Convergence rate for some quasilinear eigenvalues homogenization problems

Julián Fernández Bonder *, Juan P. Pinasco, Ariel M. Salort

Departamento de Matemática, FCEN - Universidad de Buenos Aires, and IMAS - CONICET, Ciudad Universitaria, Pabellón I, (1428) Av. Cantilo s/n, Buenos Aires, Argentina

A R T I C L E   I N F O

A B S T R A C T

In this work we study the homogenization problem for nonlinear eigenvalues of some quasilinear elliptic operators. We obtain an explicit order of convergence in k and in ε for the (variational) eigenvalues.

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1. Introduction

In this paper we study the asymptotic behavior (as $\varepsilon \to 0$) of the eigenvalues of the following problems

$$\begin{cases} -\operatorname{div}(a_\varepsilon(x, \nabla u^\varepsilon)) = \lambda^\varepsilon \rho_\varepsilon |u^\varepsilon|^{p-2} u^\varepsilon & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, ε is a positive real number, and λ^ε is the eigenvalue parameter.

The weight functions $\rho_\varepsilon(x)$ are assumed to be positive and uniformly bounded away from zero and infinity

$$0 < \rho^- \le \rho_\varepsilon(x) \le \rho^+ < \infty \tag{1.2}$$

and the family of operators $a_\varepsilon(x, \xi)$ have precise hypotheses that are stated below, but the prototypical example is

$$a_\varepsilon(x, \nabla u^\varepsilon) = \left| A^\varepsilon(x) \nabla u^\varepsilon \cdot \nabla u^\varepsilon \right|^{\frac{p-2}{2}} A^\varepsilon(x) \nabla u^\varepsilon, \tag{1.3}$$

with $1 < p < +\infty$, and $A^\varepsilon(x)$ is a family of uniformly elliptic matrices (both in $x \in \Omega$ and in $\varepsilon > 0$).

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* Corresponding author.

E-mail addresses: jfbonder@dm.uba.ar (J.F. Bonder), jpinasco@dm.uba.ar (J.P. Pinasco), asalort@dm.uba.ar (A.M. Salort).
URLs: http://mate.dm.uba.ar/~jfbonder (J.F. Bonder), http://mate.dm.uba.ar/~jpinasco (J.P. Pinasco).

In this work we will deal with the problem where the operator $a_\varepsilon(x, \xi)$ is independent of $\varepsilon > 0$, i.e. $a_\varepsilon(x, \xi) = a(x, \xi)$ for every $\varepsilon > 0$.

The study of this type of problems has a long history due to its relevance in different fields of applications. The problem of finding the asymptotic behavior of the eigenvalues of (1.1) is an important part of what is called *Homogenization Theory*. Homogenization Theory is applied in composite materials in which the physical parameters such as conductivity and elasticity are oscillating. Homogenization Theory tries to get a good approximation of the macroscopic behavior of the heterogeneous material by letting the parameter $\varepsilon \to 0$. The main references for the homogenization theory of periodic structures are the books by Bensoussan–Lions–Papanicolaou [4], Sanchez–Palencia [21], Oleǐnik–Shamaev–Yosifian [18] among others.

In the linear setting (i.e., $a_\varepsilon(x, \xi)$ as in (1.3) with $p = 2$) this problem is well understood. It is known that, up to a subsequence, there exists a limit operator $a_h(x, \xi) = A^h(x)\xi$ and a limit function $\bar\rho$ such that the spectrum of (1.1) converges to that of the limit problem.

$$\begin{cases} -\text{div}(a_h(x, \nabla u)) = \lambda \bar\rho |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Let us recall that the convergence of eigenvalues in the multidimensional linear case was studied in 1976 by Boccardo and Marcellini [5] for general bounded matrices. Kesavan [16] studied the problem in an periodic frame. This is an important case of homogenization, i.e. when $\rho_\varepsilon(x) = \rho(\frac{x}{\varepsilon})$ and $A_\varepsilon(x) = A(\frac{x}{\varepsilon})$ where $\rho(x)$ and $A(x)$ are Q-periodic functions, Q being the unit cube in $\mathbb{R}^N$ and in this case, the limit problem can be fully characterized and so the entire sequence $\varepsilon \to 0$ is convergent. See [16,17].

In the general nonlinear setting, Baffico, Conca and Rajesh [3], relying on the G-convergence results of Chiadó Piat, Dal Maso and Defranceschi [9] for monotone operators, study the convergence problem of the principal eigenvalue of (1.1). The concept of G-convergence of linear elliptic second order operators was introduced by Spagnolo in [23].

The convergence problem for higher (variational) nonlinear eigenvalues was addressed by T. Champion and L. De Pascale in [8] where by means of the Γ-convergence methods they are able to show convergence of the (variational) eigenvalues of (1.1) to those of the limit equation (1.4).

Now, we turn our attention to the order of convergence of the eigenvalues that is the focus of this work. To this end, we restrict ourselves to the problems

$$\begin{cases} -\text{div}(a(x, \nabla u^\varepsilon)) = \lambda^\varepsilon \rho_\varepsilon |u^\varepsilon|^{p-2} u^\varepsilon & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where the family of weight functions ρ_ε are given in terms of a single bounded Q-periodic function ρ in the form $\rho_\varepsilon(x) := \rho(x/\varepsilon)$, Q being the unit cube of $\mathbb{R}^N$.

The limit problem is then given by

$$\begin{cases} -\text{div}(a(x, \nabla u)) = \lambda \bar\rho |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $\bar\rho$ is the average of ρ in Q, i.e. $\bar\rho = \int_Q \rho(x)\, dx$.

The first estimate of the order of convergence of the eigenvalues, for the linear case, can be found in Chapter III, Section 2 of [18]. By estimating the eigenvalues of the inverse operator, which is compact, and using tools from functional analysis in Hilbert spaces, it is proved that

$$\left| \lambda_k^\varepsilon - \lambda_k \right| \leq \frac{C \lambda_k^\varepsilon (\lambda_k)^2}{1 - \lambda_k \beta_k^\varepsilon} \varepsilon^{\frac{1}{2}}.$$

Here, C is a positive constant, and β_ε^k satisfies

$$0 \leq \beta_\varepsilon^k < \lambda_k^{-1},$$

with

$$\lim_{\varepsilon \to 0} \beta_\varepsilon^k = 0$$

for each $k \geq 1$.

Santosa and Vogelius in [22] show that the difference between λ_k^ε and λ_k is of order ε. Moreover, when λ_k is a simple eigenvalue they obtain the limit of the incremental quotient under certain hypotheses on the domain.

The problem, again in the linear setting and in dimension $N = 1$, with $a = 1$, was recently studied by Castro and Zuazua in [6,7]. In those articles the authors, using the so-called WKB method which relays on asymptotic expansions of the solutions of the problem, and the explicit knowledge of the eigenfunctions and eigenvalues of the constant coefficient limit problem, proved

$$\left| \sqrt{\lambda_k^\varepsilon} - \sqrt{\lambda_k} \right| \leq Ck^2 \varepsilon,$$

equivalently, since the Weyl's formula implies that $\lambda_k^\varepsilon \sim k^2$,

$$\left| \lambda_k^\varepsilon - \lambda_k \right| \leq Ck^3 \varepsilon$$

and they also presented a variety of results on correctors for the eigenfunction approximation.

In the linear problem, in any space dimension, Kenig, Lin and Shen [14] (allowing an ε dependence in the diffusion matrix of the elliptic operator) proved that for Lipschitz domains Ω one has

$$\left| \lambda_k^\varepsilon - \lambda_k \right| \leq C\varepsilon \left| \log(\varepsilon) \right|^{\frac{1}{2} + \sigma}$$

for any $\sigma > 0$, C depending on k and σ.

Moreover, the authors show that if the domain Ω is more regular ($C^{1,1}$ is enough) they can get rid of the logarithmic term in the above estimate. However, no explicit dependence of C on k is obtained in that work.

