Research Article

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An optimal mass transport approach for limits of eigenvalue problems for the fractional p-Laplacian

Abstract: We find an interpretation using optimal mass transport theory for eigenvalue problems obtained as limits of the eigenvalue problems for the fractional p-Laplacian operators as $p \to +\infty$. We deal both with Dirichlet and Neumann boundary conditions.

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1 Introduction

Our main goal in this paper is to use tools from mass transport theory to study eigenvalue problems that are obtained taking limits as $p \to +\infty$ in eigenvalue problems that involve fractional spaces $W^{s,p}$ (with $0 < s < 1$ and $1 < p \leq +\infty$). We deal both with Dirichlet and Neumann boundary conditions.

Along this paper we let U be a smooth bounded domain in $\mathbb{R}^n$ with $1 < p < +\infty$ and $0 < s < 1$. We also fix a distance $d(\cdot\,,\cdot)$ in $\mathbb{R}^n$ equivalent to the Euclidean one.

Let $\lambda^D_{s,p}$ be the first eigenvalue of the fractional p-Laplacian of order s in U with Dirichlet boundary conditions, that is, let us consider

$$\lambda^D_{s,p} := \inf\Big\{[u]^p_{s,p} : u \in \widetilde{W}^{s,p}(U),\ \int\limits_U |u|^p\,dx = 1\Big\},$$

where

$$[u]^p_{s,p} := \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy$$

is the seminorm of $W^{s,p}(\mathbb{R}^n)$ and

$$\widetilde{W}^{s,p}(U) := \{u \in W^{s,p}(\mathbb{R}^n) : u = 0 \text{ in } \mathbb{R}^n \setminus U\}.$$

For this problem, Lindgren and Lindqvist proved in [19] that

$$\Lambda^D_{s,\infty} := \lim_{p\to+\infty}(\lambda^D_{s,p})^{1/p} = \frac{1}{R^s},$$

where

$$R := \max_{x\in\overline{U}} \operatorname{dist}(x,\partial U) = \max_{x\in\overline{U}}\min_{y\in\partial U}|x-y|.$$

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Moreover, via a subsequence, the eigenfunctions u_p suitably normalized (a minimizer for $\lambda^D_{s,p}$) converge uniformly to a minimizer for $\Lambda^D_{s,\infty}$.

Our first purpose in this work is to relate $\Lambda^D_{s,\infty}$ to an optimal mass transport problem with cost function $c(x,y) = |x-y|^s$. We prove the following result.

Theorem 1.1. *There holds that*

$$\frac{1}{\Lambda^D_{s,\infty}} = \sup_{\mu \in P(\overline{U})} W_s(\mu, P(\partial U)),$$

where

$$W_s(\mu,\nu) := \inf_\pi \int_U \int_U |x-y|^s \, d\pi(x,y).$$

Here, $P(A)$ is the set of probability measures on A and $\pi \in P(\overline{U} \times \overline{U})$ is a measure with marginals μ and ν.

Note that $W_s(\mu,\nu)$ is the total cost when we have to transport the measure μ onto ν using the Euclidean distance to the power s, that is $|x-y|^s$, as the cost for transporting one unit of mass from position x to position y. We refer to [24] and to Section 2 for precise definitions, notation and properties of optimal mass transport theory. Hence, our result says that the eigenvalue $\Lambda^D_{s,\infty}$ is related to the problem of finding a probability measure supported inside U, μ that is far (in terms of the transport cost) from the set of probability measures supported on the boundary ∂U. One easy solution to this problem is the following: take a ball $B_R(x_0)$ with maximum radius R inside U and let $y_0 \in \partial U \cap \partial B_R(x_0)$ (there exists such y_0 due to the maximality of R). Then, $\mu = \delta_{x_0}$ (with $\nu = \delta_{y_0}$) solves $\sup_{\mu\in P(\overline{U})} W_s(\mu, P(\partial U))$. Observe that from Theorem 1.1 we can recover that $\Lambda^D_{s,\infty} = 1/R^s$.

Now, let us turn our attention to the case of the first nontrivial eigenvalue for Neumann boundary conditions, that is, let us consider

$$\lambda^N_{s,p} := \inf\{[\![u]\!]^p_{s,p} : u \in \mathcal{C}\},$$

where

$$[\![u]\!]^p_{s,p} := \int_U \int_U \frac{|u(x)-u(y)|^p}{d(x,y)^{n+sp}} \, dx\, dy \tag{1.1}$$

and

$$\mathcal{C} := \Big\{ u \in W^{s,p}(U) : \|u\|_{L^p(U)} = 1, \int_U |u|^{p-2}u \, dx = 0 \Big\}.$$

For this problem, in the case $d(x,y) = |x-y|$, Del Pezzo and Salort proved in [8] that

$$\Lambda^N_{s,\infty} := \lim_{p\to+\infty} (\lambda^N_{s,p})^{1/p} = \frac{2}{(\operatorname{diam}(U))^s},$$

where $\operatorname{diam}(U)$ is the extrinsic diameter, that is,

$$\operatorname{diam}(U) := \max_{x,y\in\overline{U}} |x-y|.$$

Their proof actually extends to the case in which we consider $[\![u]\!]^p_{s,p}$ with $d(x,y)$ any distance as above (for instance, for the geodesic distance in U). In this case, it holds that

$$\Lambda^N_{s,\infty} = \lim_{p\to+\infty} (\lambda^N_{s,p})^{1/p} = \frac{2}{(\operatorname{diam}_d(U))^s}, \tag{1.2}$$

where $\operatorname{diam}_d(U)$ is the diameter of U according to d, that is,

$$\operatorname{diam}_d(U) = \max_{x,y\in\overline{U}} d(x,y).$$

Moreover, as happens for the Dirichlet problem, via a subsequence, the normalized eigenfunctions u_p (a minimizer for $\lambda^N_{s,p}$) converge uniformly to a minimizer for $\Lambda^N_{s,\infty}$.

In order to introduce the mass transport interpretation, we need the following notation. We denote by $M(\overline{U})$ the space of finite Borel measures over $\overline{U}$. Given $\sigma \in M(\overline{U})$, we denote its positive and negative part by σ^+ and σ^-, so that $\sigma = \sigma^+ - \sigma^-$ and $|\sigma| = \sigma^+ + \sigma^-$. Then, we have the following theorem.

Theorem 1.2. *There holds*

$$\frac{2}{\Lambda^N_{s,\infty}} = \max\{W_s(\sigma^+, \sigma^-) : \sigma \in M(\overline{U}),\ \sigma^+(\overline{U}) = \sigma^-(\overline{U}) = 1\}, \tag{1.3}$$

where W_s is as in Theorem 1.1.

