A TWO PHASE FREE BOUNDARY PROBLEM FROM COMBUSTION: UNIQUENESS AND AGREEMENT OF SOLUTION

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Abstract. We consider the following two phase free boundary problem: find a function $u(x,t)$, defined in a domain $\mathcal{D} \subset \mathbb{R}^N \times (0,T)$, satisfying that

$$\begin{aligned} &\Delta u + \sum a_i(x,t)\, u_{x_i} - u_t = 0 \quad \text{in} \quad \mathcal{D} \cap \{u \neq 0\}, \\ &u = 0, \quad |\nabla_x u^+|^2 - |\nabla_x u^-|^2 = 2M \quad \text{on} \quad \mathcal{D} \cap \partial\{u > 0\}, \end{aligned}$$

where $M > 0$ is a fixed constant. In addition, initial data are specified, as well as either Dirichlet or Neumann data on the parabolic boundary of $\mathcal{D}$.

The problem admits *classical* solutions only for good data and for small times. Different generalized concepts of solution have been proposed, among them the concepts of *limit* solution and *viscosity* solution. We discuss conditions under which the three concepts agree and produce a unique solution.

1. Introduction

In this paper we consider the following two phase free boundary problem: find a function $u(x,t)$, defined in a domain $\mathcal{D} \subset \mathbb{R}^N \times (0,T)$, satisfying that

$$(\mathcal{P}) \qquad \begin{aligned} &\Delta u + \sum a_i(x,t)\, u_{x_i} - u_t = 0 \quad \text{in} \quad \mathcal{D} \cap \{u \neq 0\}, \\ &u = 0, \quad |\nabla u^+|^2 - |\nabla u^-|^2 = 2M \quad \text{on} \quad \mathcal{D} \cap \partial\{u > 0\}, \end{aligned}$$

where $u^+ = \max(u,0)$, $u^- = \max(-u,0)$, the gradients are spatial side-derivatives, M is a positive constant and a_i are bounded. In addition, initial data are specified, as well as either Dirichlet or Neumann data on the parabolic boundary of $\mathcal{D}$.

In the particular case when $u \geq 0$, we obtain the corresponding one phase free boundary problem, where the free boundary condition reads

$$u = 0, \quad |\nabla u^+|^2 = 2M \quad \text{on} \quad \mathcal{D} \cap \partial\{u > 0\}.$$

This free boundary problem arises in several contexts (cf. [20]). The most important motivation to date has come from combustion theory, where it appears as a limit situation in the description of the propagation of premixed equi-diffusional deflagration flames. In this case, u is the limit, as $\varepsilon \to 0$, of solutions u^ε to

$$(\mathcal{P}_\varepsilon) \qquad \Delta u^\varepsilon + \sum a_i(x,t)\, u^\varepsilon_{x_i} - u^\varepsilon_t = \beta_\varepsilon(u^\varepsilon),$$

with $\varepsilon > 0$, $\beta_\varepsilon \geq 0$, $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon})$, support β=[0,1] and $\int \beta(s)\, ds = M$.

Problem $\mathcal{P}$ admits *classical* solutions only for good data and for small times, since singularities can arise in finite time even in the one phase problem. Different generalized concepts of solution have been proposed, among them the concepts of *limit* solution (that is, $u = \lim u^\varepsilon$) and *viscosity* solution, cf. [10], [9].

In this work, we investigate conditions under which the concepts of classical, viscosity and limit solution agree and produce a unique solution.

2. Passage to the limit $\mathcal{P}_\varepsilon \to \mathcal{P}$

Problem $\mathcal{P}_\varepsilon$ appears in combustion theory, for small ε, as the high activation energy range of the equation for equidiffusional flames, after convenient simplifications. Here we have $u^\varepsilon(x,t) = T_f - T(x,t)$, with T the temperature of the reactive mixture and T_f the flame temperature. The function $\beta_\varepsilon(u)$ represents the exothermic chemical reaction. The parameter $\varepsilon > 0$ is essentially the inverse of the activation energy of the chemical reaction. The vector $(a_1, \cdots, a_N)$ represents the transport velocity of the reactive mixture, which is sometimes taken to be zero in the literature. For further details on the model see [4], [22].

The study of the limit $\mathcal{P}_\varepsilon \to \mathcal{P}$ as $\varepsilon \to 0$ was proposed in the 30's by Zeldovich and Frank-Kamenetski [23] and has been much discussed in the combustion literature.