Later on, in [15] the authors obtain the precise dependence on k showing that

$$\left| \lambda_k^\varepsilon - \lambda_k \right| \leq Ck^{\frac{3}{N}} \varepsilon \left| \log(\varepsilon) \right|^{\frac{1}{2} + \sigma}$$

for any $\sigma > 0$, C depending on σ. Again, when the domain Ω is smooth, the logarithmic term can be removed.

In this paper we analyze the order of convergence of eigenvalues of (1.5) to the ones of (1.6) and we prove that,

$$\left| \lambda_k^\varepsilon - \lambda_k \right| \leq Ck^{\frac{p+1}{N}} \varepsilon$$

with C independent of k and ε. In this result, by λ_k^ε and λ_k we refer to the variational eigenvalues of problems (1.5) and (1.6) respectively.

Some remarks are in order:

(1) Classical Weyl's estimates on the eigenvalues of second order N-dimensional problems show that λ_k and λ_k^ε behaves like $ck^{\frac{2}{N}}$, with c depending only on the coefficients of the operator and N. Hence, the order of growth of the right-hand side in the estimate of [18] is

$$\frac{\lambda_k^\varepsilon(\lambda_k)^2\varepsilon^{\frac{1}{2}}}{1-\lambda_k\beta_k^\varepsilon} \sim \frac{k^{\frac{6}{N}}\varepsilon^{\frac{1}{2}}}{1-\lambda_k\beta_k^\varepsilon} \leq Ck^{\frac{6}{N}}\varepsilon^{\frac{1}{2}}.$$

Moreover, the constant involved in their bound is unknown.

(2) If we specialize our result to the one dimensional linear case, we recover the estimate obtained in [7]. Moreover, we are considering more general weights ρ since very low regularity is needed and the estimate is valid for any k. On the other hand, no corrector results are presented here.

(3) In our result no regularity assumptions on Ω are needed. Any bounded open set will do.

(4) The constant C entering in our estimate, can be computed *explicitly* in terms of the weight ρ, the diffusion operator $a(x,\xi)$ and the Poincaré constant in the unit cube of $\mathbb{R}^N$ (see Theorem 3.6).

In the one dimensional problem, we can provide a extremely elementary proof of Theorems 3.5 and 3.6, generalizing the estimates obtained in [7]. Moreover, in this case an oscillating coefficient can be allowed.

Let us mention that in the nonlinear case considered in this paper there are no quantitative estimates on the convergence of the eigenfunctions. However, it is possible to prove in one spatial dimension that the zeros of the eigenfunctions converge to the zeros of the ones of the limit problem and we also find an explicit estimate of the rate of convergence of the nodal domain.

Organization of the paper The rest of the paper is organized as follows: In Section 2, we collect some preliminary results on monotone operators that are needed in order to deal with (1.1) and some facts about the eigenvalue problem. In Section 3 we prove the convergence of the kth-variational eigenvalue of problem (1.5) to the kth-variational eigenvalue of the limit problem (1.6), and we study the rates of convergence. In Section 4 we study the one dimensional problem, and we show in Section 5 that in this case, an oscillating coefficient can be allowed. In Section 6 we deal with the zeros of eigenfunctions, and we close the paper in Section 7 with some numerical results.

2. Preliminary results

In this section we review some results gathered from the literature, enabling us to clearly state our results and making the paper self-contained.

2.1. Monotone operators

We consider $\mathcal{A}: W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ given by

$$\mathcal{A}u := -\mathrm{div}\big(a(x, \nabla u)\big),$$

where $a: \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies, for every $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$, the following conditions:

(H0) *measurability*: $a(\cdot, \cdot)$ is a Carathéodory function, i.e. $a(x, \cdot)$ is continuous a.e. $x \in \Omega$, and $a(\cdot, \xi)$ is measurable for every $\xi \in \mathbb{R}^N$.

(H1) *monotonicity*: $0 \leq (a(x, \xi_1) - a(x, \xi_2)) \cdot (\xi_1 - \xi_2)$.

(H2) *coercivity*: $\alpha|\xi|^p \leq a(x, \xi) \cdot \xi$.

(H3) *continuity*: $|a(x,\xi)| \le \beta|\xi|^{p-1}$.

(H4) *p-homogeneity*: $a(x,t\xi) = t^{p-1}a(x,\xi)$ for every $t > 0$.

(H5) *oddness*: $a(x,-\xi) = -a(x,\xi)$.

Let us introduce $\Psi(x,\xi_1,\xi_2) = a(x,\xi_1) \cdot \xi_1 + a(x,\xi_2) \cdot \xi_2$ for all $\xi_1, \xi_2 \in \mathbb{R}^N$, and all $x \in \Omega$; and let $\delta = \min\{p/2, (p-1)\}$.

(H6) *equi-continuity*:

$$\left|a(x,\xi_1) - a(x,\xi_2)\right| \le c\Psi(x,\xi_1,\xi_2)^{(p-1-\delta)/p}\left[\left(a(x,\xi_1) - a(x,\xi_2)\right) \cdot (\xi_1 - \xi_2)\right]^{\delta/p}$$

(H7) *cyclical monotonicity*: $\sum_{i=1}^{k} a(x,\xi_i) \cdot (\xi_{i+1} - \xi_i) \le 0$, for all $k \ge 1$, and $\xi_1, \ldots, \xi_{k+1}$, with $\xi_1 = \xi_{k+1}$.

(H8) *strict monotonicity*: let $\gamma = \max(2,p)$, then

$$\alpha|\xi_1 - \xi_2|^{\gamma}\Psi(x,\xi_1,\xi_2)^{1-(\gamma/p)} \le \left(a(x,\xi_1) - a(x,\xi_2)\right) \cdot (\xi_1 - \xi_2).$$

See [3], Section 3.4 where a detailed discussion on the relation and implications of every condition (H0)–(H8) is given.

In particular, under these conditions, we have the following proposition:

Proposition 2.1. *(See [3], Lemma 3.3.) Given $a(x,\xi)$ satisfying* (H0)–(H8) *there exists a unique Carathéodory function Φ which is even, p-homogeneous strictly convex and differentiable in the variable ξ satisfying*

$$\alpha|\xi|^p \le \Phi(x,\xi) \le \beta|\xi|^p \tag{2.1}$$

for all $\xi \in \mathbb{R}^N$ a.e. $x \in \Omega$ such that

$$\nabla_\xi\Phi(x,\xi) = pa(x,\xi)$$

and normalized such that $\Phi(x,0) = 0$.

Remark 2.2. When the operator $a(x,\xi)$ is given by (1.3), the corresponding potential function Φ is given by

$$\Phi(x,\xi) = \left|A(x)\xi \cdot \xi\right|^{\frac{p}{2}}.$$

2.2. The nonlinear eigenvalue problem

In this section we review some properties of the spectrum of (1.1) for fixed ε. That is, we study

$$\begin{cases} -\mathrm{div}(a(x,\nabla u)) = \lambda\rho|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where $0 < \rho_- \le \rho(x) \le \rho_+$ and $0 < \alpha \le a(x) \le \beta$ for some constants ρ_-, ρ_+, α and β.

We denote by

$$\Sigma := \left\{\lambda \in \mathbb{R}: \text{there exists } u \in W_0^{1,p}(\Omega), \text{ nontrivial solution to } (2.2)\right\}$$

the spectrum of (2.2).

By means of the critical point theory of Ljusternik–Schnirelmann it is straightforward to obtain a discrete sequence of variational eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ tending to $+\infty$. We denote by Σ_{var} sequence of variational eigenvalues.

The kth-variational eigenvalue is given by

$$\lambda_k = \inf_{C\in\Gamma_k} \sup_{v\in C} \frac{\int_\Omega \Phi(x,\nabla v)\,dx}{\int_\Omega \rho(x)|v|^p\,dx} \tag{2.3}$$

where Φ is the potential function given by Proposition 2.1 and

$$\Gamma_k = \left\{ C \subset W_0^{1,p}(\Omega) : C \text{ compact}, C = -C, \ \gamma(C) \geq k \right\}$$

and $\gamma(C)$ is the Kranoselskii genus, see [20] for the definition and properties of γ.