Here, we relate $\Lambda^N_{s,\infty}$ to the problem of finding two probability measures σ^+ and σ^- supported in $\overline{U}$ such that the cost of transporting one into the other is maximized. To obtain a solution to this problem, one can argue as follows: take two points x_0 and y_0 in $\overline{U}$ that realize the diameter, that is, we have $d(x_0, y_0) = \operatorname{diam}_d(U)$. Then, take $\sigma^+ = \delta_{x_0}$ and $\sigma^- = \delta_{y_0}$ as a solution to $\max\{W_s(\sigma^+, \sigma^-) : \sigma \in M(\overline{U}),\ \sigma^+(\overline{U}) = \sigma^-(\overline{U}) = 1\}$. Note that we can recover (1.2) from Theorem 1.2.

A different concept of Neumann boundary condition for fractional operators was recently introduced in [10]. More precisely, for the fractional p-Laplacian $(-\Delta)^s_p$ given by

$$(-\Delta)^s_p u(x) = \text{P.V.} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{d(x, y)^{n+sp}}\, dy,$$

where the symbol P.V. stands for the principal value of the integral, we consider the nonlocal nonlinear fractional normal derivative

$$\mathcal{N}_{s,p} u(x) = \int_U \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{d(x, y)^{n+sp}}\, dy, \quad x \in \mathbb{R}^n \setminus \overline{U}.$$

Associated with this operator, we consider the eigenvalue problems

$$\begin{cases} (-\Delta)^s_p u = \lambda |u|^{p-2} u & \text{in } U, \\ \mathcal{N}_{s,p} u = 0 & \text{in } \mathbb{R}^n \setminus \overline{U}. \end{cases} \tag{1.4}$$

Before stating our main result concerning these problems, we need to introduce some notation. Let $\mathcal{W}^{s,p}(U)$ be the set of measurable functions with finite norm

$$\|u\|^p_{\mathcal{W}^{s,p}(U)} := \|u\|^p_{L^p(U)} + \mathcal{H}_{s,p}(u),$$

where

$$\mathcal{H}_{s,p}(u) := \iint_{\mathbb{R}^{2n} \setminus (U^c)^2} \frac{|u(x) - u(y)|^p}{d(x, y)^{n+ps}}\, dx\, dy$$

and $(U^c)^2 = U^c \times U^c$. Let us also introduce

$$\mathcal{H}_{s,\infty}(u) := \sup\Big\{\frac{|u(x) - u(y)|}{d(x, y)^s} : (x, y) \in \mathbb{R}^{2n} \setminus (U^c)^2\Big\}.$$

Then, we have the following result for (1.4).

Theorem 1.3. *The first nonzero eigenvalue of* (1.4) *is given by*

$$\lambda_{s,p} = \inf\Big\{\frac{\mathcal{H}_{s,p}(v)}{2\|v\|^p_{L^p(U)}} : v \in \mathcal{W}^{s,p}(U) \setminus \{0\},\ \int_U |v|^{p-2} v\, dx = 0\Big\}.$$

Concerning the limit as $p \to +\infty$ of these eigenvalues, we have

$$\lim_{p \to +\infty} (\lambda_{s,p})^{1/p} = \frac{2}{(\operatorname{diam}_d(U))^s} = \Lambda_{s,\infty} := \inf\Big\{\frac{\mathcal{H}_{s,\infty}(u)}{\|u\|_{L^\infty(U)}} : u \in \mathcal{A}\Big\},$$

where

$$\mathcal{A} := \Big\{v \in \mathcal{W}^{s,\infty}(U) \setminus \{0\} : \sup_{x \in U} u(x) + \inf_{x \in U} u(x) = 0\Big\}.$$

Moreover, if u_p is a minimizer of $\lambda_{s,p}$ normalized by $\|u_p\|_{L^p(U)} = 1$, then, up to a subsequence, u_p converges in $C(\overline{U})$ to some minimizer $u_\infty \in W^{s,\infty}(U)$ of $\Lambda^N_{s,\infty}$.

Note that since the limit $\Lambda_{s,\infty}$ of $(\lambda_{s,p})^{1/p}$ coincides with $\Lambda^N_{s,\infty}$ (given in (1.2)), we get the same interpretation in terms of optimal mass transportation given in Theorem 1.2.

To end this introduction, let us briefly comment on previous results. The limit as $p \to +\infty$ of the first eigenvalue λ^D_p of the usual local p-Laplacian with Dirichlet boundary condition was studied in [15, 16] (see also [3] for an anisotropic version). In those papers, the authors prove that

$$\lambda^D_\infty := \lim_{p\to+\infty} (\lambda^D_p)^{1/p} = \inf\left\{ \frac{\|\nabla v\|_{L^\infty(U)}}{\|v\|_{L^\infty(U)}} : v \in W^{1,\infty}_0(\Omega) \right\} = \frac{1}{R},$$

where, as before, R is the largest possible radius of a ball contained in U. In addition, the authors show the existence of extremals, that is, functions where the above infimum is attained. These extremals can be constructed taking the limit as $p \to +\infty$ in the eigenfunctions of the p-Laplacian eigenvalue problems (see [16]) and are viscosity solutions of the eigenvalue problem (called the infinity eigenvalue problem in the literature)

$$\begin{cases} \min\{|Du| - \lambda^D_\infty u,\ \Delta_\infty u\} = 0 & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

The limit operator Δ_∞ that appears here is the ∞-Laplacian given by $\Delta_\infty u = -\langle D^2 u Du, Du\rangle$. Remark that solutions to $\Delta_p v_p = 0$ with Dirichlet data $v_p = f$ on ∂U converge as $p \to +\infty$ to the viscosity solution to $\Delta_\infty v = 0$ with $v = f$ on ∂U (see [2, 4, 6]). This operator appears naturally when one considers absolutely minimizing Lipschitz extensions in U of a boundary data f (see [1, 2, 14]).

Recently in [5], the authors related λ^D_∞ to the Monge–Kantorovich distance W_1. Recall that the Monge–Kantorovich distance $W_1(\mu, \nu)$ between two probability measures μ and ν over $\overline{U}$ is defined by

$$W_1(\mu, \nu) := \max\left\{ \int_U v\,(d\mu - d\nu) : v \in W^{1,\infty}(U),\ \|\nabla v\|_{L^\infty(U)} \le 1 \right\}. \tag{1.5}$$

In [5], it was proved that

$$\frac{1}{\lambda^D_\infty} = \sup_{\mu\in P(U)} W_1(\mu, P(\partial U)).$$

Notice that this result is the analogue to Theorem 1.1 in the local case.