In the one phase case, Berestycki and his collaborators have rigorously studied the convergence problem for traveling waves and, more generally in the elliptic stationary case, cf. [2] and its references.

The study of the limit in the one phase evolution case for the heat operator has been performed in [10]. In the two phase case, uniform estimates were obtained in [5] and [6], and the passage to the limit was studied in [8] and [9]. In [17] the regularity of the free boundary was studied for the two phase elliptic case. In [7] uniform estimates were obtained for a more general operator. Further results for the one phase evolution case were obtained in [11]. See also [18].

3. Classical solutions to problem $\mathcal{P}$

In the one phase case, classical solutions to problem $\mathcal{P}$ in one space dimension are relatively easy to construct, cf. [21]. In several space dimensions, classical solutions to problem $\mathcal{P}$ are more difficult to obtain, cf. [19] and [1] for the one phase case and two phase case respectively.

Classical solutions to problem $\mathcal{P}$ are also obtained in [13] and [3], and there the stability of traveling waves for problem $\mathcal{P}$ is studied. In [12], classical solutions to problem $\mathcal{P}$ are constructed under the assumption of radial symmetry.

Some of these works are based on an elliptic-parabolic formulation of the problem. In the other cases the approach is based on reformulating the free boundary problem as a fully nonlinear parabolic problem in a fixed domain.

We point out that, in general, classical solutions to $\mathcal{P}$ exist only locally in time, since singularities can arise in finite time, cf. [20].

4. Uniqueness of solution. Assumptions

In the sequel we will present results obtained in [14] and [16], where we investigate conditions under which the concepts of classical, viscosity and limit solution to problem $\mathcal{P}$ agree and produce a unique solution.

Our assumptions are the following. We take as spatial domain a cylinder of the form $\Omega = I \times \Sigma$ with $\Sigma \subset \mathbb{R}^{N-1}$ a smooth domain, and $I = \mathbb{R}$ (a full cylinder), $I = (0, \infty)$ or $I = (-\infty, d)$ (a semi-cylinder) or $I = (0, d)$, $d > 0$ (a bounded cylinder), and we put homogeneous Neumann conditions on the lateral boundary $\overline{I} \times \partial\Sigma$. We require monotonicity of the initial data in the direction of the cylinder axis, but we make no requirement of monotonicity of the solution in time.

In the family of problems $\mathcal{P}_\varepsilon$ we assume that the functions β_ε are defined by scaling of a single function $\beta : \mathbb{R} \to \mathbb{R}$ satisfying:

- β is a Lipschitz continuous function,
- $\beta > 0$ in $(0, 1)$ and $\beta \equiv 0$ otherwise,
- $\int \beta(s)ds = M$.

We then define $\beta_\varepsilon(s) = 1/\varepsilon\beta(s/\varepsilon)$.

The coefficients a_i of the first-order terms in the differential operator are assumed to be independent of x_1, the direction of the cylinder axis, and belong to $C^{\alpha,\frac{\alpha}{2}}(\overline{\Sigma} \times [0, T])$.

5. Uniqueness of classical and viscosity solutions

The results we present in this section say that, under suitable assumptions, a classical solution is the unique viscosity solution to the initial and boundary value problem associated to $\mathcal{P}$ and, in particular, it is the unique classical solution.

For the sake of simplicity we state the results for the two phase case. Analogous results also hold for the one phase case. For the proof of these results see [14] and [16].

We start the section by giving precise definitions of the concepts of classical and viscosity solution and also some consequences.

Definition. *Let Q be a space-time domain of the form $\Omega \times (T_1, T_2)$, with $\Omega \subset \mathbb{R}^N$. Let v be a continuous function in $\overline{Q}$. Then v is called a classical subsolution (supersolution) to $\mathcal{P}$ in Q if*

