsilon \rho_\varepsilon |u^\varepsilon|^{p-2}u^\varepsilon & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{in } \partial\Omega. \end{cases} \tag{1.5}$$

- Neumann problem

$$N_\varepsilon(\Omega) : \begin{cases} -div(a(x,\nabla u^\varepsilon)) + V_\varepsilon |u^\varepsilon|^{p-2}u^\varepsilon = \lambda^\varepsilon \rho_\varepsilon |u^\varepsilon|^{p-2}u^\varepsilon & \text{in } \Omega, \\ a(x,\nabla u^\varepsilon)\nu = 0 & \text{in } \partial\Omega. \end{cases} \tag{1.6}$$

- Robin problem

$$R_\varepsilon(\Omega) : \begin{cases} -div(a(x,\nabla u^\varepsilon)) + V_\varepsilon |u^\varepsilon|^{p-2}u^\varepsilon = \lambda^\varepsilon \rho_\varepsilon |u^\varepsilon|^{p-2}u^\varepsilon & \text{in } \Omega, \\ a(x,\nabla u^\varepsilon)\nu + \beta |u^\varepsilon|^{p-2}u^\varepsilon = 0 & \text{in } \partial\Omega. \end{cases} \tag{1.7}$$

- Non-flux problem:

$$(1.8)\qquad P_\varepsilon(\Omega):\begin{cases}-div(a(x,\nabla u^\varepsilon))+V_\varepsilon|u^\varepsilon|^{p-2}u^\varepsilon=\lambda^\varepsilon\rho_\varepsilon|u^\varepsilon|^{p-2}u^\varepsilon & \text{in } \Omega,\\ u^\varepsilon=\text{constant} & \text{in } \partial\Omega,\\ \int_{\partial\Omega}a(x,\nabla u^\varepsilon)\nu\,dS=0. & \end{cases}$$

- Eigenvalue dependent boundary condition

$$(1.9)\qquad B_\varepsilon(\Omega):\begin{cases}-div(a(x,\nabla u^\varepsilon))+V_\varepsilon|u^\varepsilon|^{p-2}u^\varepsilon=\lambda^\varepsilon\rho_\varepsilon|u^\varepsilon|^{p-2}u^\varepsilon & \text{in } \Omega,\\ a(x,\nabla u^\varepsilon)\nu=\lambda^\varepsilon|u^\varepsilon|^{p-2}u^\varepsilon & \text{in } \partial\Omega.\end{cases}$$

- Steklov problem

$$(1.10)\qquad S_\varepsilon(\Omega):\begin{cases}-div(a(x,\nabla u^\varepsilon))+V_\varepsilon|u^\varepsilon|^{p-2}u^\varepsilon=0 & \text{in } \Omega,\\ a(x,\nabla u^\varepsilon)\nu=\lambda^\varepsilon|u^\varepsilon|^{p-2}u^\varepsilon & \text{in } \partial\Omega.\end{cases}$$

We consider Ω to be a bounded domain in $\mathbb{R}^N$, the smoothness will be precised in each case. The operator $a(x,\xi)$ satisfy properties (H0)–(H8) given in §2. Observe that in problems (1.5)–(1.10) the operator $a(x,\xi)$ does not depends on ε, the dependence on the parameter appears only in the weights and potential functions. By ν we denote the outer unit normal vector with respect to $\partial\Omega$. The parameter β in the Robin problem is in $[0,\infty)$. We observe that when $\beta=0$ it corresponds to the Neumann problem and when $\beta=\infty$ it corresponds with the Dirichlet problem. In the boundary condition problem (1.9) and in the Steklov problem (1.10) we require the potential function to be strictly positive, i.e., there exists $V^->0$ such that $V^-\le V(x)$ a.e. in Ω. Observe that this requirement is not necessary in (1.5)–(1.8) since the hypothesis of V being bounded below away from zero can be assumed without loss of generality.

In (1.5)–(1.10) the natural limit problems (as $\varepsilon\to 0$) are the analogous ones with the weights ρ_ε and the potentials V_ε replaced by their averages in the unit cube Q, i.e.

$$\bar\rho=\int_Q\rho(y)\,dy,\qquad \bar V=\int_Q V(y)\,dy.$$

It is not difficult to see, for any of the problems (1.5)–(1.10), that if λ^ε is a convergent sequence of eigenvalues as $\varepsilon\to 0$ then $\lambda=\lim_{\varepsilon\to 0}\lambda^\varepsilon$ is an eigenvalue of the corresponding limit problem and, up to some subsequence, the associated eigenfunctions u^ε converge weakly to an associated eigenfunction u of the corresponding limit problem.

For the Dirichlet problem (1.5) this fact was proved in [2] (see also [8] for a simplified proof of this result). The proofs for the others problems (1.6)–(1.10) are analogous.

Our aim is to study the order of convergence of the eigenvalues of problems (1.5)–(1.10) to those of the limit equations.