For the Neumann problem for the local p-Laplacian, we refer to [12, 23], where the authors prove the local analogue to Theorem 1.2. In this local case, the distance that appears in the limit is the geodesic distance inside U. This is in contrast to the nonlocal case studied here, where we can consider any distance d equivalent to the Euclidean one (see (1.1)).

For references concerning nonlocal fractional problems, we refer to [10, 11, 17, 19–22] and the references therein. For limits as $p \to +\infty$ in nonlocal p-Laplacian problems and their relation to optimal mass transport, we refer to [17] (note that eigenvalue problems were not considered in [17]).

The case of a Steklov boundary condition has also been investigated recently. Indeed, the authors in [13] (see also [18] for a slightly different problem) studied the behavior as $p \to +\infty$ of the so-called variational eigenvalues $\lambda^S_{k,p}$, $k \ge 1$, of the p-Laplacian with a Steklov boundary condition. In particular, they proved that

$$\lim_{p\to+\infty} (\lambda^S_{1,p})^{1/p} = 1$$

and

$$\lambda^S_{2,\infty} := \lim_{p\to+\infty} (\lambda^S_{2,p})^{1/p} = \frac{2}{\operatorname{diam}(U)},$$

and also identified the limit variational problem defining $\lambda^S_{2,\infty}$.

The present paper is organized as follows. In Section 2, we collect some preliminary results concerning optimal mass transport with cost $d(x, y)^s$ and, in particular, we provide a statement of the Kantorovich duality result that will be used in the proofs of our results. In Section 3 we deal with the Dirichlet problem and prove Theorem 1.1. In Section 4, we study the Neumann case (Theorem 1.2). Finally, in Section 5, we deal with problem (1.4) and we prove Theorem 1.3.

2 Kantorovich duality for the cost $c(x, y) = d(x, y)^s$

In this section, we follow [24]. We first recall the definitions of c-concavity and c-transform.

Definition 2.1 ([24, Definitions 5.2 and 5.7]). Let X, Y be two sets and $c : X \times Y \to \mathbb{R} \cup \{+\infty\}$. A function $\psi : X \to \mathbb{R} \cup \{+\infty\}$ is said to be c-convex if $\psi \not\equiv +\infty$ and there exists $\phi : Y \to \mathbb{R} \cup \{\pm\infty\}$ such that

$$\psi(x) = \sup_{y \in Y} \phi(y) - c(x, y) \quad \text{for all } x \in X. \tag{2.1}$$

Its c-transform is the function ψ^c defined by

$$\psi^c(y) = \inf_{x \in X} \psi(x) + c(x, y) \quad \text{for all } y \in Y.$$

A function $\phi : Y \to \mathbb{R} \cup \{-\infty\}$ is c-concave if $\phi \not\equiv -\infty$ and $\phi = \psi^c$ for some function $\psi : X \to \mathbb{R} \cup \{\pm\infty\}$. Then, its c-transform ϕ^c is

$$\phi^c(x) = \sup_{y \in Y} \phi(y) - c(x, y) \quad \text{for all } x \in X.$$

We have the following proposition.

Proposition 2.2 ([24, Proposition 5.8]). *For any $\psi : X \to \mathbb{R} \cup \{+\infty\}$, there holds $\psi^c = \psi^{ccc}$ and ψ is c-convex if and only if $\psi = \psi^{cc}$.*

In the case where the cost function is $c(x, y) = d(x, y)^s$, we have the following characterization of c-convex functions.

Lemma 2.3. *Let $c(x, y) = d(x, y)^s$ and $X = Y = \overline{U}$. Any c-convex function ψ satisfies $\psi^c = \psi$ and*

$$|\psi(x) - \psi(\tilde{x})| \le d(x, \tilde{x})^s \quad \textit{for all } x, \tilde{x} \in \overline{U}. \tag{2.2}$$

Proof. Notice that

$$\psi^c(y) = \inf_{x \in \overline{U}} \psi(x) + d(x, y)^s \le \psi(y)$$

and that the opposite inequality holds if (2.2) holds. We now verify (2.2). Let $\phi : \overline{U} \to \mathbb{R} \cup \{\pm\infty\}$ such that $\psi = \phi^c$ as in (2.1). Since $s \in (0, 1)$, we have $d(x, y)^s \le d(x, \tilde{x})^s + d(y, \tilde{x})^s$ for any $x, \tilde{x}, y \in \overline{U}$. It follows that

$$\psi(x) = \phi^c(x) = \sup_{y \in \overline{U}} \phi(y) - d(x, y)^s \ge \sup_{y \in \overline{U}} \phi(y) - d(y, \tilde{x})^s - d(x, \tilde{x})^s = \psi(\tilde{x}) - d(x, \tilde{x})^s,$$

that is, $\psi(\tilde{x}) - \psi(x) \le d(x, \tilde{x})^s$. The opposite inequality holds as well by switching x and $\tilde{x}$. As a result, we get that (2.2) holds. □

We recall the following result, see [24, Theorem 5.9].

Theorem 2.4. *Let (X, μ) and (Y, ν) be two Polish probability spaces (that is, metric, complete and separable) and let $c : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function such that*

$$c(x, y) \ge a(x) + b(y) \quad \textit{for all } (x, y) \in X \times Y$$

for some real-valued upper semicontinuous functions $a \in L^1(\mu)$ and $b \in L^1(\nu)$. Then, letting

$$J(\phi, \psi) := \int_Y \phi \, d\nu - \int_X \psi \, d\mu,$$

we have

$$W_c(\mu, \nu) := \min_{\pi \in \Pi(\mu,\nu)} \int_{X \times Y} c(x, y) \, d\pi(x, y) = \sup_{(\psi,\phi) \in L^1(\mu) \times L^1(\nu),\ \phi - \psi \le c} J(\phi, \psi) = \sup_{\psi \in L^1(\mu)} J(\psi^c, \psi)$$

and in the above sup, *one might as well impose ψ to be c-convex. Moreover, if c is real-valued, $W_c(\mu, \nu) < \infty$ and*

$$c(x, y) \le c_X(x) + c_Y(y) \quad \textit{for all } (x, y) \in X \times Y$$

for some $c_X \in L^1(\nu)$ and $c_Y \in L^1(\mu)$, then the above sup *is a* max *and one might as well impose ψ to be c-convex.*

In the particular case $c(x,y) = d(x,y)^s$, $X = Y = \overline{U}$ with U bounded, we obtain in view of Lemma 2.3 the following result.