(i) $\mathcal{L}v \geq 0 \quad (\leq 0) \qquad$ *in* $Q^+ := Q \cap \{v > 0\}$,

(ii) $\mathcal{L}v \geq 0 \quad (\leq 0) \qquad$ *in* $Q^- := Q \cap \{v \leq 0\}^\circ$,
(iii) $v \in C^1(\overline{Q^+}) \cap C^1(\overline{Q^-})$, $\nabla v \in C^{\alpha,\frac{\alpha}{2}}(\overline{Q^+}) \cap C^{\alpha,\frac{\alpha}{2}}(\overline{Q^-})$,
(iv) *For any* $(x,t) \in \{v = 0\} \cap \partial\{v > 0\}$, *we have* $\nabla v^+(x,t) \neq 0$ *and*

$$\left(\frac{\partial v^+}{\partial \nu}\right)^2 - \left(\frac{\partial v^-}{\partial \nu}\right)^2 \geq 2M \qquad (\leq 2M),$$

where $\nu := -\frac{\nabla v^+}{|\nabla v^+|}$. *That is,*

$$|\nabla v^+|^2 - |\nabla v^-|^2 \geq 2M \qquad (\leq 2M).$$

We say that v *is a classical solution to* $\mathcal{P}$ *in* Q *if it is both a classical subsolution and a classical supersolution to* $\mathcal{P}$.

Definition. *Let* $u \in C(\overline{Q})$; u *is called a viscosity subsolution (supersolution) to* $\mathcal{P}$ *in* Q *if, for every space-time subcylinder* $Q' \subset Q$ *and for every* v *bounded classical supersolution (subsolution) to* $\mathcal{P}$ *in* Q', *with* $Q' \cap \partial\{v > 0\}$ *bounded,*

$$\begin{aligned} &u \leq v \quad (u \geq v) \quad \text{on } \partial_p Q' \text{ and} \\ &v > 0 \quad \text{on } \overline{\{u > 0\}} \cap \partial_p Q' \\ &(u > 0 \quad \text{on } \overline{\{v > 0\}} \cap \partial_p Q') \end{aligned}$$

implies that $u \leq v$ $(u \geq v)$ *in* Q'.

The function u *is called a viscosity solution to* $\mathcal{P}$ *if it is both a viscosity supersolution and a viscosity subsolution to* $\mathcal{P}$.

The following result gives the consistency between both concepts of solution (with Dirichlet boundary data).

Proposition. *If* u *is a bounded classical supersolution (subsolution) to* $\mathcal{P}$ *in* Q *with* $Q \cap \partial\{u > 0\}$ *bounded, then* u *is a viscosity supersolution (subsolution) to* $\mathcal{P}$ *in* Q.

Definition. *Let* $\Omega \subset \mathbb{R}^N$ *be a domain and let* $Q = \Omega \times (0,T)$. *Let* Γ_N *be an open* C^1 *subset of* $\partial\Omega$ *and let* $\partial_N Q = \Gamma_N \times (0,T)$. *We say that* $u \in C(\overline{Q})$ *is a viscosity solution to* $\mathcal{P}$ *in* Q *with* $\frac{\partial u}{\partial \eta} = 0$ *on* $\partial_N Q$, *if the following holds: for every space-time subcylinder* $Q' \subset Q$ *and for every* v *bounded classical supersolution (subsolution) to* $\mathcal{P}$ *in* Q', *with* $Q' \cap \partial\{v > 0\}$ *bounded, such that* $\frac{\partial v}{\partial \eta} = 0$ *on* $\partial_p Q' \cap \partial_N Q$,

$$\begin{aligned} &u \leq v \quad (u \geq v) \quad \text{on } \partial_p Q' \setminus \partial_N Q \text{ and} \\ &v > 0 \quad \text{on } \overline{\{u > 0\}} \cap \overline{\partial_p Q' \setminus \partial_N Q} \\ &\big(u > 0 \quad \text{on } \overline{\{v > 0\}} \cap \overline{\partial_p Q' \setminus \partial_N Q}\big) \end{aligned}$$

implies that $u \leq v$ $(u \geq v)$ *in* Q'.

We also have the consistency between both concepts of solution, with Neumann boundary data.

Proposition. *Let $\Omega = \mathbb{R} \times \Sigma$, $Q = \Omega \times (0,T)$ and $\partial_N Q = \mathbb{R} \times \partial\Sigma \times (0,T)$.*

Let u be a bounded classical solution to $\mathcal{P}$ in Q with $Q \cap \partial\{u > 0\}$ bounded and $\frac{\partial u}{\partial \eta} = 0$ on $\partial_N Q$. Then u is a viscosity solution to $\mathcal{P}$ in Q with $\frac{\partial u}{\partial \eta} = 0$ on $\partial_N Q$.

Analogous results hold for $\Omega = (0,d) \times \Sigma$, $\Omega = (-\infty, d) \times \Sigma$ and $\Omega = (0,+\infty) \times \Sigma$.