Using results concerning to oscillating integrals, we prove our main results:

Theorem 1.1. *Let λ_k^ε be the k–th variational eigenvalue associated to any of the problems* (1.5)–(1.9)*, respectively. Let λ_k be the k–th variational eigenvalue associated to the correspondent limit problem. Then there exists a constant $C>0$ independent of the parameters ε and k such that*

$$|\lambda_k^\varepsilon-\lambda_k|\le Ck^{\frac{p+1}{N}}\varepsilon.$$

Theorem 1.2. *Let λ_k^ε be the k–th variational eigenvalue associated to equation* (1.10)*. Let λ_k be the k–th variational eigenvalue associated to the correspondent limit problem. Then there exists a constant $C > 0$ independent of the parameters ε and k such that*

$$|\lambda_k^\varepsilon - \lambda_k| \leq Ck^{\frac{p-1}{N-1}}\varepsilon.$$

Remark 1.3. Let us note that, for problem (1.9), the bound can be improved when $p < N$, and we get

$$|\lambda_k^\varepsilon - \lambda_k| \leq Ck^{\frac{2(p-1)}{N-1}}\varepsilon.$$

The rest of the paper is organized as follows. In Section §2 we introduce the class of operators considered and the hypotheses on the functions $a(x,\xi)$, ρ, V, and the notation which will be used. In Section §3 we analyze the eigenvalue problems and we show the relationships between them. Section §4 is devoted to oscillatory integrals, and in Section §5 we prove the main results.

2. Preliminary results

2.1. **Monotone operators.** We start this section by making the precise assumptions on the function $a(x,\xi)$

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$ be a bounded domain. We consider $a\colon \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ that satisfies the following conditions:

- (H0) *measurability:* $a(\cdot,\cdot)$ is a Carathéodory function, i.e. $a(x,\cdot)$ is continuous a.e. $x \in \Omega$, and $a(\cdot,\xi)$ is measurable for every $\xi \in \mathbb{R}^N$.
- (H1) *monotonicity:* $0 \leq (a(x,\xi_1) - a(x,\xi_2))(\xi_1 - \xi_2)$.
- (H2) *coercivity:* $\alpha|\xi|^p \leq a(x,\xi)\xi$.
- (H3) *continuity:* $a(x,\xi) \leq \beta|\xi|^{p-1}$.
- (H4) *$p-$homogeneity:* $a(x,t\xi) = t^{p-1}a(x,\xi)$ for every $t > 0$.
- (H5) *oddness:* $a(x,-\xi) = -a(x,\xi)$.

Let us introduce $\Psi(x,\xi_1,\xi_2) = a(x,\xi_1)\xi_1 + a(x,\xi_2)\xi_2$ for all $\xi_1,\xi_2 \in \mathbb{R}^N$, and all $x \in \Omega$; and let $\delta = min\{p/2,(p-1)\}$.

- (H6) *equi-continuity:*

$$|a(x,\xi_1) - a(x,\xi_2)| \leq c\Psi(x,\xi_1,\xi_2)^{(p-1-\delta)/p}(a(x,\xi_1) - a(x,\xi_2))(\xi_1 - \xi_2)^{\delta/p}$$

- (H7) *cyclical monotonicity:* $\sum_{i=1}^k a(x,\xi_i)(\xi_{i+1} - \xi_i) \leq 0$, for all $k \geq 1$, and $\xi_1,\ldots,\xi_{k+1}$, with $\xi_1 = \xi_{k+1}$.
- (H8) *strict monotonicity:* let $\gamma = \max(2,p)$, then

$$\alpha|\xi_1 - \xi_2|^\gamma \Psi(x,\xi_1,\xi_2)^{1-(\gamma/p)} \leq (a(x,\xi_1) - a(x,\xi_2))(\xi_1 - \xi_2).$$

Hypotheses (H1)–(H3) are necessary to ensure the $G-$convergence of the operators associated to $a(x,\xi)$. On the other hand, hypotheses (H4)–(H7) are all important in the context of a well-posed eigenvalue problem. We assume (H8) for technical reasons.

We add that the conditions (H0)–(H8) are not completely independent of each other. It can be seen easily that (H8) implies (H1)–(H2) and that (H4) implies (H3) in addition to the continuity of the coefficient, for details see [2].

In particular, under these conditions, we have the following Proposition due to Baffico, Conca and Rajesh [2]

Proposition 2.1. *Given $a(x,\xi)$ satisfying conditions* (H0)–(H8)*, there exists a unique Carathéodory function Φ which is even, $p-$homogeneous strictly convex and differentiable in the variable ξ satisfying*

$$\alpha|\xi|^p \leq \Phi(x,\xi) \leq \beta|\xi|^p \tag{2.1}$$

for all $\xi \in \mathbb{R}^N$ a.e. $x \in \Omega$ such that

$$\nabla_\xi \Phi(x,\xi) = pa(x,\xi)$$

and normalized such that $\Phi(x,0) = 0$.