Theorem 2.5. *For any $\mu, \nu \in P(\overline{U})$, there holds*

$$\min_{\pi\in\Pi(\mu,\nu)} \int_{\overline{U}\times\overline{U}} d(x,y)^s \, d\pi(x,y) = \max_{|\psi(x)-\psi(y)|\le d(x,y)^s} \int_{\overline{U}} \psi \, d\nu - \int_{\overline{U}} \psi \, d\mu.$$

Proof. In view of Lemma 2.3 and the previous theorem, we can write that

$$W_c(\mu,\nu) = \max_{\psi\in L^1(\mu)\ c\text{-convex}} J(\psi^c,\psi) \le \max_{|\psi(x)-\psi(y)|\le d(x,y)^s} J(\psi,\psi) \le \sup_{\substack{(\psi,\phi)\in L^1(\mu)\times L^1(\nu)\\ \phi-\psi\le d(x,y)^s}} J(\phi,\psi) = W_c(\mu,\nu)$$

from which we deduce the result. □

3 The Dirichlet case

In this section, we borrow ideas from [5]. Let us consider the functionals $G_p, G_\infty : C(\overline{U}) \times M(\overline{U}) \to \mathbb{R} \cup \{+\infty\}$ given by

$$G_p(v,\sigma) = \begin{cases} -\int_U v\sigma \, dx & \text{if } \sigma \in L^{p'}(U),\ \|\sigma\|_{L^{p'}(U)} \le 1,\ \text{and } v \in \widetilde{W}^{s,p}(U),\ [v]_{s,p} \le (\lambda_{s,p}^D)^{1/p}, \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$G_\infty(v,\sigma) = \begin{cases} -\int_U v \, d\sigma & \text{if } \sigma \in M(\overline{U}),\ |\sigma|(U) \le 1,\ \text{and } v \in \widetilde{W}^{s,\infty}(U),\ |v(x)-v(y)| \le \Lambda_{s,\infty}^D |x-y|^s, \\ +\infty & \text{otherwise.} \end{cases}$$

We consider the weak convergence of measures in the space $M(\overline{U})$ and the uniform convergence in the space $C(\overline{U})$.

First, we have that G_∞ is the limit of G_p as $p \to +\infty$ in the Γ-limit sense (we refer to [7] for the definition of Γ-convergence).

Lemma 3.1. *The functionals G_p Γ-converge as $p \to +\infty$ to G_∞.*

Proof. It follows as in [5]. □

Now, we let $f_p : \mathbb{R}^n \to \mathbb{R}$ be defined as

$$f_p(x) := (u_p(x))^{p-1},$$

where u_p is a nonnegative eigenfunction associated to $\lambda_{s,p}^D(U)$ such that $\|u_p\|_{L^p(U)} = 1$. When we consider f_p as an element of $M(\overline{U})$ together with u_p, we obtain a minimizer for G_p. The proof of this fact is immediate.

Lemma 3.2. *The pair (f_p, u_p) minimizes G_p in $C(\overline{U}) \times M(\overline{U})$ with*

$$G_p(f_p, u_p) = -1.$$

Now, let us show that we can extract a subsequence $p_n \to +\infty$ such that f_{p_n} and u_{p_n} converge.

Lemma 3.3. *There exists a sequence $p_n \to +\infty$ such that*

$$u_{p_n} \to u_\infty$$

uniformly in $\mathbb{R}^n$. This limit u_∞ verifies

$$|u_\infty(x) - u_\infty(y)| \le \Lambda_{s,\infty}^D |x-y|^s, \quad x, y \in \mathbb{R}^n.$$

Moreover, we have

$$f_{p_n} \overset{*}{\rightharpoonup} f_\infty$$

weakly-$*$ in $M(\overline{U})$ and f_∞ is a nonnegative measure that verifies $f_\infty(\overline{U}) \le 1$.

Proof. The convergence of u_p, via a subsequence, is contained in [19]. Concerning f_{p_n}, the conclusion follows from the inequality

$$\int_U f_p\,dx \le \Big(\int_U (u_p)^p\,dx\Big)^{(p-1)/p} |U|^{1/p} = |U|^{1/p} \tag{3.1}$$

that implies that f_p is bounded in $M(\overline{U})$ and, hence, we can extract a sequence $p_n \to +\infty$ such that $f_{p_n} \overset{*}{\rightharpoonup} f_\infty$ weakly-$*$ in $M(\overline{U})$. The fact that the limit f_∞ is a nonnegative measure that verifies $f_\infty(\overline{U}) \le 1$ also follows from (3.1). □

We obtain the following corollary from the main property of Γ-convergence.

Corollary 3.4. *The pair (f_∞, u_∞) minimizes G_∞ with*

$$G_\infty(f_\infty, u_\infty) = -1.$$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. As (f_∞, u_∞) minimizes G_∞, we obtain that

$$\Big(f_\infty, \frac{u_\infty}{\Lambda^D_{\infty,s}}\Big)$$

minimizes

$$-\int_U v\,d\sigma$$

with (v,σ) belonging to

$$A := \{(v,\sigma) \in \widetilde{W}^{s,\infty}(U) \times M(\overline{U}) : |\sigma|(U) \le 1,\ |v(x) - v(y)| \le d(x,y)^s\}.$$

Then,

$$\frac{1}{\Lambda^D_{s,\infty}} = \frac{1}{\Lambda^D_{s,\infty}} \int_U u_\infty\,df_\infty = \max_{(v,\sigma)\in A} \int_U v\,d\sigma = \max_{\mu\in P(\overline{U})} \max_{|w(x)-w(y)|\le d(x,y)^s} \int_U w\,d\mu = \max_{\mu\in P(\overline{U})} W_s(\mu, P(\partial U))$$

as we wanted to show. □

4 The Neumann case

Again, we follow ideas from [5] (see also [23]). Let u_p be an extremal for $\lambda^N_{p,s}$ (that is, a minimizer for (1.1)) normalized by $\|u_p\|_{L^p(U)} = 1$. Then, $f_p := |u_p|^{p-2}u_p \in L^{p'}(U)$ satisfies

$$\|f_p\|_{L^{p'}(U)} = 1 \quad \text{and} \quad \int_U f_p\,dx = 0, \tag{4.1}$$

where $p' = p/(p-1)$. The first step consists in extracting from $\{f_p\}_{p>1}$ a subsequence converging weakly to some measure $f_\infty \in M(\overline{U})$, the weak convergence meaning here that

$$\lim_{p\to+\infty} \int_{\overline{U}} \phi f_p\,dx = \int_{\overline{U}} \phi\,df_\infty$$

for any $\phi \in C(\overline{U})$.