Our first uniqueness result is the following.

Theorem 5.1. *Let $\Omega = (0,d) \times \Sigma$, $Q = \Omega \times (0,T)$, $\partial_N Q = (0,d) \times \partial\Sigma \times (0,T)$ and $\partial_D Q = \partial_p Q \setminus \partial_N Q$.*

Let u be a bounded classical solution to $\mathcal{P}$ in Q, with $\frac{\partial u}{\partial \eta} = 0$ on $\partial_N Q$, such that $u_{x_1} < 0$ on $\partial_D Q$.

Assume that $u(0,x',t) > 0$ for $(x',t) \in \overline{\Sigma} \times [0,T]$ with $u(0,x',t) \in C^{2,1}(\overline{\Sigma} \times [0,T])$ and $u(d,x',t) < 0$ for $(x',t) \in \overline{\Sigma} \times [0,T]$ with $u(d,x',t) \in C^{2,1}(\overline{\Sigma} \times [0,T])$.

Let $v \in C(\overline{Q})$ be a viscosity solution to $\mathcal{P}$ in Q with $\frac{\partial v}{\partial \eta} = 0$ on $\partial_N Q$.

If $v = u$ on $\partial_D Q$, then $v = u$ in $\overline{Q}$.

For two classical solutions we have the following uniqueness result.

Corollary. *Let Ω, Q, $\partial_N Q$, $\partial_D Q$ and u as in Theorem 5.1. Let v be a bounded classical solution to $\mathcal{P}$ in Q with $\frac{\partial v}{\partial \eta} = 0$ on $\partial_N Q$, such that $v = u$ on $\partial_D Q$. Then, $v = u$ in $\overline{Q}$.*

The next theorem gives the uniqueness of viscosity solution under different assumptions from those in Theorem 5.1. As in the corollary above, uniqueness of classical solutions follows.

Theorem 5.2. *The result of Theorem 5.1 holds if we let instead $\partial_N Q = \emptyset$ so that $\partial_D Q = \partial_p Q$. Moreover, the result of Theorem 5.1 also holds if we let $\Omega = \mathbb{R} \times \Sigma$ with $\partial_N Q = \mathbb{R} \times \partial\Sigma \times (0,T)$ or $\partial_N Q = \emptyset$, as long as $||u||_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q})} < \infty$. In this case we make no assumptions on u on $x_1 = 0, d$.*

Analogous results hold with $\Omega = (-\infty, d) \times \Sigma$ and $\Omega = (0,+\infty) \times \Sigma$.

6. Uniqueness of the limit solution

The results we present in this section say that, under certain assumptions, a classical solution to the initial and boundary value problem associated to $\mathcal{P}$ is the uniform limit of any family of solutions to $\mathcal{P}_\varepsilon$ with corresponding boundary data. This in particular implies that such limit exits and is unique.

In particular, under the assumptions of this section our classical solution is the unique classical solution and also the unique viscosity solution (by the results of Section 5).

For the sake of simplicity we state the results for the two phase case. Analogous results also hold for the one phase case. For the proof of these results see [14] and [16].

We first obtain existence and uniqueness of the limit solution in a bounded cylinder.

Theorem 6.1. *Let $\Omega = (0,d) \times \Sigma$, $Q = \Omega \times (0,T)$, $\partial_N Q = (0,d) \times \partial\Sigma \times (0,T)$ and $\partial_D Q = \partial_p Q \setminus \partial_N Q$.*

Let u be a bounded classical solution to $\mathcal{P}$ in Q, with $\frac{\partial u}{\partial \eta} = 0$ on $\partial_N Q$, such that $u_{x_1} < 0$ on $\partial_D Q$.

Assume that $u(0,x',t) > 0$ for $(x',t) \in \overline{\Sigma} \times [0,T]$ with $u(0,x',t) \in C^{2,1}(\overline{\Sigma} \times [0,T])$ and $u(d,x',t) < 0$ for $(x',t) \in \overline{\Sigma} \times [0,T]$ with $u(d,x',t) \in C^{2,1}(\overline{\Sigma} \times [0,T])$.