For the one dimensional p-Laplace operator in $J = (0,\ell)$,

$$-\left(|w'|^{p-2}w'\right)' = \mu|w|^{p-2}w \tag{2.4}$$

with zero Dirichlet boundary conditions $w(0) = w(\ell) = 0$, we have

$$\mu_k = \inf_{C\in\mathcal{C}_k} \sup_{v\in C} \frac{\int_J |v'|^p\,dx}{\int_J |v|^p\,dx}, \tag{2.5}$$

with $v \in W_0^{1,p}(J)$.

Here, all the eigenvalues and eigenfunctions can be found explicitly:

Theorem 2.3. *(See Del Pino, Drabek and Manasevich, [10].) The eigenvalues μ_k and eigenfunctions w_k of Eq. (2.4) on the interval J are given by*

$$\mu_k = \frac{\pi_p^p k^p}{\ell^p},$$
$$w_k(x) = S_p(\pi_p k x/\ell).$$

Remark 2.4. It was proved in [11] that they coincide with the variational eigenvalues given by Eq. (2.5). However, let us observe that the notation is different in both papers.

The function $S_p(x)$ is the solution of the initial value problem

$$\begin{cases} -(|v'|^{p-2}v')' = |v|^{p-2}v \\ v(0) = 0, \qquad v'(0) = 1, \end{cases}$$

and is defined implicitly as

$$x = \int_0^{S_p(x)} \left(\frac{p-1}{1-t^p}\right)^{1/p} dt.$$

Moreover, its first zero is π_p, given by

$$\pi_p = 2\int_0^1 \left(\frac{p-1}{1-t^p}\right)^{1/p} dt.$$

We will call C_p the derivative of S_p.

In [2], problem (2.2) in one space dimension with $a(x, u') = |u'|^{p-2}u'$ is studied and, among other things, it is proved that any eigenfunction associated to λ_k has exactly k nodal domains. As a consequence of this fact follows the simplicity of every variational eigenvalue.

The exact same proof of [2] works for general $a(x, u')$, and so we obtain the following:

Theorem 2.5. *Every eigenfunction of* (2.2) *in dimension $N = 1$ corresponding to the kth-eigenvalue λ_k has exactly $k - 1$ zeroes. Moreover, for every k, λ_k is simple, consequently the eigenvalues are ordered as $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k \nearrow +\infty$.*

Now, using the same ideas as in [12] is easy to prove, for the one dimensional problem, that the spectrum of (2.2) coincides with the variational spectrum. In fact, we have:

Theorem 2.6. $\Sigma = \Sigma_{\mathrm{var}}$, *i.e., every solution of problem* (2.4) *is given by* (2.5).

Proof. The proof of this theorem is completely analogous to that of Theorem 1.1 in [12]. $\quad\square$

3. Rates of convergence

In this section we prove that kth-variational eigenvalue of problem (1.5) converges to the kth-variational eigenvalue of the limit problem (1.6).

Moreover, our goal is to estimate the rate of convergence between the eigenvalues. That is, we want to find explicit bounds for the error $|\lambda_k^\varepsilon - \lambda_k|$.

We begin this section by proving some auxiliary results that are essential in the remaining of the paper.

We first prove a couple of lemmas in order to prove Theorem 3.4 which is a generalization for $p \neq 2$ of Oleinik's Lemma [18].

We start with an easy Lemma that computes the Poincaré constant on the cube of side ε in terms of the Poincaré constant of the unit cube. Although this result is well known and its proof follows directly by a change of variables, we choose to include it for the sake of completeness.

Lemma 3.1. *Let Q be the unit cube in $\mathbb{R}^N$ and let c_q be the Poincaré constant in the unit cube in L^q, $q \geq 1$, i.e.*

$$\|u - \bar{u}\|_{L^q(Q)} \leq c_q \|\nabla u\|_{L^q(Q)}, \quad \text{for every } u \in W^{1,q}(Q),$$

where $\bar{u}$ is the average of u in Q. Then, for every $u \in W^{1,q}(Q_\varepsilon)$ we have

$$\|u - \bar{u}\|_{L^q(Q_\varepsilon)} \leq c_q \varepsilon \|\nabla u\|_{L^q(Q_\varepsilon)},$$

where $Q_\varepsilon = \varepsilon Q$.

Proof. Let $u \in W^{1,q}(Q_\varepsilon)$. We can assume that $\bar{u} = 0$. Now, if we denote $u_\varepsilon(y) = u(\varepsilon y)$, we have that $u_\varepsilon \in W^{1,q}(Q)$ and by the change of variables formula, we get

$$\int_{Q_\varepsilon} |u|^q = \int_Q |u_\varepsilon|^q \varepsilon^N \leq c_q^q \varepsilon^N \int_Q |\nabla u_\varepsilon|^q = c_q^q \varepsilon^q \int_{Q_\varepsilon} |\nabla u|^q.$$

The proof is now complete. $\quad\square$

Remark 3.2. In [1], Acosta and Duran show that for convex domains U, one has

$$\|u - \bar{u}\|_{L^1(U)} \leq \frac{d}{2} \|\nabla u\|_{L^1(U)},$$

where d is the diameter of U. When we apply this result to the unit cube, we get

$$c_1 \leq \frac{\sqrt{N}}{2}. \tag{3.1}$$

The next Lemma is the final ingredient in the estimate of Theorem 3.4.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and denote by Q the unit cube in $\mathbb{R}^N$. Let $g \in L^\infty(\mathbb{R}^N)$ be a Q-periodic function such that $\bar{g} = 0$. Then the inequality*

$$\left| \int_\Omega g\left(\frac{x}{\varepsilon}\right) v \right| \leq \|g\|_{L^\infty(\mathbb{R}^N)} c_1 \varepsilon \|\nabla v\|_{L^1(\Omega)}$$

holds for every $v \in W_0^{1,1}(\Omega)$, where c_1 is the Poincaré constant given in (3.1).

Proof. Denote by I^ε the set of all $z \in \mathbb{Z}^N$ such that $Q_{z,\varepsilon} \cap \Omega \neq \emptyset$, $Q_{z,\varepsilon} := \varepsilon(z + Q)$. Given $v \in W_0^{1,1}(\Omega)$ we extended by 0 outside Ω and consider the piecewise function $\bar{v}_\varepsilon$ given by the formula

$$\bar{v}_\varepsilon|_{Q_{z,\varepsilon}} = \frac{1}{\varepsilon^N} \int_{Q_{z,\varepsilon}} v(y) dy \quad \text{in } Q_{z,\varepsilon}.$$

We denote by $\Omega_1 = \bigcup_{z \in I^\varepsilon} Q_{z,\varepsilon} \supset \Omega$ and by $g_\varepsilon(x) = g(\frac{x}{\varepsilon})$. Then we have

$$\int_\Omega g_\varepsilon v = \int_{\Omega_1} g_\varepsilon(v - \bar{v}_\varepsilon) + \int_{\Omega_1} g_\varepsilon \bar{v}_\varepsilon. \tag{3.2}$$

Now, by Lemma 3.1 we get

$$\|v - \bar{v}_\varepsilon\|_{L^1(\Omega_1)} = \sum_{z \in I^\varepsilon} \int_{Q_{z,\varepsilon}} |v - \bar{v}_\varepsilon| dx$$

$$\leq c_1 \varepsilon \sum_{z \in I^\varepsilon} \int_{Q_{z,\varepsilon}} |\nabla v(x)| dx$$

$$\leq c_1 \varepsilon \|\nabla v\|_{L^1(\Omega)}. \tag{3.3}$$

Finally, since $\bar{g} = 0$ and since g is Q-periodic, we get

$$\int_{\Omega_1} g_\varepsilon \bar{v}_\varepsilon = \sum_{z \in I^\varepsilon} \bar{v}_\varepsilon|_{Q_{z,\varepsilon}} \int_{Q_{z,\varepsilon}} g_\varepsilon = 0. \tag{3.4}$$

Now, combining (3.3) and (3.4) we can bound (3.2) by

$$\left| \int_\Omega g_\varepsilon v \right| \leq \|g\|_{L^\infty(\mathbb{R}^N)} c_1 \varepsilon \|\nabla v\|_{L^1(\Omega)}.$$

This finishes the proof. $\quad \square$

The next Theorem is essential to estimate the rate of convergence of the eigenvalues since it allows us to replace an integral involving a rapidly oscillating function with one that involves its average in the unit cube.