Lemma 4.1. *Up to a subsequence, the measures f_p converge weakly in measure in $\overline{U}$ to some measure f_∞ supported in $\overline{U}$ satisfying*

$$f_\infty(\overline{U}) = 0 \quad \textit{and} \quad |f_\infty|(\overline{U}) = 1. \tag{4.2}$$

Proof. We claim that

$$\lim_{p\to+\infty} \int_U |f_p|\, dx = 1. \tag{4.3}$$

First, in view of (4.1), we have that

$$\int_U |f_p|\, dx \le \|f_p\|_{L^{p'}(U)} |U|^{1-1/p'} = |U|^{1-1/p'} \to 1 \quad \text{as } p \to +\infty$$

and then, recalling that $u_p \to u$ in $C(\overline{U})$ with $\|u\|_{L^\infty(U)} = 1$,

$$1 = \int_U u_p f_p\, dx \le \|u_p\|_{L^\infty(U)} \|f_p\|_{L^1(U)} = (1 + o(1)) \|f_p\|_{L^1(U)}.$$

In particular, it follows that the measures $|f_p|$ are bounded in $M(\overline{U})$ independently of p. Since $\overline{U}$ is compact, we can then extract from this sequence a subsequence converging weakly to some measure $f_\infty \in M(\overline{U})$. Passing to the limit in (4.1) and (4.3) gives (4.2). □

Consider the functionals $G_p, G_\infty : C(\overline{U}) \times M(\overline{U}) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$G_p(v, \sigma) = \begin{cases} -\int_U v\sigma\, dx & \text{if } \sigma \in L^{p'}(U),\ \|\sigma\|_{L^{p'}(U)} \le 1,\ \int_U \sigma\, dx = 0, \text{ and } v \in W^{s,p}(U),\ [\![v]\!]_{s,p} \le (\lambda_{p,s}^N)^{1/p}, \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$G_\infty(v, \sigma) = \begin{cases} -\int_U v\, d\sigma, & \text{if } \sigma \in M(\overline{U}),\ |\sigma|(\overline{U}) \le 1,\ \sigma(\overline{U}) = 0, \text{ and } v \in W^{s,\infty}(U),\ |v(x) - v(y)| \le \Lambda_{\infty,s}^N d(x, y)^s, \\ +\infty & \text{otherwise.} \end{cases}$$

Remark that these functionals are similar to the ones considered for the Dirichlet case but the spaces involved are different. In fact, here we consider $W^{s,p}(U)$ instead of $\widetilde{W}^{s,p}(U)$ (that encodes the fact that we are considering functions that vanish outside U when dealing with the Dirichlet problem).

As for the Dirichlet case, we can prove as in [5, 23] that G_∞ is the limit of G_p in the sense of Γ-convergence.

Lemma 4.2. *The functionals G_p converge in the sense of Γ-convergence to G_∞.*

The proof is similar to that of [5, Proposition 3.7] and hence we omit it. As a corollary, we obtain the following lemma.

Lemma 4.3. *Let u_p be an extremal for $\lambda_{p,s}^N$. Then, (u_p, f_p) is a minimizer for G_p and any limit (u_∞, f_∞) along a subsequence $p_j \to +\infty$ is a minimizer for G_∞ with*

$$G_\infty(u_\infty, f_\infty) = \lim_{p\to+\infty} G_p(u_p, f_p) = -1.$$

Proof. Notice that the pair (u_p, f_p) is a minimizer of G_p. Indeed, given a pair (v, σ) admissible for G_p, take $\bar{v} \in \mathbb{R}$ such that

$$\int_U |v - \bar{v}|^{p-2}(v - \bar{v})\, dx = 0.$$

Then, recalling that $\int_U \sigma\, dx = 0$ and the definition of $\lambda_{p,s}^N$, we have

$$G_p(v, \sigma) = -\int_U (v - \bar{v})\sigma\, dx \ge -\|v - \bar{v}\|_{L^p(U)} \|\sigma\|_{L^{p'}(U)} \ge -(\lambda_{p,s}^N)^{-1/p} [\![v - \bar{v}]\!]_{s,p} \ge -1 = G_p(u_p, f_p).$$

Moreover, $(u_p, f_p) \to (u_\infty, f_\infty)$ along a sequence $p_j \to +\infty$. Then, it follows that

$$\liminf_{p\to+\infty}(\inf\ G_p) = \liminf_{p\to+\infty} G_p(u_p, f_p) \geq G_\infty(u_\infty, f_\infty) \geq \inf_B G_\infty,$$

where B is the set of all pairs $(v, \sigma) \in W^{s,\infty}(U) \times M(\overline{U})$ such that

$$|\sigma|(\overline{U}) \leq 1, \quad \sigma(\overline{U}) = 0 \quad \text{and} \quad |v(x) - v(y)| \leq \Lambda^N_{\infty,s} d(x,y)^s.$$

Moreover, the lim sup property implies that

$$\limsup_{p\to+\infty}\Big(\inf_B\ G_p\Big) \leq \inf_B\ G_\infty.$$

Hence,

$$\lim_{p\to+\infty}\ \inf_B G_p = \lim_{p\to\infty} G_p(u_p, f_p) = G_\infty(u_\infty, f_\infty) = \inf_B G_\infty. \qquad \square$$

We can now relate $\Lambda^N_{s,\infty}$ to W_s. Recall that if $\sigma \in M(\overline{U})$, then $\sigma^\pm \in M(\overline{U})$ denote the positive and negative parts of σ. In particular, $\sigma = \sigma^+ - \sigma^-$ and $|\sigma| = \sigma^+ + \sigma^-$.