Let $u^\varepsilon \in C(\overline{Q})$ with $\nabla u^\varepsilon \in C(Q \cup \partial_N Q) \cap L^2_{\mathrm{loc}}(\overline{Q})$ be a family of bounded weak solutions to $\mathcal{P}_\varepsilon$ in Q, with $\frac{\partial u^\varepsilon}{\partial \eta} = 0$ on $\partial_N Q$, such that $u^\varepsilon \to u$ uniformly on $\partial_D Q$.

Then $u^\varepsilon \to u$ uniformly in $\overline{Q}$.

A similar result holds for a full cylinder as spatial domain, under suitable monotonicity assumptions at $x_1 = \pm\infty$.

Theorem 6.2. *Let $\Omega = \mathbb{R} \times \Sigma$, $Q = \Omega \times (0,T)$, $\partial_N Q = \mathbb{R} \times \partial\Sigma \times (0,T)$ and $\partial_D Q = \partial_p Q \setminus \partial_N Q$.*

Let u be a bounded classical solution to $\mathcal{P}$ in Q, with $\frac{\partial u}{\partial \eta} = 0$ on $\partial_N Q$ and $||u||_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q})} < \infty$, such that $u\big|_{\partial_D Q}$ has a bounded, nonempty free boundary.

Assume that $u_{x_1} < 0$ on $\partial_D Q$ and $u_{x_1}(x,0) \le -c_1 e^{-c_2 |x_1|}$ for $|x_1| \ge a$ for some constants c_1, c_2, $a > 0$.

Let $u^\varepsilon \in C(\overline{Q})$ with $\nabla u^\varepsilon \in C(Q \cup \partial_N Q) \cap L^2_{\mathrm{loc}}(\overline{Q})$ be a family of bounded weak solutions to $\mathcal{P}_\varepsilon$ in Q, with $\frac{\partial u^\varepsilon}{\partial \eta} = 0$ on $\partial_N Q$, such that $u^\varepsilon \to u$ uniformly on $\partial_D Q$, with $|u^\varepsilon(x,0) - u(x,0)| \le k_1\, e^{-k_2\, x_1^2}$ for $|x_1| \ge a$ for some constants k_1, $k_2 > 0$.

Then $u^\varepsilon \to u$ uniformly in $\overline{Q}$.

With the ideas of Theorems 6.1 and 6.2 we can prove the following theorem in a semicylinder.

Theorem 6.3. *Let $\Omega = (0,\infty) \times \Sigma$ (resp. $(-\infty,d) \times \Sigma$), $Q = \Omega \times (0,T)$, $\partial_N Q = (0,\infty) \times \partial\Sigma \times (0,T)$ (resp. $\partial_N Q = (-\infty,d) \times \partial\Sigma \times (0,T)$) and $\partial_D Q = \partial_p Q \setminus \partial_N Q$.*

Let u be a bounded classical solution to $\mathcal{P}$ in Q, with $\frac{\partial u}{\partial \eta} = 0$ on $\partial_N Q$ and $||u||_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q})} < \infty$, such that $u\big|_{\partial_D Q}$ has a bounded, nonempty free boundary.

Assume that $u_{x_1} < 0$ on $\partial_D Q$ and $u_{x_1}(x,0) \le -c_1 e^{-c_2 |x_1|}$ for $x_1 \ge a$ (resp. for $x_1 \le -a$) for some constants c_1, c_2, $a > 0$. Also, $u(0,x',t) \in C^{2,1}(\overline{\Sigma} \times [0,T])$ with $u(0,x',t) > 0$ for $x' \in \overline{\Sigma}$, $t \in [0,T]$ (resp. $u(d,x',t) \in C^{2,1}(\overline{\Sigma} \times [0,T])$ with $u(d,x',t) < 0$ for $x' \in \overline{\Sigma}$, $t \in [0,T]$).

Let $u^\varepsilon \in C(\overline{Q})$ with $\nabla u^\varepsilon \in C(Q \cup \partial_N Q) \cap L^2_{\mathrm{loc}}(\overline{Q})$ be a family of bounded weak solutions to $\mathcal{P}_\varepsilon$ in Q, with $\frac{\partial u^\varepsilon}{\partial \eta} = 0$ on $\partial_N Q$, such that $u^\varepsilon \to u$ uniformly on $\partial_D Q$, with $|u^\varepsilon(x,0) - u(x,0)| \le k_1\, e^{-k_2\, x_1^2}$ for $x_1 \ge a$ (resp. for $x_1 \le -a$) for some constants k_1, $k_2 > 0$.