Proof. See Lemma 3.3 in [2]. □

2.2. **General hypotheses and notation.** Throughout the paper the following hypotheses and notation will be used:

- By Q we always mean the unit cube in $\mathbb{R}^N$, i.e. $Q = [0,1]^N$.
- The functions g, ρ, V will always refer to functions in $L^\infty(\mathbb{R}^N)$ that are $Q-$periodic.
- For $\varepsilon > 0$ we denote
$$g_\varepsilon(x) = g(\tfrac{x}{\varepsilon}), \qquad \rho_\varepsilon(x) = \rho(\tfrac{x}{\varepsilon}), \qquad V_\varepsilon(x) = V(\tfrac{x}{\varepsilon}).$$
- The average on Q will be denoted by
$$\bar{g} = \int_Q g(y)\,dy, \qquad \bar{\rho} = \int_Q \rho(y)\,dy, \qquad \bar{V} = \int_Q V(y)\,dy.$$
- Observe that $g_\varepsilon \rightharpoonup \bar{g}$, $\rho_\varepsilon \rightharpoonup \bar{\rho}$ and $V_\varepsilon \rightharpoonup \bar{V}$ (as $\varepsilon \to 0$) weakly * in L^∞.
- On the weight function ρ we assume that it is bounded away from zero. That is, there exist constants $0 < \rho^- < \rho^+ < \infty$ such that
$$\rho^- \leq \rho(x) \leq \rho^+.$$
- On the potential function V, for the problems (1.9) and (1.10), we assume that the potential function is strictly positive, i.e., there exists $V^- > 0$ such that
$$V^- \leq V(x) \text{ a.e. in } \Omega.$$
- Lipschitz regularity of the domain Ω is assumed in problems (1.5) and (1.8), and C^1 regularity in all other cases.

3. Eigenvalues

In this section we define the variational eigenvalues of problems (1.5)–(1.10) and collect some properties and relations between them. We focus only on the properties of the eigenvalues that will be used later, a more detailed study can be found in the following references: the Dirichlet problem (1.5) was studied by García–Azorero and Peral Alonso in [10]; the Neumann problem can be found in the work of Huang [12]; the Robin problem (1.7) can be found in [15], together with the Non-flux problem (1.8); both generalize periodic and separated boundary conditions in classical Sturm Liouville problems, studied by several authors, see the paper of Binding and Rynne [4] among others; the problem with Eigenvalue dependent boundary conditions (1.9) was studied by Binding, Browne and Watson [3] in the one dimensional case, and the Steklov problem (1.10) was considered by Fernández Bonder and Rossi in [9]. Finally, more general monotone operators than the $p-$Laplacian, like the ones we will consider here, were studied by Kawohl, Lucia and Prashanth, see [13].

First of all, we observe that by replacing λ^ε by $\lambda^\varepsilon + \|V\|_\infty + V_-$ in (1.5)–(1.8) we can assume that the potential function verifies that

$$V(x) \geq V_- > 0$$

In problems (1.9)–(1.10) this has to be imposed on V.

By means of the Ljusternik-Schnirelmann theory (see [22] for instance) we know that the variational spectrum of these problems consists in countable sequences of positive eigenvalues tending to $+\infty$. Define the followings functionals

$$\begin{aligned} F(u,\rho) &= \int_\Omega \rho |u|^p, \\ G(u) &= \int_\Omega \Phi(x, \nabla u), \\ H(u) &= \int_{\partial\Omega} |u|^p, \end{aligned}$$

where $\Phi(x,\xi)$ is the potential function given in Proposition 2.1. Then, for each fixed $\varepsilon > 0$ we can give the characterization of the k–th variational eigenvalues of (1.5)–(1.10) as follow:

$$\begin{aligned} \lambda_k^D &= \inf_{C\in\tilde{\Gamma}_k} \sup_{u\in C} \frac{G(u)+F(u,V)}{F(u,\rho)}, \\ \lambda_k^N &= \inf_{C\in\Gamma_k} \sup_{u\in C} \frac{G(u)+F(u,V)}{F(u,\rho)}, \\ \lambda_k^R &= \inf_{C\in\Gamma_k} \sup_{u\in C} \frac{\beta H(u)+G(u)+F(u,V)}{F(u,\rho)}, \\ \lambda_k^P &= \inf_{C\in\bar{\Gamma}_k} \sup_{u\in C} \frac{G(u)+F(u,V)}{F(u,\rho)}, \\ \lambda_k^B &= \inf_{C\in\Gamma_k} \sup_{u\in C} \frac{G(u)+F(u,V)}{H(u)+F(u,\rho)}, \\ \lambda_k^S &= \inf_{C\in\Gamma_k} \sup_{u\in C} \frac{G(u)+F(u,V)}{H(u)}. \end{aligned} \tag{3.1}$$

Here,

$$\begin{aligned} \Gamma_k &= \{C \subset W^{1,p}(\Omega) : C \text{ compact}, \ C = -C, \ \gamma(C) \geq k\}, \\ \tilde{\Gamma}_k &= \{C \subset W_0^{1,p}(\Omega) : C \text{ compact}, \ C = -C, \ \gamma(C) \geq k\}, \\ \bar{\Gamma}_k &= \{C \subset W_0^{1,p}(\Omega) \oplus \mathbb{R} : C \text{ compact}, \ C = -C, \ \gamma(C) \geq k\} \end{aligned}$$

and $\gamma(C)$ is the Kranoselskii genus (see for instance [19, 7] for definition and properties).