Theorem 3.4. *Let $g \in L^\infty(\mathbb{R}^N)$ be a Q-periodic function such that $0 < g^- \leq g \leq g^+ < \infty$. Then,*

$$\left| \int_\Omega (g_\varepsilon(x) - \bar{g})|u|^p \right| \leq pc_1 \|g - \bar{g}\|_{L^\infty(\mathbb{R}^N)} \varepsilon \|u\|_{L^p(\Omega)}^{p-1} \|\nabla u\|_{L^p(\Omega)}$$

for every $u \in W_0^{1,p}(\Omega)$. Here, c_1 is the optimal constant in Poincaré's inequality in $L^1(Q)$ given by (3.1).

Proof. Let $\varepsilon > 0$ be fixed. Now, denote by $h_\varepsilon(x) = h(\frac{x}{\varepsilon}) = g(\frac{x}{\varepsilon}) - \bar{g}$ and so, by Lemma 3.3 we obtain

$$\left| \int_\Omega h_\varepsilon |u|^p \right| \leq \|h\|_{L^\infty(\mathbb{R}^N)} c_1 \varepsilon \left\| \nabla \left(|u|^p \right) \right\|_{L^1(\Omega)}. \tag{3.5}$$

An easy computation shows that

$$\left\| \nabla \left(|u|^p \right) \right\|_{L^1(\Omega)} \leq p\|u\|_{L^p(\Omega)}^{p-1} \|\nabla u\|_{L^p(\Omega)}. \tag{3.6}$$

Finally, combining (3.5) and (3.6) we obtain the desired result. $\quad \square$

Now we are ready to prove the main result of this section.

Theorem 3.5. *Let λ_k^ε be the kth-variational eigenvalue associated to Eq. (1.5) and let be λ_k be the kth-variational eigenvalue associated to the limit problem (1.6). Then*

$$\left| \lambda_k^\varepsilon - \lambda_k \right| \leq c_1 p \|\rho - \bar{\rho}\|_\infty \left(\frac{\rho^+}{\alpha} \right)^{1/p} \frac{1}{\rho^-} \varepsilon \max\{\lambda_k, \lambda_k^\varepsilon\}^{1+\frac{1}{p}}.$$

Proof. Let $\delta > 0$ and let $G_\delta^k \subset W_0^{1,p}(\Omega)$ be a compact, symmetric set of genus k such that

$$\lambda_k = \inf_{G \in \Gamma_k} \sup_{u \in G} \frac{\int_\Omega \Phi(x, \nabla u)}{\bar{\rho} \int_\Omega |u|^p} = \sup_{u \in G_\delta^k} \frac{\int_\Omega \Phi(x, \nabla u)}{\bar{\rho} \int_\Omega |u|^p} + O(\delta).$$

We use now the set G_δ^k, which is admissible in the variational characterization of the kth-eigenvalue of (1.5), in order to found a bound for it as follows,

$$\lambda_k^\varepsilon \leq \sup_{u \in G_\delta^k} \frac{\int_\Omega \Phi(x, \nabla u)}{\int_\Omega \rho_\varepsilon |u|^p} = \sup_{u \in G_\delta^k} \frac{\int_\Omega \Phi(x, \nabla u)}{\bar{\rho} \int_\Omega |u|^p} \frac{\bar{\rho} \int_\Omega |u|^p}{\int_\Omega \rho_\varepsilon |u|^p}. \tag{3.7}$$

To bound λ_k^ε we look for bounds of the two quotients in (3.7). For every function $u \in G_\delta^k$ we have that

$$\frac{\int_\Omega \Phi(x, \nabla u)}{\bar{\rho} \int_\Omega |u|^p} \leq \sup_{v \in G_\delta^k} \frac{\int_\Omega \Phi(x, \nabla v)}{\bar{\rho} \int_\Omega |v|^p} = \lambda_k + O(\delta). \tag{3.8}$$

Since $u \in G_\delta^k \subset W_0^{1,p}(\Omega)$, by Theorem 3.4 we obtain that

$$\frac{\bar{\rho} \int_\Omega |u|^p}{\int_\Omega \rho_\varepsilon |u|^p} \leq 1 + c_1 p \varepsilon \|\rho - \bar{\rho}\|_\infty \frac{\|u\|_{L^p(\Omega)}^{p-1} \|\nabla u\|_{L^p(\Omega)}}{\int_\Omega \rho_\varepsilon |u|^p}. \tag{3.9}$$

Now, by (1.2), (2.1) together with (3.8), we have

$$\frac{\|u\|_{L^p(\Omega)}^{p-1} \|\nabla u\|_{L^p(\Omega)}}{\int_\Omega \rho_\varepsilon |u|^p} \leq \frac{\bar{\rho}^{1/p}}{\rho^-} \left(\frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega \bar{\rho} |u|^p} \right)^{1/p}$$

$$\leq \left(\frac{\bar{\rho}}{\alpha} \right)^{1/p} \frac{1}{\rho^-} \left(\frac{\int_\Omega \Phi(x, \nabla u)}{\int_\Omega \bar{\rho} |u|^p} \right)^{1/p}$$

$$\leq \left(\frac{\bar{\rho}}{\alpha} \right)^{1/p} \frac{1}{\rho^-} (\lambda_k + O(\delta))^{1/p}. \tag{3.10}$$

Then combining (3.7), (3.8), (3.9) and (3.10) we find that

$$\lambda_k^\varepsilon \leq (\lambda_k + O(\delta)) \left(1 + c_1 p \|\rho - \bar{\rho}\|_\infty \left(\frac{\bar{\rho}}{\alpha} \right)^{1/p} \frac{1}{\rho^-} \varepsilon (\lambda_k + O(\delta))^{1/p} \right).$$

Letting $\delta \to 0$ we get

$$\lambda_k^\varepsilon - \lambda_k \leq c_1 p \|\rho - \bar{\rho}\|_\infty \left(\frac{\bar{\rho}}{\alpha} \right)^{1/p} \frac{1}{\rho^-} \varepsilon \lambda_k^{1 + \frac{1}{p}}. \tag{3.11}$$

In a similar way, interchanging the roles of λ_k and λ_k^ε, we obtain

$$\lambda_k - \lambda_k^\varepsilon \leq c_1 p \|\rho - \bar{\rho}\|_\infty \left(\frac{\rho^+}{\alpha} \right)^{1/p} \frac{1}{\bar{\rho}} \varepsilon (\lambda_k^\varepsilon)^{1 + \frac{1}{p}}. \tag{3.12}$$

So, from (3.11) and (3.12), we arrive at

$$\left| \lambda_k^\varepsilon - \lambda_k \right| \leq c_1 p \|\rho - \bar{\rho}\|_\infty \left(\frac{\rho^+}{\alpha} \right)^{1/p} \frac{1}{\rho^-} \varepsilon \max\{\lambda_k, \lambda_k^\varepsilon\}^{1 + \frac{1}{p}},$$

and so the proof is complete. $\quad \square$

It would be desirable to give a precise rate of convergence in terms of ε, k and the coefficients of the problem. In order to achieve this goal, we need to give explicit bounds on the eigenvalues λ_k and λ_k^ε. But this follows by comparison with the kth-variational eigenvalue of the p-Laplacian, μ_k and a refinement of the bound on μ_k proved in [13].