Then $u^\varepsilon \to u$ uniformly in $\overline{Q}$.

7. Basic ideas and tools used in the proofs

The proof of the uniqueness of classical and viscosity solution is based on the comparison of the classical solution u with suitable families of classical subsolutions and supersolutions to problem $\mathcal{P}$ converging to u.

The proof of the uniqueness of the limit solution is more delicate and is based on the following auxiliary results: under certain assumptions, a classical subsolution to problem $\mathcal{P}$ is the uniform limit of a family of subsolutions to problem $\mathcal{P}_\varepsilon$. An analogous result holds for supersolutions.

For the sake of completeness we state these results below for the two phase case. For the proofs see [16]; for the corresponding one phase results see [14].

We assume that $\Omega = \mathbb{R}\times\Sigma$ is a full cylinder (resp. $\Omega = (0, +\infty)\times\Sigma$, $\Omega = (-\infty, d)\times\Sigma$ is a semicylinder or $\Omega = (0, d) \times \Sigma$ is a bounded cylinder). We define $Q = \Omega \times (0, T)$ and we let $\partial_N Q = \mathbb{R} \times \partial\Sigma \times (0, T)$ (resp. $\partial_N Q = (0, +\infty) \times \partial\Sigma \times (0, T)$, $\partial_N Q = (-\infty, d) \times \partial\Sigma \times (0, T)$ or $\partial_N Q = (0, d) \times \partial\Sigma \times (0, T)$).

In case $\Omega = (0, +\infty) \times \Sigma$, we assume that $w(0, x', t) > 0$ for $(x', t) \in \overline{\Sigma} \times [0, T]$. In case $\Omega = (-\infty, d) \times \Sigma$, we assume that $w(d, x', t) < 0$ for $(x', t) \in \overline{\Sigma} \times [0, T]$. Finally, in case $\Omega = (0, d) \times \Sigma$, we assume that $w(0, x', t) > 0$ for $(x', t) \in \overline{\Sigma} \times [0, T]$ and that $w(d, x', t) < 0$ for $(x', t) \in \overline{\Sigma} \times [0, T]$.

Theorem 7.1. *Let w be a classical subsolution to $\mathcal{P}$ in Q, with $\frac{\partial w}{\partial \eta} = 0$ on $\partial_N Q$. Assume that there exists $\delta_0 > 0$ such that*

$$|\nabla w^+|^2 - |\nabla w^-|^2 = 2M + \delta_0 \qquad \text{on } Q \cap \partial\{w > 0\}.$$

Then, there exists a family $v^\varepsilon \in C(\overline{Q})$, with $\nabla v^\varepsilon \in L^2_{loc}(\overline{Q})$, of weak subsolutions to $\mathcal{P}_\varepsilon$ in Q, with $\frac{\partial v^\varepsilon}{\partial \eta} = 0$ on $\partial_N Q$, such that, as $\varepsilon \to 0$, $v^\varepsilon \to w$ uniformly in $\overline{Q}$.

Moreover $v^\varepsilon = w$ in $\{|w| \geq A\varepsilon\}$ and $\nabla v^\varepsilon \in C(\{|w| \leq A\varepsilon\} \cap \{t > 0\})$. Here $A = A(L, M, \delta_0) > 1$, and $L > 0$ is such that $|\nabla w| \leq L$ in a neighborhood of the free boundary $\overline{Q \cap \partial\{w > 0\}}$.

Theorem 7.2. *Let w be a classical supersolution to $\mathcal{P}$ in Q, with $\frac{\partial w}{\partial \eta} = 0$ on $\partial_N Q$. Assume that there exists $\delta_0 > 0$, with $|\nabla w^-|^2 > \delta_0$ on the free boundary $Q \cap \partial\{w > 0\}$, such that*

$$|\nabla w^+|^2 - |\nabla w^-|^2 = 2M - \delta_0 \qquad \text{on } Q \cap \partial\{w > 0\}.$$

Then, there exists a family $v^\varepsilon \in C(\overline{Q})$, with $\nabla v^\varepsilon \in L^2_{loc}(\overline{Q})$, of weak supersolutions to $\mathcal{P}_\varepsilon$ in Q, with $\frac{\partial v^\varepsilon}{\partial \eta} = 0$ on $\partial_N Q$, such that, as $\varepsilon \to 0$, $v^\varepsilon \to w$ uniformly in $\overline{Q}$.