From the variational characterization, we immediately obtain the following inequalities

$$\lambda_k^B \leq \lambda_k^N \leq \min\{\lambda_k^P, \lambda_k^R\} \leq \max\{\lambda_k^P, \lambda_k^R\} \leq \lambda_k^D \tag{3.2}$$

$$\lambda_k^B \leq \lambda_k^S \tag{3.3}$$

In the followings Lemmas we give upper bounds for the eigenvalues λ_k defined in (3.1) in terms of k and Ω. Here and in all the paper we will consider that $\Omega \subset \mathbb{R}^N$ is a bounded domain. These estimates will be useful to prove the main results since it provide us with the growth rate of the eigenvalues.

Lemma 3.1. *Let λ_k^N, λ_k^P, λ_k^D, λ_k^B and λ_k^R be the k–th variational eigenvalues defined in* (3.1). *Then*

$$\lambda_k^B \le \lambda_k^N \le \min\{\lambda_k^P, \lambda_k^R\} \le \max\{\lambda_k^P, \lambda_k^R\} \le \lambda_k^D \le Ck^{p/N}$$

where C depends only on Ω and the bounds (1.4), (2.1).

Proof. From (3.2), it is enough to prove the last inequality.

Now, from (2.1) we have

$$\frac{G(u)+F(u,V)}{F(u,\rho)} \le \frac{\max\{\beta, V^+\}}{\rho^-} \frac{\int_\Omega |\nabla u|^p + |u|^p}{\int_\Omega |u|^p},$$

from where it follows that

$$\lambda_k^D \le \frac{\max\{\beta, V^+\}}{\rho^-}\mu_k,$$

where μ_k is the $k-$th eigenvalue of

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \mu|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Observe that $u \in W_0^{1,p}(\Omega)$ is solution of (3.4) if and only if u is solution of

$$\begin{cases} -\Delta_p u = \tilde{\mu}|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\tilde{\mu} = \mu - 1$, which satisfies that (see [11])

$$\tilde{\mu}_k \le Ck^{p/N}, \tag{3.5}$$

and the result follows. □

Lemma 3.2. *Let λ_k^B and λ_k^S be the k–th variational eigenvalue of $B(\Omega)$ and $S(\Omega)$ respectively. Then*

$$\lambda_k^B \le \lambda_k^S \le Ck^{\frac{p-1}{N-1}},$$

where C is a constant depending on V^+, α and Ω.

Proof. From (2.1) and (1.4) we have

$$\frac{G(u)+F(u,V)}{H(u)} \le \max\{\tfrac{1}{\alpha}, V^+\} \frac{\int_\Omega |\nabla u|^p + \int_\Omega |u|^p}{\int_{\partial\Omega} |u|^p}$$

from where it follows that

$$\lambda_k^S \le \max\{\tfrac{1}{\alpha}, V^+\}\mu_k, \tag{3.6}$$

where μ_k is the $k-$th eigenvalue of

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \eta} = \mu|u|^{p-2}u & \text{in } \partial\Omega. \end{cases}$$

Ii is proved in [18] the following estimate for μ_k

$$\mu_k \le c k^{\frac{p-1}{N-1}}. \tag{3.7}$$

where c is a positive constant independent of k.

Finally, from (3.3), (3.6) and (3.7) the result follows. $\square$

Remark 3.3. From the previous lemmata, we have that

$$\lambda_k^B \le \min\{Ck^{\frac{p}{N}}, Ck^{\frac{p-1}{N-1}}\},$$

equivalently,

$$\lambda_k^B \le \begin{cases} Ck^{\frac{p}{N}} & p \ge N, \\ Ck^{\frac{p-1}{N-1}} & p \le N. \end{cases}$$

4. Preliminaries on oscillatory integrals.

In order to deal with the rate of convergence of the eigenvalues, the main tool that we use is the study of oscillating integrals. These will allow us to replace an integral involving a rapidly oscillating function with one that involves its average in the unit cube.

Let ρ be a Q-periodic weight. It is well-known that $\rho(\frac{x}{\varepsilon})$ converges weakly* in L^∞ to its average over Q. We are interested in the rate of the convergence in terms of ε. In [8] it is proved that

Theorem 4.1 ([8], Theorem 3.4)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let $g \in L^\infty(\mathbb{R}^N)$ be a $Q-$periodic function, Q being the unit cube in $\mathbb{R}^N$. Then, for every $u \in W_0^{1,p}(\Omega)$*

$$\Big|\int_\Omega (g(\tfrac{x}{\varepsilon}) - \bar g)|u|^p\Big| \le pc_1\|g-\bar g\|_{L^\infty(Q)}\varepsilon\|u\|_{L^p(\Omega)}^{p-1}\|\nabla u\|_{L^p(\Omega)},$$

where $\bar g$ is the average of g over Q and c_1 is the optimal constant in Poincaré's inequality in $L^1(Q)$.