Proof of Theorem 1.2. The conditions $\sigma(\overline{U}) = 0$ and $|\sigma|(\overline{U}) = 1$ are equivalent to

$$\sigma^+(\overline{U}) = \sigma^-(\overline{U}) = \frac{1}{2}.$$

We can therefore rewrite the fact that the pair (u_∞, f_∞) is a minimizer of G_∞ as

$$1 = \max_{\sigma\in M_{1/2}}\ \max_{v\in F_{\Lambda^N_{s,\infty}}} \int_U v\, d(\sigma^+ - \sigma^-),$$

where

$$M_t = \{\sigma \in M(\overline{U}) : \sigma^+(U) = \sigma^+(U) = t\}$$

and

$$F_R = \{v \in W^{s,\infty}(U) : |v(x) - v(y)| \leq R\, d(x,y)\},$$

that is,

$$\frac{2}{\Lambda^N_{\infty,s}} = \max_{\sigma\in M_1}\max_{v\in F_1} \int_U v\, d(\sigma^+ - \sigma^-).$$

Then, we obtain the conclusion (1.3), recalling the definition of W_s given by (1.5). $\square$

5 Eigenvalue problems with a different Neumann boundary condition

In this section, we prove Theorem 1.3. For this purpose, we first present some previous results.

Theorem 5.1. *The spaces*

$$\mathcal{W}^{s,p}(U) := \{u : \mathbb{R}^n \to \mathbb{R}\ \textit{measurable} \mid \|u\|^p_{L^p(U)} + \mathcal{H}_{s,p}(u) < +\infty\}$$

and

$$\mathcal{W}^{s,\infty}(U) := \{u : \mathbb{R}^n \to \mathbb{R}\ \textit{measurable} \mid \|u\|_{L^\infty(U)} + \mathcal{H}_{s,\infty}(u) < +\infty\}$$

are Banach spaces with the norms

$$\|u\|^p_{\mathcal{W}^{s,p}(U)} := \|u\|^p_{L^p(U)} + \mathcal{H}_{s,p}(u)$$

and

$$\|u\|_{\mathcal{W}^{s,\infty}(U)} := \|u\|_{L^\infty(U)} + \mathcal{H}_{s,\infty}(u),$$

respectively.

The proof follows exactly as in the proof of [10, Proposition 3.1].

Remark 5.2. It holds that $\mathcal{W}^{s,p}(U) \subset W^{s,p}(U)$.

Remark 5.3. The operator $I : \mathcal{W}^{s,p}(U) \to E = L^p(U) \times L^p(\mathbb{R}^{2n} \setminus (U^c)^2)$ given by

$$I(u) := \left(u, \frac{u(x) - u(y)}{d(x,y)^{(n/p)+s}}\right)$$

is an isometry. Then, $I(\mathcal{W}^{s,p}(U))$ is a closed subspace of E due to the fact that $\mathcal{W}^{s,p}(U)$ is a Banach space. Hence, $I(\mathcal{W}^{s,p}(U))$ is reflexive since E is reflexive. Then, $\mathcal{W}^{s,p}(U)$ is reflexive.

Following the proofs of [10, Lemma 3.2 and Lemma 3.7], we have the following result.

Lemma 5.4. *Let u and v be bounded C^2 functions in $\mathbb{R}^n$. Then, the following formulae hold.*

- *Divergence theorem*

$$\int_U (-\Delta)_p^s u(x)\,dx = -\int_{\mathbb{R}\setminus U} \mathcal{N}_{s,p} u(x)\,dx.$$

- *Integration by parts formula*

$$\frac{1}{2}\mathcal{H}_{s,p}(u,v) = \int_U v(x)(-\Delta)_p^s u(x)\,dx + \int_{\mathbb{R}\setminus U} v(x)\mathcal{N}_{s,p} u(x)\,dx,$$

  where

$$\mathcal{H}_{s,p}(u,v) := \iint_{\mathbb{R}^{2n}\setminus(U^c)^2} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{d(x,y)^{n+ps}}\,dx\,dy.$$

This result leads us to the following definition.

Definition 5.5. A function $u \in \mathcal{W}^{s,p}(U)$ is a weak solution of (1.4) if

$$\frac{1}{2}\mathcal{H}_{s,p}(u,v) = \lambda \int_U |u|^{p-2} uv\,dx \tag{5.1}$$

for all $v \in \mathcal{W}^{s,p}(U)$.

In this context we have the following definition.

Definition 5.6. We say that λ is a fractional Neumann p-eigenvalue provided there exists a nontrivial weak solution $u \in \mathcal{W}^{s,p}(U)$ of (1.4). The function u is a corresponding eigenfunction.

Let us observe the following: if $\lambda > 0$ is an eigenvalue and u is an eigenfunction associated to λ, then, taking $v \equiv 1$ as a test function in (5.1), we have

$$\int_U |u|^{p-2} u\,dx = 0.$$

In fact, we have that $\lambda = 0$ is the first eigenvalue of our problem.

Lemma 5.7. *It holds that $\lambda = 0$ is an eigenvalue of* (1.4) *(with $u = 1$ as eigenfunction) and it is isolated and simple.*

Proof. Let u be an eigenfunction corresponding to $\lambda = 0$ in problem (1.4). From (5.1), taking $v = u$ as a test function, we obtain that u is constant in U.

Now, if we have a sequence of eigenvalues $\lambda_k \to 0$, then the corresponding eigenfunctions u_k, normalized by $\|u_k\|_{L^p(U)} = 1$, converge to some u. It is not difficult to show that u is an eigenfunction corresponding to $\lambda = 0$ (consequently, $u \equiv$ constant) with $\|u\|_{L^p(U)} = 1$ and $\int_U |u|^{p-2}u\,dx = 0$, a contradiction that shows that $\lambda = 0$ is an isolated eigenvalue. □

Thus, the existence of the first nonzero eigenvalue of (1.4) is related to the problem of minimizing the nonlocal quotient

$$\frac{\mathcal{H}_{s,p}(v)}{2\|v\|_{L^p(U)}^p}$$

among all functions $v \in \mathcal{W}^{s,p}(U) \setminus \{0\}$ such that $\int_U |v|^{p-2} v\,dx = 0$.

We are now ready to prove Theorem 1.3. For simplicity, we divide the proof of this theorem into three parts contained in the following lemmas.

First, by a standard compactness argument and using that $\mathcal{W}^{s,p}(U) \subset W^{s,p}(U)$, we have that $\lambda_{s,p}$ is the first nonzero eigenvalue of (1.4).

Lemma 5.8. *It holds that $\lambda_{s,p}$ is the first nonzero eigenvalue of* (1.4).

Remark 5.9. Since $\mathcal{W}^{s,p}(U) \subset W^{s,p}(U)$ and

$$[\![u]\!]_{s,p}^p \le \mathcal{H}_{s,p}(u) \quad \text{for all } u \in \mathcal{W}^{s,p}(U),$$

we have that

$$\lambda_{s,p}^N \le 2\lambda_{s,p}.$$

Our next result shows the asymptotic behavior of $(\lambda_{s,p})^{1/p}$.