In fact, from (2.1) we have

$$\frac{\alpha}{\bar{\rho}} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p} \leq \frac{\int_\Omega \Phi(x, \nabla u)}{\int_\Omega \bar{\rho}|u|^p} \leq \frac{\beta}{\bar{\rho}} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p},$$

$$\frac{\alpha}{\rho^+} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p} \leq \frac{\int_\Omega \Phi(x, \nabla u)}{\int_\Omega \rho_\varepsilon |u|^p} \leq \frac{\beta}{\rho^-} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p},$$

from where it follows that

$$\max\{\lambda_k, \lambda_k^\varepsilon\} \leq \frac{\beta}{\rho^-} \mu_k.$$

Now, in [13], it is shown that

$$\mu_k \leq \tilde{\mu}_1 \left(\frac{k}{|\Omega|}\right)^{p/N}$$

where $\tilde{\mu}_1$ is the first Dirichlet eigenvalue for the p-laplacian in the unit cube. Finally, in [12] an estimate for $\tilde{\mu}_1$ by comparing with the first eigenvalue of the *pseudo* p-laplacian is obtained, namely

$$\tilde{\mu}_1 \leq \max\{N^{(p-2)/2}, 1\} N \pi_p^p,$$

where π_p is defined in Section 2.2.

Combining all of these facts, we immediately prove

Theorem 3.6. *With the same assumptions and notations as in Theorem 3.5 we have*

$$\left|\lambda_k - \lambda_k^\varepsilon\right| \leq C\varepsilon k^{\frac{p+1}{N}},$$

where the constant C is given by

$$C = \frac{\sqrt{N}}{2} p \|\rho - \bar{\rho}\|_\infty \left(\frac{\beta^{p+1}}{\alpha}\right)^{\frac{1}{p}} \frac{1}{(\rho^-)^2} \left(\frac{\rho^+}{\rho^-}\right)^{\frac{1}{p}} \pi_p^{p+1} N^{\frac{p+1}{p}} \max\{N^{\frac{p-2}{2}}, 1\}^{\frac{p+1}{p}}.$$

Remark 3.7. As we mentioned in the introduction, in the linear case and in one space dimension Castro and Zuazua [7] prove that, for $k < C\varepsilon^{-1}$,

$$\left|\lambda_k^\varepsilon - \lambda_k\right| \leq Ck^3 \varepsilon.$$

If we specialize our result to this case, we get the same bound. The advantage of our method is that very low regularity on ρ is needed (only L^∞). However, the method in [7], making use of the linearity of the problem, gives precise information about the behavior of the eigenfunctions u_k^ε.

Moreover, in the one dimensional linear case with diffusion coefficient equal to 1, we can simplify the constant and obtain

$$\left|\lambda_k^\varepsilon - \lambda_k\right| \leq \frac{\|\rho - \bar{\rho}\|_\infty}{(\rho^-)^2} \sqrt{\frac{\rho^+}{\rho^-}} (\pi k)^3 \varepsilon$$

Remark 3.8. In [14,15], Kenig, Lin and Shen studied the linear case in any space dimension (allowing a periodic oscillation diffusion matrix) and prove the bound

$$\left|\lambda_k^\varepsilon - \lambda_k\right| \leq C\varepsilon |\log \varepsilon|^{1+\sigma} \lambda_k^{\frac{3}{2}}.$$

for some $\sigma > 0$ and C depending on σ and k (The authors can get rid off the logarithmic term assuming more regularity on Ω). If we specialize our result to this case, we cannot treat an ε dependence on the operator, we get the same dependence on k but without any regularity assumption on Ω we get the optimal dependence on ε.

4. The one dimensional problem

In this section we consider the following one dimensional problems

$$\begin{cases} -(|u_\varepsilon'|^{p-2}u_\varepsilon')' = \lambda^\varepsilon \rho(\tfrac{x}{\varepsilon})|u_\varepsilon|^{p-2}u_\varepsilon & \text{in } I := (0,1) \\ u_\varepsilon(0) = u_\varepsilon(1) = 0, \end{cases} \tag{4.1}$$

and the limit problem

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda\bar\rho|u|^{p-2}u & \text{in } I \\ u(0) = u(1) = 0, \end{cases} \tag{4.2}$$

where ρ is 1-periodic and $\bar\rho$ is the average of ρ in the unit interval.

In order to prove the rate of convergence, let us assume first that $\bar\rho = 0$ and let $R(x) = \int_0^x \rho(t)\, dt$. Then R is 1-periodic and if we denote $R_\varepsilon(x) = R(\tfrac{x}{\varepsilon})$, we get

$$\varepsilon R_\varepsilon'(x) = \rho\left(\frac{x}{\varepsilon}\right).$$

Hence, if $v \in W_0^{1,1}(I)$,

$$\int_0^1 \rho\left(\frac{x}{\varepsilon}\right) v(x)\, dx = \varepsilon \int_0^1 R_\varepsilon'(x) v(x)\, dx = -\varepsilon \int_0^1 R_\varepsilon(x) v'(x)\, dx.$$

So

$$\left| \int_0^1 \rho\left(\frac{x}{\varepsilon}\right) v(x)\, dx \right| \leq \varepsilon \|R\|_\infty \int_0^1 |v'(x)|\, dx.$$

It is easy to see that

$$\|R\|_\infty \leq \left\|\max\{\rho, 0\}\right\|_1 = \frac{1}{2}\|\rho\|_1,$$

since $\bar\rho = 0$.

We have proved

Lemma 4.1. *Let $v \in W_0^{1,1}(I)$ and $\rho \in L^1(I)$ be such that $\bar\rho := \int_0^1 \rho(x)\, dx = 0$. Then*

$$\left| \int_0^1 \rho\left(\frac{x}{\varepsilon}\right) v(x)\, dx \right| \leq \frac{1}{2}\|\rho\|_1 \varepsilon \|v'\|_1.$$

Then we get the following corollary

Corollary 4.2. *Let* $u \in W_0^{1,p}(I)$ *and* $\rho \in L^1(I)$. *Then*

$$\left| \int_0^1 \left(\rho\left(\frac{x}{\varepsilon}\right) - \bar{\rho} \right) |u(x)|^p \, dx \right| \leq \frac{p}{2} \varepsilon \|\rho - \bar{\rho}\|_1 \|u\|_p^{p-1} \|u'\|_p.$$

Proof. It follows just by noticing that

$$\int_0^1 \left| (|u|^p)' \right| dx = p \int_0^1 |u|^{p-1} |u'| \, dx \leq p \|u\|_p^{p-1} \|u'\|_p$$

and applying Lemma 4.1. □

Now if we argue exactly as in Theorem 3.5 but use Corollary 4.2 instead of Theorem 3.4, we get

$$\left| \lambda_k^\varepsilon - \lambda_k \right| \leq \frac{p}{2} \|\rho - \bar{\rho}\|_1 \frac{\rho_+^{\frac{1}{p}}}{\rho_-} \varepsilon \max\{\lambda_k, \lambda_k^\varepsilon\}^{1 + \frac{1}{p}}.$$

The bound for $\lambda_k, \lambda_k^\varepsilon$ follows directly from Theorem 2.3. In fact,

$$\lambda_k, \lambda_k^\varepsilon \leq \frac{1}{\rho_-} \mu_k = \frac{1}{\rho_-} (\pi_p k)^p.$$

So we have proved:

Theorem 4.3. *The following estimate holds*

$$\left| \lambda_k^\varepsilon - \lambda_k \right| \leq \frac{p}{2} \frac{\|\rho - \bar{\rho}\|_1}{\rho_-^2} \left(\frac{\rho_+}{\rho_-} \right)^{\frac{1}{p}} \varepsilon (\pi_p k)^{p+1}.$$

Remark 4.4. If we replace the unit interval $I = (0,1)$ by $I_\ell = (0,\ell)$ by a simple change of variables, the estimates of Theorem 4.3 are modified as

$$\left| \lambda_k^\varepsilon(I_\ell) - \lambda_k(I_\ell) \right| = \ell^p \left| \lambda_k^\varepsilon(I) - \lambda_k(I) \right|. \tag{4.3}$$

Remark 4.5. This theorem gives almost the same results as Theorem 3.5. Nevertheless, we wish to provide an alternative and simpler proof exploiting the one dimensional nature of the problem.