Remark 4.2. In [1], the authors show the estimate

$$c_1 \le \frac{\sqrt{N}}{2}.$$

With the aid of Theorem 4.1, in [8] we were able to analyze the Dirichlet boundary condition case. To deal with different boundary conditions, we need a similar Theorem that allows us to include the function space $W^{1,p}(\Omega)$.

The fact of enlarge the set of test functions is reflected in the need for more regularity on the domain Ω. In [20] the following result is proved.

Theorem 4.3 ([20], Theorem 4.3)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary and let $g \in L^\infty(\mathbb{R}^N)$ be a $Q-$periodic function, Q being the unit cube in $\mathbb{R}^N$. Then for every $u \in W^{1,p}(\Omega)$ there exists a constant C depending only on p, Ω and $\|g\|_{L^\infty(\mathbb{R}^N)}$ such that*

$$\Big|\int_\Omega (g(\tfrac{x}{\varepsilon}) - \bar g)u\Big| \le C\varepsilon\|u\|_{W^{1,p}(\Omega)}.$$

Now, we need a couple of simple technical lemmas which are used in the proof of our main results.

Lemma 4.4. *Let $u \in W^{1,p}(\Omega)$. Then*

$$\||u|^p\|_{W^{1,1}(\Omega)} \leq p\|u\|^p_{W^{1,p}(\Omega)} \leq C(G(u) + F(u,\bar\rho))$$

where ρ is an arbitrary weight satisfying (1.4) and C is a constant depending on p, ρ and the bounds of (2.1).

Proof. By using Young's inequality

$$\begin{aligned} \||u|^p\|_{W^{1,1}(\Omega)} &= \||u|^p\|_{L^1(\Omega)} + p\||u|^{p-1}\nabla u\|_{L^1(\Omega)} \\ &\leq p\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega)} \\ &\leq p\|u\|^p_{W^{1,p}(\Omega)}. \end{aligned} \tag{4.1}$$

Moreover, by (2.1)

$$\begin{aligned} \|u\|^p_{W^{1,p}(\Omega)} &\leq \frac{1}{\bar\rho}\left(\bar\rho\int_\Omega |u|^p + \frac{\bar\rho}{\alpha}\int_\Omega \Phi(x,\nabla u)\right) \\ &\leq \frac{1}{\bar\rho}\max\{\tfrac{\bar\rho}{\alpha}, 1\}(G(u) + F(u,\bar\rho)). \end{aligned} \tag{4.2}$$

From (4.1) and (4.2), the Lemma follows. □

Lemma 4.5. *Let $u \in W^{1,p}(\Omega)$ and $\rho, V \in L^\infty(\mathbb{R}^N)$ be two $Q-$periodic functions satisfying* (1.4)*. Then there exists a constant c depending only on p, $\|\rho\|_{L^\infty(\mathbb{R}^N)}$, $\|V\|_{L^\infty(\mathbb{R}^N)}$, the constants in* (1.4) *and* (2.1) *such that*

$$\frac{F(u,\bar\rho)}{F(u,\rho_\varepsilon)} \leq 1 + c\varepsilon\frac{F(u,\bar V) + G(u)}{F(u,\bar\rho)}$$

and

$$\frac{F(u,\rho_\varepsilon)}{F(u,\bar\rho)} \leq 1 + c\varepsilon\frac{F(u,V_\varepsilon) + G(u)}{F(u,\rho_\varepsilon)}.$$

Proof. Applying Theorem 4.3 we obtain that

$$\frac{\bar\rho\int_\Omega |u|^p}{\int_\Omega \rho_\varepsilon |u|^p} \leq 1 + C\varepsilon\frac{\||u|^p\|_{W^{1,1}(\Omega)}}{\int_\Omega \rho_\varepsilon|u|^p}, \tag{4.3}$$

By Lemma 4.4 and (1.4) we bound (4.3) as

$$1 + C\varepsilon\frac{\bar\rho}{\rho^-}\frac{F(u,\bar V) + G(u)}{F(u,\bar\rho)}. \tag{4.4}$$

Similarly, by Lemma 4.4 and (1.4) we get

$$\begin{aligned} \frac{\int_\Omega \rho_\varepsilon|u|^p}{\bar\rho\int_\Omega |u|^p} &\leq 1 + C\varepsilon\frac{\||u|^p\|_{W^{1,1}(\Omega)}}{\bar\rho\int_\Omega |u|^p} \\ &\leq 1 + C\varepsilon\max\{1,\bar V\}\frac{\rho^-}{\bar\rho}\frac{1}{V^-}\frac{G(u) + F(u,V_\varepsilon)}{F(u,\rho_\varepsilon)}. \end{aligned}$$

This completes the proof of the Lemma. □

5. Main results.

The proofs of Theorems 1.1 and 1.2 follow the same general lines of the ones of Theorem 5.6 in [8]. The fundamental tool to estimate the rates of convergence of the eigenvalues is the error bound of oscillating integrals given in Theorems 4.1 and 4.3. Observe that the regularity of the domain Ω considered in equations (1.5)–(1.10) are the necessary to apply Theorems 4.3 and 4.1, i.e., Lipschitz regularity in equation (1.5) and C^1 regularity in all other cases.