Lemma 5.10. *We have*

$$\lim_{p\to+\infty} (\lambda_{s,p})^{1/p} = \frac{2}{(\operatorname{diam}_d(U))^s} = \Lambda_{s,\infty} := \inf\left\{\frac{\mathcal{H}_{s,\infty}(u)}{\|u\|_{L^\infty(U)}} : u \in \mathcal{A}\right\},$$

where

$$\mathcal{A} := \left\{v \in \mathcal{W}^{s,\infty}(U) \setminus \{0\} : \sup_{x\in U} u(x) + \inf_{x\in U} u(x) = 0\right\}.$$

Proof. For the reader's convenience, we split the proof into four steps.

Step 1. We begin by showing that

$$\Lambda_{s,\infty} \le \frac{2}{(\operatorname{diam}_d(U))^s}.$$

Let $x_0, y_0 \in \overline{U}$ such that $d(x_0, y_0) = \operatorname{diam}_d(U)$. Let $u : \mathbb{R}^n \to \mathbb{R}$ be given by

$$u(x) := -1 + \frac{2}{\operatorname{diam}_d(U)} d(x, y_0)^s.$$

Observe that

$$\sup_{x\in U} u(x) = -\inf_{x\in U} u(x) = 1$$

and

$$\frac{|u(x) - u(y)|}{d(x, y)^s} = \frac{2}{(\operatorname{diam}_d(U))^s} \frac{|d(x, y_0)^s - d(y, y_0)^s|}{d(x, y)^s} \le \frac{2}{(\operatorname{diam}_d(U))^s}$$

for all $x, y \in \mathbb{R}^n$. Then, $u \in \mathcal{A}$, $\|u\|_{L^\infty(U)} = 1$ and

$$\mathcal{H}_{s,\infty}(u) \le \frac{2}{(\operatorname{diam}_d(U))^s}.$$

Therefore,

$$\Lambda_{s,\infty} \le \mathcal{H}_{s,\infty}(u) \le \frac{2}{(\operatorname{diam}_d(U))^s}.$$

Step 2. We now prove that

$$\Lambda_{s,\infty} \ge \frac{2}{(\operatorname{diam}_d(U))^s}.$$

If $u \in \mathcal{A}$, then

$$\begin{aligned} 2\|u\|_{L^\infty(U)} &= \sup_{x\in U} u(x) - \inf_{x\in U} u(x) \\ &= \sup\{|u(x) - u(y)| : x, y \in U\} \\ &\le (\operatorname{diam}_d(U))^s \sup\left\{\frac{|u(x) - u(y)|}{d(x, y)^s} : x, y \in U\right\} \\ &\le (\operatorname{diam}_d(U))^s \mathcal{H}_{s,\infty}(u). \end{aligned}$$

Thus,

$$\frac{2}{(\operatorname{diam}_d(U))^s} \leq \frac{\mathcal{H}_{s,\infty}(u)}{\|u\|_{L^\infty(U)}}$$

for any $u \in \mathcal{A}$, that is,

$$\Lambda_{s,\infty} \geq \frac{2}{(\operatorname{diam}_d(U))^s}.$$

Step 3. We show that

$$\frac{2}{(\operatorname{diam}_d(U))^s} \leq \liminf_{p\to+\infty}(\lambda_{s,p})^{1/p}.$$

By (1.2) and Remark 5.9, we have that

$$\frac{2}{(\operatorname{diam}_d(U)^s)} \leq \lim_{p\to+\infty}(\lambda_{s,p}^N)^{1/p} \leq \liminf_{p\to+\infty} 2^{1/p}(\lambda_{s,p}(U))^{1/p} = \liminf_{p\to+\infty}(\lambda_{s,p}(U))^{1/p}.$$

Step 4. Finally, we prove that

$$\limsup_{p\to+\infty}(\lambda_{s,p})^{1/p} \leq \frac{2}{(\operatorname{diam}_d(U))^s}.$$

As in Step 1, let $x_0, y_0 \in \overline{U}$ be such that $d(x_0, y_0) = \operatorname{diam}_d(U)$. Set $\delta = \operatorname{diam}_d(U)$,

$$U_\delta := \Big\{x \in \mathbb{R}^n : \inf_{y\in U} d(x,y) \leq \delta\Big\}$$

and

$$u(x) := \begin{cases} d(x, y_0) & \text{if } x \in U_\delta, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus U_\delta. \end{cases}$$

Let $\varepsilon > 0$. Then,

$$\begin{aligned}\mathcal{H}_{s,p}(u) &\leq 2 \int\limits_{U\times U_\delta} \frac{|d(x,y_0) - d(y,y_0)|^p}{d(x,y)^{n+sp}}\,dx\,dy \;+\; 2 \int\limits_{U\times(\mathbb{R}^n\setminus U_\delta)} \frac{d(x,y_0)^p}{d(x,y)^{n+sp}}\,dx\,dy \\ &\leq 2 \int\limits_{U\times U_\delta} \frac{d(x,y)^{p(1-s)-\varepsilon}}{d(x,y)^{n-\varepsilon}}\,dx\,dy \;+\; 2 \int\limits_{U\times(\mathbb{R}^n\setminus U_\delta)} \frac{d(x,y_0)^p}{d(x,y)^{n+\varepsilon+sp-\varepsilon}}\,dx\,dy.\end{aligned}$$

Thus, since d is a distance equivalent to the Euclidean one, if

$$p > \max\Big\{\frac{\varepsilon}{(1-s)}, \frac{\varepsilon}{s}\Big\},$$

we get that $u \in \mathcal{W}^{s,p}(U)$ and

$$\mathcal{H}_{s,p}(u) \leq C(\operatorname{diam}_d(U))^{p(1-s)}\{(\operatorname{diam}_d(U))^{-\varepsilon} + (\operatorname{diam}_d(U))^{\varepsilon}\}, \tag{5.2}$$

where C is a constant independent of p.