5. The general one dimensional equation

In this section we consider, for the one dimensional problem the case where an oscillating coefficient in the equation is allowed. i.e., the problem

$$\begin{cases} -(a(\frac{x}{\varepsilon})|u'|^{p-2}u')' = \lambda^\varepsilon \rho(\frac{x}{\varepsilon})|u|^{p-2}u & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{5.1}$$

We will show that this case can be reduced to Theorem 4.3 by a suitable change of variables. In fact, if we define

$$P_\varepsilon(x) = \int\limits_0^x \frac{1}{a_\varepsilon(s)^{1/(p-1)}} ds = \varepsilon \int\limits_0^{x/\varepsilon} \frac{1}{a(s)^{1/(p-1)}} ds = \varepsilon P\left(\frac{x}{\varepsilon}\right)$$

and perform the change of variables

$$(x, u) \to (y, v)$$

where

$$y = P_\varepsilon(x) = \varepsilon P\left(\frac{x}{\varepsilon}\right), \qquad v(y) = u(x).$$

By simple computations we get

$$\begin{cases} -(|\dot v|^{p-2}\dot v)^{\cdot} = \lambda^\varepsilon Q_\varepsilon(y)|v|^{p-2}v, & y \in [0, L_\varepsilon] \\ v(0) = v(L_\varepsilon) = 0 \end{cases}$$

where

$$\cdot = d/dy,$$

with

$$L_\varepsilon = \int\limits_0^1 \frac{1}{a_\varepsilon(s)^{1/(p-1)}} ds \to L = a^{\overline{\frac{-1}{p-1}}},$$

and

$$Q_\varepsilon(y) = a_\varepsilon(x)^{1/(p-1)}\rho_\varepsilon(x)$$
$$= a\left(P^{-1}\left(\frac{y}{\varepsilon}\right)\right)^{1/(p-1)} \rho\left(P^{-1}\left(\frac{y}{\varepsilon}\right)\right)$$
$$= Q\left(\frac{y}{\varepsilon}\right).$$

Observe that Q is an L-periodic function.

Moreover, it is easy to see that

$$|L_\varepsilon - L| \le \varepsilon L \tag{5.2}$$

and that $L_\varepsilon = L$ if $\varepsilon = 1/j$ for some $j \in \mathbb{N}$.

In order to apply Theorem 4.3 we need to rescale to the unit interval. So we define

$$w(z) = v(L_\varepsilon z), \quad z \in I$$

and get

$$\begin{cases} -(|\dot w|^{p-2}\dot w)^{\cdot} = L_\varepsilon^p \lambda^\varepsilon Q_\varepsilon(L_\varepsilon z)|w|^{p-2}w & \text{in } I \\ w(0) = w(1) = 0 \end{cases}$$

So if we denote $\delta = \varepsilon L/L_\varepsilon$, $\mu^\delta = L_\varepsilon^p \lambda^\varepsilon$ and $g(z) = Q(Lz)$, we get that g is a 1-periodic function and that w verifies

$$\begin{cases} -(|\dot{w}|^{p-2}\dot{w})^{\cdot} = \mu^\delta g(\frac{z}{\delta})|w|^{p-2}w & \text{in } I \\ w(0) = w(1) = 0 \end{cases}$$

Now we can apply Theorem 4.3 to the eigenvalues μ^δ to get

$$\left| \mu_k^\delta - \mu_k \right| \leq \frac{p}{2} \frac{\|g - \bar{g}\|_1}{g_-^2} \left(\frac{g_+}{g_-} \right)^{\frac{1}{p}} \delta(\pi_p k)^{p+1}. \tag{5.3}$$

In the case where $\varepsilon = 1/j$ with $j \in \mathbb{N}$ we directly obtain

$$\left| \lambda_k^\varepsilon - \lambda_k \right| \leq \frac{1}{L^p} \frac{p}{2} \frac{\|g - \bar{g}\|_1}{g_-^2} \left(\frac{g_+}{g_-} \right)^{\frac{1}{p}} \varepsilon(\pi_p k)^{p+1}.$$

In the general case, one has to measure the defect between L and L_ε. So,

$$\left| \lambda_k^\varepsilon - \lambda_k \right| \leq \frac{1}{L^p} \left(\left| \mu_k^\delta - \mu_k \right| + \lambda_k^\varepsilon |L_\varepsilon^p - L^p| \right) \leq \frac{1}{L^p} \left(\left| \mu_k^\delta - \mu_k \right| + \frac{\beta}{\rho_-} \pi_p^p k^p |L_\varepsilon^p - L^p| \right). \tag{5.4}$$

From (5.2) it is easy to see that

$$\left| \left(\frac{L_\varepsilon}{L} \right)^p - 1 \right| \leq p(1 + \varepsilon)^{p-1}\varepsilon,$$

so

$$\left| L_\varepsilon^p - L^p \right| = L^p \left| \left(\frac{L_\varepsilon}{L} \right)^p - 1 \right| \leq pL^p(1 + \varepsilon)^{p-1}\varepsilon. \tag{5.5}$$

Finally, using (5.3), (5.4) and (5.5) we obtain:

Theorem 5.1. *Let λ_k^ε be the kth-eigenvalue of*

$$\begin{cases} -(a(\frac{x}{\varepsilon})u')' = \lambda^\varepsilon \rho(\frac{x}{\varepsilon})|u|^{p-2}u & \text{in } I = (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

and let λ_k be the kth-eigenvalue of the homogenized limit problem

$$\begin{cases} -(a_p^*|u'|^{p-2}u')' = \lambda \bar{\rho}|u|^{p-2}u & \text{in } I \\ u(0) = u(1) = 0, \end{cases}$$

where $a_p^* = (\int_0^1 a^{-\frac{1}{p-1}} dt)^{-(p-1)}$.
Then, if $\varepsilon = 1/j$ for some $j \in \mathbb{N}$,

$$\left| \lambda_k^\varepsilon - \lambda_k \right| \leq \frac{1}{L^p} \frac{p}{2} \frac{\|g - \bar{g}\|_1}{g_-^2} \left(\frac{g_+}{g_-} \right)^{\frac{1}{p}} \varepsilon(\pi_p k)^{p+1}$$

and if $\varepsilon \neq 1/j$,

$$\left| \lambda_k^\varepsilon - \lambda_k \right| \leq \frac{1}{L^p} \frac{p}{2} \frac{\|g - \bar{g}\|_1}{g_-^2} \left(\frac{g_+}{g_-} \right)^{\frac{1}{p}} \frac{\varepsilon}{1 - \varepsilon}(\pi_p k)^{p+1} + \frac{\beta}{\rho_-} pL^p(1 + \varepsilon)^{p-1}\varepsilon(\pi_p k)^p.$$

6. Convergence of nodal domains

In this section we prove the following result about the convergence of the nodal sets and of the zeros of the eigenfunctions.

Theorem 6.1. *Let* $(\lambda_k^\varepsilon, u_k^\varepsilon)$ *and* (λ_k, u_k) *be eigenpairs associated to Eqs.* (4.1) *and* (4.2) *respectively. We denote by* $\mathcal{N}_k^\varepsilon$ *and* $\mathcal{N}_k$ *the nodal domains of* u_k^ε *and* u_k *respectively. Then*

$$\left|\mathcal{N}_k^\varepsilon\right| \to |\mathcal{N}_k| \quad as \ \varepsilon \to 0$$

and we have the estimate

$$\left|\left|\mathcal{N}_k^\varepsilon\right|^{-p} - |\mathcal{N}_k|^{-p}\right| \le c\varepsilon\big(k^{p+1} + 1\big)$$

Proof. By using Theorem 3.6, together with (4.3) and the explicit form of the eigenvalues of the limit problem we obtain that

$$\lambda_k^\varepsilon(I) = \lambda_1^\varepsilon\big(\mathcal{N}_k^\varepsilon\big) \le \lambda_1\big(\mathcal{N}_k^\varepsilon\big) + c\big|\mathcal{N}_k^\varepsilon\big|^{p-1}\varepsilon \le \frac{\pi_p^p}{\bar\rho|\mathcal{N}_k^\varepsilon|^p} + c\varepsilon, \tag{6.1}$$

where $\lambda_k^\varepsilon(a, b)$ denotes the $k-$th variational eigenvalue of problem (4.1) posed on the interval (a, b); analogously for $\lambda_k(a, b)$.