5.1. **Proof of Theorem 1.1.** The Dirichlet boundary condition case, i.e. problem $D_\varepsilon(\Omega)$, was treated in [8, Theorem 3.6]. We prove the result in detail for problem $N_\varepsilon(\Omega)$. For $R_\varepsilon(\Omega)$, $P_\varepsilon(\Omega)$and $B_\varepsilon(\Omega)$ the proofs are very similar and we will make a sketch highlighting only the differences.

For simplicity we will denote λ_k^ε and λ_k (without the superindex N) to the k–th variational eigenvalue of $N_\varepsilon(\Omega)$ and its limit problem obtained as $\varepsilon \to 0$.

Let $\delta > 0$ and let $G_\delta^k \subset W^{1,p}(\Omega)$ be a compact, symmetric set of genus k such that

$$\lambda_k = \sup_{u\in G_\delta^k} \frac{G(u) + F(u,\bar V)}{F(u,\bar\rho)} + O(\delta). \tag{5.1}$$

We use now the set G_δ^k, which is admissible in the variational characterization of the k–th eigenvalue λ_k^ε, in order to find a bound for it as follows,

$$\lambda_k^\varepsilon \le \sup_{u\in G_\delta^k} \frac{G(u) + F(u,V_\varepsilon)}{F(u,\bar\rho)} \frac{F(u,\bar\rho)}{F(u,\rho_\varepsilon)}. \tag{5.2}$$

Now, we look for bounds of the two quotients in (5.2).

For every function $u \in G_\delta^k \subset W^{1,p}(\Omega)$ we can apply Theorem 4.3 and we obtain that

$$\frac{G(u) + F(u,V_\varepsilon)}{F(u,\bar\rho)} \le \frac{G(u) + F(u,\bar V)}{F(u,\bar\rho)} + C\varepsilon \frac{\||u|^p\|_{W^{1,1}(\Omega)}}{F(u,\bar\rho)}. \tag{5.3}$$

By using Lemma 4.4, we have for each $u \in G_\delta^k$ there exists some constant $c > 0$ such that

$$\begin{aligned} \frac{\||u|^p\|_{W^{1,1}(\Omega)}}{F(u,\bar\rho)} &\le C\frac{G(u) + F(u,\bar V)}{F(u,\bar\rho)} \\ &\le C \sup_{v\in G_\delta^k} \frac{G(v) + F(v,\bar V)}{F(v,\bar\rho)} \\ &= C(\lambda_k + O(\delta)). \end{aligned} \tag{5.4}$$

Since $u \in G_\delta^k \subset W^{1,p}(\Omega)$, by applying Lemma 4.5 and (5.4) we obtain that

$$\begin{aligned} \frac{F(u,\bar\rho)}{F(u,\rho_\varepsilon)} &\le 1 + C\varepsilon \frac{\||u|^p\|_{W^{1,1}(\Omega)}}{F(u,\bar\rho)} \\ &\le 1 + C\varepsilon(\lambda_k + O(\delta)). \end{aligned} \tag{5.5}$$

Then, combining (5.3), (5.4) and (5.5) we find that

$$\lambda_k^\varepsilon \le (\lambda_k + O(\delta) + C\varepsilon(\lambda_k + O(\delta)))\,(1 + C\varepsilon(\lambda_k + O(\delta)))\,.$$

Letting $\delta \to 0$ we get

$$\lambda_k^\varepsilon - \lambda_k \le C\varepsilon(\lambda_k^2 + \lambda_k). \tag{5.6}$$

In a similar way, interchanging the roles of λ_k and λ_k^ε, we obtain

$$\lambda_k - \lambda_k^\varepsilon \le C\varepsilon((\lambda_k^\varepsilon)^2 + \lambda_k^\varepsilon). \tag{5.7}$$

So, from (5.6) and (5.7), we arrive at

$$|\lambda_k^\varepsilon - \lambda_k| \le C\varepsilon \max\{\lambda_k^2 + \lambda_k, (\lambda_k^\varepsilon)^2 + \lambda_k^\varepsilon\}.$$

In order to complete the proof of the Theorem, we need an estimate on λ_k and λ_k^ε. By Lemma 3.1 we can compare them with the k–th variational eigenvalue of the $p-$Laplacian obtaining

$$|\lambda_k^\varepsilon - \lambda_k| \le C\varepsilon k^{2p/N},$$

and the proof is complete. □

In the remaining of this subsection, we highlight the difference between the Neumann case and the rest of the boundary conditions with the exception of the Steklov problem that has a separate treatment.