We now choose $c_p \in \mathbb{R}$ such that

$$w_p(x) = u(x) - c_p$$

satisfies

$$\int\limits_U |w_p|^{p-2} w_p\,dx = 0.$$

Hence, if $p > \max\{\varepsilon/(1-s), \varepsilon/s\}$, by (5.2), we have that

$$\lambda_{s,p} \leq \frac{\mathcal{H}(w_p)}{2\|w_p\|^p_{L^p(U)}} = \frac{\mathcal{H}(u)}{2\|w_p\|^p_{L^p(U)}} \leq \frac{C}{2\|w_p\|^p_{L^p(U)}}(\operatorname{diam}_d(U))^{p(1-s)}\{(\operatorname{diam}_d(U))^{-\varepsilon} + (\operatorname{diam}_d(U))^{\varepsilon}\}$$

and, therefore,

$$\limsup_{p\to+\infty}(\lambda_{s,p})^{1/p} \leq \frac{(\operatorname{diam}_d(U))^{1-s}}{\liminf\limits_{p\to+\infty}\|w_p\|_{L^p(U)}}. \tag{5.3}$$

On the other hand, in [12] it is proved that

$$\liminf_{p\to+\infty} \|w_p\|_{L^p(U)} \geq \frac{2}{\operatorname{diam}_d(U)}. \tag{5.4}$$

Thus, by (5.3) and (5.4), we get

$$\limsup_{p\to+\infty} (\lambda_{s,p})^{1/p} \leq \frac{2}{(\operatorname{diam}_d(U))^s}.$$

This concludes the proof. □

Remark 5.11. By (1.2) and Lemma 5.10, we have that

$$\Lambda^N_{s,\infty} = \lim_{p\to+\infty} (\lambda^N_{s,p})^{1/p} = \frac{2}{(\operatorname{diam}_d(U))^s} = \lim_{p\to+\infty} (\lambda_{s,p})^{1/p} = \Lambda_{s,\infty}.$$

Concerning the convergence of the eigenfunctions as $p \to +\infty$, we have the following result.

Lemma 5.12. *If u_p is a minimizer of $\lambda_{s,p}$, normalized with $\|u_p\|_{L^p(U)} = 1$, then, up to a subsequence, u_p converges in $C(\overline{U})$ to some minimizer $u_\infty \in W^{s,\infty}(U)$ of $\Lambda^N_{s,\infty}$.*

Proof. For any $p \in (1,\infty)$, we consider $u_p \in \mathcal{W}^{s,p}(U)$ such that

$$\|u_p\|_{L^p(U)} = 1, \quad \int_U |u_p|^{p-2} u_p \, dx = 0 \quad \text{and} \quad \frac{1}{2}\mathcal{H}_{s,p}(u_p) = \lambda_{s,p}.$$

Then, by Lemma 5.10, there exists a constant C, independent of p, such that

$$\left(\frac{\mathcal{H}_{s,p}(u_p)}{2}\right)^{1/p} \leq C \tag{5.5}$$

for all $p \in (1,\infty)$.

Let us fix $q \in (1,\infty)$ such that $sq > 2n$. If $p > q$, then, by Hölder's inequality, we have that

$$\|u_p\|_{L^q(\Omega)} \leq |U|^{1/q-1/p} \|u_p\|_{L^p(\Omega)} \leq |U|^{1/q-1/p} \quad \text{for all } p \geq q \tag{5.6}$$

and taking $r = s - n/q \in (0,1)$, again by Hölder's inequality, we get

$$\begin{aligned}
[\![u_p]\!]^q_{r,q} &= \int_U \int_U \frac{|u_p(x) - u_p(y)|^q}{d(x,y)^{sq}} \, dx \, dy \\
&\leq |U|^{2(1-q/p)} \left(\int_U \int_U \frac{|u_p(x) - u_p(y)|^p}{d(x,y)^{sp}} \, dx \, dy\right)^{q/p} \\
&\leq 2^{q/p} (\operatorname{diam}_d(U))^{nq/p} |U|^{2(1-q/p)} \left(\frac{\mathcal{H}_{s,p}(u_p)}{2}\right)^{q/p}.
\end{aligned} \tag{5.7}$$

Then, by (5.5), we get

$$[\![u_p]\!]_{r,q} \leq 2^{1/p} (\operatorname{diam}_d(U))^{n/p} |U|^{2(1/q-1/p)} C^q \quad \text{for all } p \geq q,$$

where C is a constant independent of p. Hence, $\{u_p\}_{p\geq q}$ is a bounded sequence in $W^{r,q}(U)$. Then, since $rq = sq - n > n$, by fractional compact embedding theorems (see [9, Theorem 4.54]), there exist a function $u_\infty \in C(\overline{U})$ and a subsequence $\{u_{p_j}\}_{j\in\mathbb{N}}$ of $\{u_p\}_{p\geq q}$ such that

$$\begin{aligned}
u_{p_j} &\to u_\infty \quad \text{uniformly in } \overline{U}, \\
u_{p_j} &\rightharpoonup u_\infty \quad \text{weakly in } W^{r,q}(U).
\end{aligned}$$

Hence, by (5.6), we have $\|u_\infty\|_{L^q(\Omega)} \leq |U|^{1/q}$, and by (5.7) and Remark 5.11, we get

$$[\![u_\infty]\!]_{r,q} \leq \liminf_{j\to\infty} [\![u_{p_j}]\!]_{r,q} \leq \liminf_{j\to\infty} 2^{1/p_j} (\operatorname{diam}_d(U))^{n/p_j} |U|^{2(1-1/p_j)} \left(\frac{\mathcal{H}_{s,p_j}(u_{p_j})}{2}\right)^{1/p_j} = |U|^{2/q} \Lambda^N_{s,\infty}.$$

Letting $q \to +\infty$, we obtain

$$\|u_\infty\|_{L^\infty(\Omega)} \le 1$$

and

$$[\![u_\infty]\!]_{s,\infty} \le \Lambda_{s,\infty}^N. \tag{5.8}$$

On the other hand,

$$1 = \|u_{p_j}\|_{L^{p_j}(U)} \le |U|^{1/p_j}\|u_{p_j}\|_{L^\infty(U)} \quad \text{for all } j \in \mathbb{N}$$

and, as a result,

$$1 \le \|u_\infty\|_{L^\infty(U)}.$$

Hence,

$$\|u_\infty\|_{L^\infty(U)} = 1$$

and by (5.8), we get

$$\frac{[\![u_\infty]\!]_{s,\infty}}{\|u_\infty\|_{L^\infty(U)}} \le \Lambda_{s,\infty}^N. \tag{5.9}$$

Finally, in [12, 23] it was proved that the condition

$$\int_U |u_{p_j}|^{p_j-2}u_{p_j}\,dx = 0$$

leads to

$$\sup_{x\in U} u_\infty(x) + \inf_{x\in U} u_\infty(x) = 0$$

in the limit as $p \to +\infty$. Then, using (5.9), we have that u_∞ is a minimizer of $\Lambda_{s,\infty}^N$. □

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