Also,

$$\lambda_k^\varepsilon(I) \ge \lambda_k(I) - c\varepsilon k^{p+1} = \frac{k^p \pi_p^p}{\bar\rho} - c\varepsilon k^{p+1}. \tag{6.2}$$

As $w_k(x) = \sin_p(k\pi_p x)$ (see Theorem 2.3: the notation used in the theorem is different from that used here) has k nodal domain in I we must have $|\mathcal{N}_k| = k^{-1}$. Then by (6.1) and (6.2) we get

$$\frac{\pi_p^p}{\bar\rho|\mathcal{N}_k|^p} - c\varepsilon k^{p+1} \le \frac{1}{|\mathcal{N}_k^\varepsilon|^p}\frac{\pi_p^p}{\bar\rho} + c\varepsilon$$

it follows that

$$|\mathcal{N}_k|^{-p} - \left|\mathcal{N}_k^\varepsilon\right|^{-p} \le c\varepsilon\big(k^{p+1} + 1\big). \tag{6.3}$$

Similarly we obtain that

$$\frac{\pi_p^p}{\bar\rho|\mathcal{N}_k|^p} = \lambda_1(\mathcal{N}_k) = \lambda_k(I) \ge \lambda_k^\varepsilon(I) - c\varepsilon k^{p+1} \ge \lambda_1^\varepsilon\big(\mathcal{N}_k^\varepsilon\big) - c\varepsilon k^{p+1}$$

and using again Theorem 3.6 we get

$$\lambda_1^\varepsilon\big(\mathcal{N}_k^\varepsilon\big) \ge \lambda_1\big(\mathcal{N}_k^\varepsilon\big) - c\varepsilon = \frac{\pi_p^p}{\bar\rho|\mathcal{N}_k^\varepsilon|^p} - c\varepsilon$$

it follows that

$$\left|\mathcal{N}_k^\varepsilon\right|^{-p} - |\mathcal{N}_k|^{-p} \le c\varepsilon\big(k^{p+1} + 1\big). \tag{6.4}$$

Combining (6.3) and (6.4) the result follows. □

Finally, as a corollary of Theorem 6.1 we are able to prove the individual convergence of the zeroes of the eigenfunctions of (4.1) to those of the limit problem (4.2).

Corollary 6.2. *Let* $(\lambda_k^\varepsilon, u_k^\varepsilon)$ *and* (λ_k, u_k) *be eigenpairs associated to Eqs.* (4.1) *and* (4.2) *respectively. Denote* x_j^ε *and* x_j, $0 \le j \le k$ *their respective zeroes. Then for each* $1 < j < k$

$$x_j^\varepsilon \to x_j \quad when \ \varepsilon \to 0$$

and

$$\left| x_j^\varepsilon - x_j \right| \le jc\varepsilon \left(k^{p+1} + 1 \right).$$

In particular $x_0^\varepsilon = x_0 = 0$ *and* $x_k^\varepsilon = x_k = 1$ *by the boundary condition.*

Proof. With the notation of Theorem 6.1 we have that $|\mathcal{N}_k^\varepsilon| \to |\mathcal{N}_k|$. For the first pair of nodal domains we get

$$\left| x_1^\varepsilon - x_1 \right| = \left| x_1^\varepsilon - x_0^\varepsilon - x_1 + x_0 \right| = \left| |\mathcal{N}_{k,1}^\varepsilon| - |\mathcal{N}_{k,1}| \right| \le c\varepsilon \left(k^{p+1} + 1 \right)$$

for the second couple

$$\left| \left(x_2^\varepsilon - x_2 \right) - \left(x_1^\varepsilon - x_1 \right) \right| = \left| |\mathcal{N}_{k,2}^\varepsilon| - |\mathcal{N}_{k,2}| \right| \le c\varepsilon \left(k^{p+1} + 1 \right)$$

then

$$\left| x_2^\varepsilon - x_2 \right| \le c\varepsilon \left(k^{p+1} + 1 \right) + \left| x_1^\varepsilon - x_1 \right| \le 2c\varepsilon \left(k^{p+1} + 1 \right).$$

We iterate the reasoning for $j < k$,

$$\left| x_j^\varepsilon - x_j \right| \le jc\varepsilon \left(k^{p+1} + 1 \right)$$

and the proof is complete. $\square$
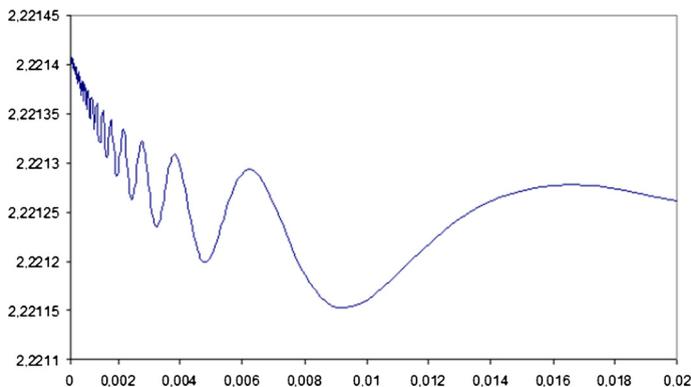
7. Some examples and numerical results

We define the following Prüfer transformation:

$$\begin{cases} \left(\frac{\lambda \rho(x)}{p-1} \right)^{1/p} u(x) = r(x) S_p(\varphi(x)), \\ u'(x) = r(x) C_p(\varphi(x)) \end{cases} \tag{7.1}$$

As in [19], we can show that $r(x)$ and $\varphi(x)$ are continuously differentiable functions, whenever $\rho \in C^1(I)$, satisfying

$$\begin{cases} \varphi'(x) = \left(\frac{\lambda \rho(x)}{p-1} \right)^{\frac{1}{p}} + \frac{1}{p} \frac{\rho'(x)}{\rho(x)} |C_p(\varphi(x))|^{p-2} C_p(\varphi(x)) S_p(\varphi(x)) \\ r'(x) = \frac{1}{p} \frac{\rho'(x)}{\rho(x)} r(x) |S_p(\varphi(x))|^p \end{cases} \tag{7.2}$$

and we obtain that

Fig. 1. The square root of the first eigenvalue as a function of ε when $\rho(x) = 2 + \sin(2\pi x)$.

$$u_k(x) = \left(\frac{\lambda_k \rho(x)}{p-1}\right)^{-1/p} r_k(x) S_p(\varphi_k(x)), \quad k \geq 1$$

is a eigenfunction of problem (2.4) corresponding to λ_k with zero Dirichlet boundary conditions.