Dirichlet: As we mentioned before, this problem was addressed in [8, Theorem 3.6]. Here, functions are taken in $W_0^{1,p}(\Omega)$ instead $W^{1,p}(\Omega)$ in the variational characterization of the eigenvalues. This leads to use Theorem 4.1 instead of Theorem 4.3 to estimate the oscillating integrals. Now, for each function $u \in W_0^{1,p}(\Omega)$ we can apply Theorem 4.1 and obtain an analogous equation to (5.3)

$$\frac{G(u) + F(u,\rho_\varepsilon)}{F(u,\bar\rho)} \le \frac{G(u) + F(u,\bar V)}{F(u,\bar\rho)} + C\varepsilon \frac{\|u\|_{L^p(\Omega)}^{p-1}\|\nabla u\|_{L^p(\Omega)}}{F(u,\bar\rho)}.$$

First observe that one can easily show that the quotients $\|u\|_{L^p(\Omega)}/F(u,\bar\rho)^{1/p}$ and $\|u\|_{L^p(\Omega)}/F(u,\rho_\varepsilon)^{1/p}$ are bounded uniformly on ε and now, the difference with the Neumann case is the way we bound the quotients $\|\nabla u\|_{L^p(\Omega)}/F(u,\bar\rho)^{1/p}$ and $\|\nabla u\|_{L^p(\Omega)}/F(u,\rho_\varepsilon)^{1/p}$.

By (1.4) and (2.1) we get

$$\begin{aligned}\frac{\|\nabla u\|_{L^p(\Omega)}^p}{F(u,\rho_\varepsilon)} &\le \frac{\bar\rho}{\rho^-}\frac{\|\nabla u\|_{L^p(\Omega)}^p}{F(u,\bar\rho)}\\ &\le \frac{\bar\rho}{\rho^-}\frac{1}{\alpha}\frac{G(u)+F(u,\bar V)}{F(u,\bar\rho)}\end{aligned}$$

and

$$\frac{\|\nabla u\|_{L^p(\Omega)}^p}{F(u,\bar\rho)} \le \frac{\rho^+}{\bar\rho}\frac{\|\nabla u\|_{L^p(\Omega)}^p}{F(u,\rho_\varepsilon)}.$$

Taking into account this changes, the proof is analogous to the Neumann one.

Non-Flux: Let $u \in W_0^{1,p}(\Omega) \oplus \mathbb{R}$, then $u = v + c$ where $v \in W_0^{1,p}(\Omega)$ and c is a constant depending on u. It follows that $u - c \in W_0^{1,p}(\Omega)$. Observe that if $u \in W_0^{1,p}(\Omega) \oplus \mathbb{R}$ then $|u|^p \in W_0^{1,1}(\Omega) \oplus \mathbb{R}$ and then $|u|^p - c \in W_0^{1,1}(\Omega)$. By using Theorem 4.1 and Theorem 4.3 together with (1.4) we observe that

$$\begin{aligned}\int_\Omega (\rho_\varepsilon - \bar\rho)|u|^p &= \int_\Omega (\rho_\varepsilon - \bar\rho)(|u|^p - c) + \int_\Omega (\rho_\varepsilon - \bar\rho)c\\ &\le C\varepsilon\|\nabla u\|_{L^p(\Omega)}^p + C\varepsilon\|c\|_{W^{1,p}(\Omega)}\\ &\le C\varepsilon(\|\nabla u\|_{L^p(\Omega)}^p + |c|).\end{aligned} \tag{5.8}$$

Observe that $u \in W_0^{1,p}(\Omega) \oplus \mathbb{R} = \{u \in W^{1,p}(\Omega) : u = c \text{ on } \partial\Omega \text{ with } c \in \mathbb{R}\}$. If $u = v + c \in W_0^{1,p}(\Omega) \oplus \mathbb{R}$, by the Sobolev trace inequality it follows that

$$\|u\|_{W^{1,p}(\Omega)}^p \geq c_T \|u\|_{L^p(\partial\Omega)}^p = c_T |c|^p |\partial\Omega|^p. \tag{5.9}$$

Moreover,

$$\|u\|_{W^{1,p}(\Omega)}^p \geq \|\nabla u\|_{L^p(\Omega)}^p. \tag{5.10}$$

Then, by (5.9) and (5.10) it follows that

$$\|\nabla u\|_{L^p(\Omega)}^p + |c| \leq C\|u\|_{W^{1,p}(\Omega)}^p. \tag{5.11}$$

From (5.8), (5.11) and Lemma 4.4 it follows that if $u \in W_0^{1,p}(\Omega) \oplus \mathbb{R}$ then

$$\int_\Omega (\rho_\varepsilon - \bar{\rho})|u|^p \leq C\|u\|_{W^{1,p}(\Omega)}^p \leq C(G(u) + F(u, V))$$

where V is an arbitrary weight.