Moreover, $v^\varepsilon = w^\varepsilon$ in $\{|w| \geq A\varepsilon\}$ and $\nabla v^\varepsilon \in C(\{|w| \leq A\varepsilon\} \cap \{t > 0\})$. Here $A = A(L, M, \delta_0) > 1$, and $L > 0$ is such that $|\nabla w| \leq L$ in a neighborhood of the free boundary $\overline{Q \cap \partial\{w > 0\}}$.

The proofs of the results stated above require a result on existence and regularity of a solution to equation $\mathcal{P}_\varepsilon$ in a non-cylindrical space-time domain, with mild regularity on the data. This result is proved in [16]. For the sake of completeness we state the result below. Further regularity of the solution can be derived from the results in [15].

Let $\Omega = \mathbb{R}\times\Sigma$, $\mathcal{D} = \{(x,t) \in \Omega\times(0,T)\,/\,p(x',t) < x_1 < q(x',t)\}$, where the functions p, $q \in C^1(\overline{\Sigma}\times[0,T])$ satisfy that $\nabla_{x'}p$, $\nabla_{x'}q \in C^{\alpha,\frac{\alpha}{2}}(\overline{\Sigma}\times[0,T])$, and in addition $\frac{\partial p}{\partial\eta} = 0$ and $\frac{\partial q}{\partial\eta} = 0$ on $\partial\Sigma\times(0,T)$. We also assume that there exists a constant $\mu_0 > 0$ such that $q(x',t) - p(x',t) \geq \mu_0$ in $\overline{\Sigma}\times[0,T]$. Let $\partial_N\mathcal{D} = \{(x,t) \in \mathcal{D}\,/\,x' \in \partial\Sigma\}$ and $\partial_D\mathcal{D} = \partial_p\mathcal{D}\setminus\partial_N\mathcal{D}$. As usual, $\partial_p\mathcal{D}$ denotes the parabolic boundary of the set $\mathcal{D}$.

Theorem 7.3. *Let $\mathcal{D}$, $\partial_N\mathcal{D}$ and $\partial_D\mathcal{D}$ as above. Let $a_i \in L^\infty(\mathcal{D})$ and $\beta \in L^\infty(\mathcal{D}\times\mathbb{R})$ such that $\beta(x,t,\cdot)$ is locally Lipschitz continuous in $\mathbb{R}$ uniformly for $(x,t) \in \mathcal{D}$. Let $\phi \in C^{\alpha,\frac{\alpha}{2}}(\overline{\mathcal{D}})$ such that $\psi_1(x',t) = \phi(p(x',t),x',t)$ and $\psi_2(x',t) = \phi(q(x',t),x',t)$ satisfy that $\psi_i \in C^1(\overline{\Sigma}\times[0,T])$ and $\nabla_{x'}\psi_i \in C^{\alpha,\frac{\alpha}{2}}(\overline{\Sigma}\times[0,T])$ for $i = 1,2$.*

There exists a unique function $u \in C(\overline{\mathcal{D}})$, with $\nabla u \in L^2(\mathcal{D})$ such that u is a weak solution to the following problem:

$$\begin{aligned}
\Delta u + \sum_i a_i(x,t)u_{x_i} - u_t = \beta(x,t,u) \qquad & \textit{in } \mathcal{D}, \\
\frac{\partial u}{\partial\eta} = 0 \qquad & \textit{on } \partial_N\mathcal{D}, \\
u = \phi \qquad & \textit{on } \partial_D\mathcal{D}.
\end{aligned}$$

Moreover, there exists $0 < \gamma \leq \alpha$ such that $u \in C^{\gamma,\frac{\gamma}{2}}(\overline{\mathcal{D}})$ and, even more, there holds that $\nabla u \in C(\overline{\mathcal{D}}\cap\{t > 0\})$.

8. Conclusions

Our results in [14] and [16] can be summarized as saying that, with these types of domains, coefficients and reaction function β_ε, and under suitable assumptions on the initial and boundary data, *if a classical solution of problem $\mathcal{P}$ exists in a certain time interval, then it is at the same time the unique classical solution, the unique limit solution and also the unique viscosity solution in that time interval.*

We show in particular that there is a unique limit solution independently of the choice of the function β. Moreover, we actually prove that the limit exists for any approximation of the initial datum.

We point out that all of these results hold for both the one phase case and the two phase case.

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