We propose the following algorithm to compute the eigenvalues of problem (2.4) based in the fact that the eigenfunction associate to λ_k has k nodal domains in I, so the phase function φ must vary between 0 and $k\pi_p$. It consists in a shooting method combined with a bisection algorithm (a Newton–Raphson version can be implemented too).

```
Let a < λ < b and let τ be the tolerance
Solve the ODE (7.2) and obtain φλ and ρλ
Let w(x) = (p − 1)^{1/p}(λρ(x))^{−1/p}rλ(x)Sp(φλ(x))
Let α = w(1)
while (|α| ≥ τ)
      λ = (a + b)/2
      Solve the ODE (7.2) and obtain φλ and ρλ
      Let w(x) = (p − 1)^{1/p}(λρ(x))^{−1/p}rλ(x)Sp(φλ(x))
      Let β = w(1)
      If (αβ < 0)
            b = (a + b)/2
      else
            a = (a + b)/2
end while
Then λ is the approximation of eigenvalue with error ≤ τ
```

For example, let us consider $\rho(x) = 2 + \sin(2\pi x)$. In this case we obtain that $\bar{\rho} = \int_I 2 + \sin(2\pi x)dx = 2$, and the eigenvalues of the limit problem are given by
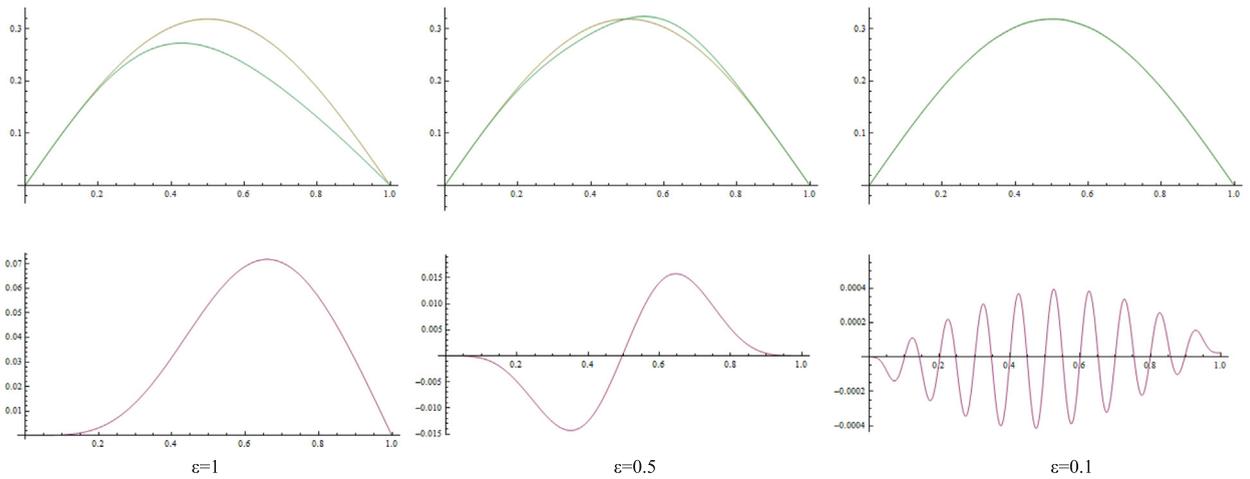
$$\lambda_k^{1/p} = \frac{k\pi_p}{2^{1/p}}.$$

When ε tends to zero the value of λ^ε tends to the limit value λ displaying oscillations.

When $p = 2$ the first limit eigenvalue is $\sqrt{\lambda_1} = \pi/\sqrt{2} \sim 2.221441469$. We see the oscillating behavior when plot $\sqrt{\lambda_1^\varepsilon}$ as function of ε in Fig. 1.
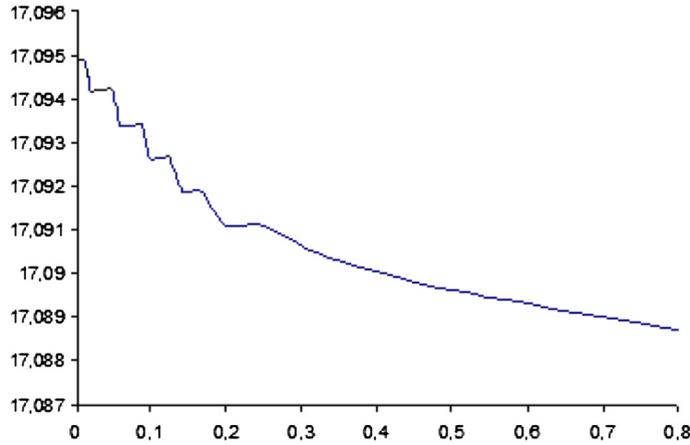
Fig. 2. The square root of the first eigenvalue as a function of ε when $\rho(x) = \frac{1}{2+\sin 2\pi x}$.

$\varepsilon=1$ $\varepsilon=0.5$ $\varepsilon=0.1$

Fig. 3. The first eigenfunctions and the difference between them for different values of ε.

A more complex behavior can be found in Fig. 2, where we considered the weight $\rho(x) = \frac{1}{2+\sin 2\pi x}$. We observe that the sequence tends to

$$\lambda_1 = \pi^2 \Big/ \int_I \frac{1}{2 + \sin 2\pi x} dx = \sqrt{3}\pi \sim 17.09465627.$$

It is not clear why the convergence of the first eigenvalue display the oscillations and the monotonicity observed (although the monotonicity is reversed for the weight $\rho(x) = 2 - \sin 2\pi x$). We believe that some Sturmian type comparison theorem with integral inequalities for the weights (instead of point-wise inequalities as usual) is involved. However, we are not able to prove it, and for higher eigenvalues it is not clear what happens.

Turning now to the eigenfunctions, with the weight $\rho(x) = 2 + \sin(2\pi x)$, the normalized eigenfunction associated to the first eigenvalue of the limit problem is given by $u_1(x) = \pi^{-1}\sin(\pi x)$. Applying the numerical algorithm we obtain that the graph of an eigenfunction associated to the first eigenvalue λ_1^ε intertwine with the graph of $u_1(x)$. When ε decreases, the number of crosses increases, and the amplitude of the difference between them decreases. In Fig. 3 we can observe this behavior and the difference between u_1 and u_1^ε for different values of ε.
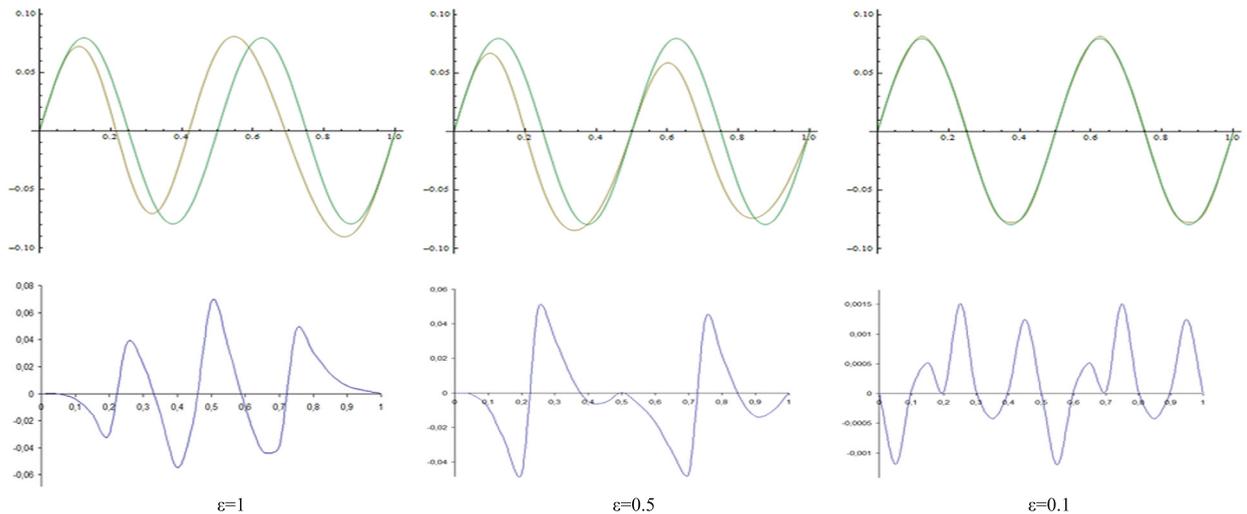
Fig. 4. The fourth eigenfunctions and the difference between them for different values of ε.

To our knowledge, it is not known any result about the number of the oscillations as ε decreases, nor it is known if those oscillations disappear for ε sufficiently small.

The same behavior seems to hold for the higher eigenfunctions, see in Fig. 4 the behavior of the fourth eigenfunction u_4^ε when the parameter ε decrease.

Here, the convergence of the nodal domains and the fact that the restriction of an eigenfunction to one of its nodal domains $\mathcal{N}$ coincides with the first eigenfunction of the problem in $\mathcal{N}$, together with the continuous dependence of the eigenfunctions on the weight and the length of the domain, suggest that the presence or not of oscillations for the higher eigenfunctions must be the same as for the first one. However, the computations show very complex patterns in the oscillations.

Acknowledgments

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