Taking into account this remark, the proof in the non-flux case is analogous to the Neumann one. Observe that for this problem, we only need the boundary $\partial\Omega$ to be Lipschitz since we have used Theorem 4.1.

Robin: The extra term $\beta H(u)$ is irrelevant in the proof. This case is completely analogous to the Neumann and it follows by means of (1.4), (2.1), Theorem 4.3, Lemma 4.4 and Lemma 4.5.

Eigenvalue depending boundary condition: From Theorem 4.3, Lemma 4.4 and (1.4) we have that for every $u \in W^{1,p}(\Omega)$

$$\begin{aligned}\frac{F(u,\bar{\rho}) + H(u)}{F(u,\rho_\varepsilon) + H(u)} &\leq 1 + C\varepsilon \frac{\|u\|_{W^{1,p}(\Omega)}}{F(u,\rho_\varepsilon) + H(u)} \\ &\leq 1 + C\varepsilon \frac{G(u) + F(u,\bar{V})}{F(u,\bar{\rho}) + H(u)}\end{aligned}$$

and

$$\begin{aligned}\frac{G(u) + F(u,V_\varepsilon)}{H(u) + F(u,\bar{\rho})} &\leq \frac{G(u) + F(u,\bar{V})}{H(u) + F(u,\bar{\rho})} + C\varepsilon \frac{\|u\|_{W^{1,p}(\Omega)}}{H(u) + F(u,\bar{\rho})} \\ &\leq (1 + C\varepsilon)\frac{G(u) + F(u,\bar{V})}{H(u) + F(u,\bar{\rho})}.\end{aligned}$$

Observe that (1.4) and (2.1) allow us to bound in the opposite sense, that is, for each $u \in W^{1,p}(\Omega)$,

$$\frac{F(u,\rho_\varepsilon) + H(u)}{F(u,\bar{\rho}) + H(u)} \leq 1 + C\varepsilon \frac{G(u) + F(u,V_\varepsilon)}{F(u,\rho_\varepsilon) + H(u)},$$

$$\frac{G(u) + F(u,\bar{V})}{H(u) + F(u,\rho_\varepsilon)} \leq (1 + C\varepsilon)\frac{G(u) + F(u,V_\varepsilon)}{H(u) + F(u,\rho_\varepsilon)}.$$

Having changed this slight detail, the proof is analogous to the Neumann case.

5.2. **Proof of Theorem 1.2.** The proof of the Steklov case is simpler due to that in this case there is only an oscillating weight in the equation $S_\varepsilon(\Omega)$.

Let $\delta > 0$ and let $G_\delta^k \subset W^{1,p}(\Omega)$ be a compact, symmetric set of genus k such that

$$\lambda_k = \sup_{u\in G_\delta^k} \frac{G(u) + F(u,\bar V)}{H(u)} + O(\delta). \tag{5.12}$$

Being G_δ^k admissible in the variational characterization of λ_k^ε, we get

$$\lambda_k^\varepsilon \le \sup_{u\in G_\delta^k} \frac{G(u) + F(u,V_\varepsilon)}{H(u)}. \tag{5.13}$$

For every function $u \in G_\delta^k \subset W^{1,p}(\Omega)$ we can apply Theorem 4.3 obtaining

$$\frac{G(u) + F(u,V_\varepsilon)}{H(u)} \le \frac{G(u) + F(u,\bar V)}{H(u)} + C\varepsilon \frac{\||u|^p\|_{W^{1,1}(\Omega)}}{H(u)}. \tag{5.14}$$

Now, for each $u \in G_\delta^k$ we can apply Lemma 4.4 to bound

$$\begin{aligned} \frac{\||u|^p\|_{W^{1,1}(\Omega)}}{H(u)} &\le c\frac{G(u) + F(u,\bar V)}{H(u)} \\ &\le c \sup_{v\in G_\delta^k} \frac{G(v) + F(v,\bar V)}{H(v)} \\ &= c(\lambda_k + O(\delta)). \end{aligned} \tag{5.15}$$

Then, combining (5.13), (5.14) and (5.15) we find that

$$\lambda_k^\varepsilon \le \lambda_k + O(\delta) + C\varepsilon(\lambda_k + O(\delta)).$$

In a similar way, interchanging the roles of λ_k and λ_k^ε, we can obtain the opposite inequality. Letting $\delta \to 0$ we arrive at

$$|\lambda_k^\varepsilon - \lambda_k| \le C\varepsilon \max\{\lambda_k, \lambda_k^\varepsilon\}.$$

By Lemma 3.2 we obtain that

$$|\lambda_k^\varepsilon - \lambda_k| \le C\varepsilon k^{\frac{p-1}{N-1}}$$

and the proof is complete.

Acknowledgements

This work was partially supported by Universidad de Buenos Aires under grant 20020100100400 and by CONICET (Argentina) PIP 5478/1438.

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