LIMIT, POINTWISE, VISCOSITY AND CLASSICAL SOLUTIONS TO A FREE BOUNDARY PROBLEM IN COMBUSTION

Claudia Lederman - Noemí Wolanski

Introduction.

In these notes we will present a survey of mathematical results on a free boundary problem in combustion which we have recently obtained in [CLW1] and [CLW2] (joint work with L. A. Caffarelli), in [LW], and in [LVW] (joint work with J. L. Vázquez). These papers deal with models of propagation of premixed flames of the following types,

$$\Delta u^\varepsilon - u^\varepsilon_t = \beta_\varepsilon(u^\varepsilon) \quad \text{or} \quad \Delta u^\varepsilon = \beta_\varepsilon(u^\varepsilon),$$

where $1/\varepsilon$ stands for the large activation energy, u^ε is a normalized temperature and $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta\big(\frac{s}{\varepsilon}\big)$. Convergence results have been obtained as $\varepsilon \to 0$ to the following FBP's:

$$\Delta u - u_t = 0 \quad \text{or} \quad \Delta u = 0,$$

in the complement of $\partial\{u > 0\}$, with free boundary conditions

$$u = 0, \quad u_\nu = \sqrt{2M} \quad \text{on } \partial\{u > 0\}$$

in the one-phase case and

$$u = 0, \quad (u^+_\nu)^2 - (u^-_\nu)^2 = 2M \quad \text{on } \partial\{u > 0\}$$

in the two-phase case. Here $M = \int \beta(s)\, ds$ and ν is the inward unit spatial normal to the free boundary $\partial\{u > 0\}$.

In the two-phase parabolic case, limits are shown to be solutions to the FBP in pointwise and viscosity senses (see [CLW1] and [CLW2]). In the elliptic case, among other results, two-phase nondegenerate limits are shown to be classical solutions (see [LW]). In the one-phase parabolic case, different notions of solution to the FBP are proved to coincide and produce a unique solution (see [LVW]).

Partially supported by UBA grants EX071, EX197 & CONICET grant PID3668/92

A. Pointwise and viscosity solutions for the limit of a two phase parabolic singular perturbation problem.

In [CLW1] and [CLW2] we are concerned with the following problem: Study the uniform properties, and the limit as $\varepsilon \to 0$, of solutions $u^\varepsilon(x,t)$ of the equation:

$$(P_\varepsilon) \qquad \Delta u^\varepsilon - u^\varepsilon_t = \beta_\varepsilon(u^\varepsilon),$$

where $\varepsilon > 0$, $\beta_\varepsilon \geq 0$, $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon})$, support β=[0,1], with β a Lipschitz continuous function and $\int \beta(s)ds = M$. Here M is a positive constant, and the functions $u^\varepsilon(x,t)$ are defined in $\mathbb{R}^N \times \mathbb{R}$, or in a subset of it.

This problem has been studied in [CV] for the one phase case, i.e., restricted to the case where $u^\varepsilon \geq 0$. It appears in combustion, in the description of laminar flames as an asymptotic limit for high activation energy. In [CV] the authors studied the initial value problem associated to P_ε. Previously, the elliptic version of this problem in the one phase case and, in particular, the convergence problem for travelling waves had been studied in [BCN].

For the two phase case, it was shown in [C1] (and in the more general work [C2]), that uniformly bounded solutions to P_ε are locally uniformly Lipschitz in space.

The purpose of [CLW1] and [CLW2] is to continue with the local study of problem P_ε: We prove further uniform estimates for uniformly bounded solutions to P_ε, we pass to the limit and we study the limit function u. As in [C1] and [C2], we are concerned with the two phase version of problem P_ε, that is, our solutions are allowed to change sign and become negative. The approach in our papers is local, also in the spirit of [C1] and [C2], since we do not force the solutions u^ε to be globally defined, nor to take on prescribed initial or boundary values.

We consider a family u^ε of solutions to P_ε in a domain $\mathcal{D} \subset \mathbb{R}^{N+1}$ which are uniformly bounded in L^∞ norm in $\mathcal{D}$. We get uniform estimates that allow us to pass to the limit as $\varepsilon \to 0$, we find properties of the limit function u in general situations, and we prove that u is a solution to the free boundary problem

$$(P) \qquad \begin{array}{ll} \Delta u - u_t = 0 & \text{in } \mathcal{D} \setminus \partial\{u > 0\} \\ u = 0 \ , \ (u_\nu^+)^2 - (u_\nu^-)^2 = 2M & \text{on } \mathcal{D} \cap \partial\{u > 0\} \end{array}$$

in a pointwise sense at regular free boundary points and in a viscosity sense when $\{u \equiv 0\}^\circ = \emptyset$. (Here ν is the inward unit spatial normal to the free boundary $\mathcal{D} \cap \partial\{u > 0\}$, $u^+ = \max(u,0)$ and $u^- = \max(-u,0)$).

First, we prove that the functions u^ε are locally uniformly Hölder continuous with exponent $\frac{1}{2}$ in time in $\mathcal{D}$. Next, we get further uniform estimates and we start our study of the limit function u. We show that u satisfies

$$\Delta u - u_t = \mu$$

where μ is a nonnegative measure which is supported on $\mathcal{D} \cap \partial\{u > 0\}$.

In order to understand the nature of the measure μ, we study the case in which the limit function u is the difference of two hyperplanes. In the case of a two phase limit we show that u satisfies the free boundary condition

$$(u_\nu^+)^2 - (u_\nu^-)^2 = 2M \quad \text{on } \partial\{u > 0\} \tag{1}$$

which is the extension to the two phase case of the free boundary condition obtained in [BCN] and [CV]. On the other hand we show that for a range of values of α with $0 < \alpha < \sqrt{2M}$ there are locally uniformly bounded solutions to P_ε converging to $\alpha x_1^+ + \alpha x_1^-$. This indicates that some extra hypotheses should be made if one intends to prove that a limit function satisfies the free boundary condition (1) in a classical sense.

Then, we study the behavior of a limit function near an arbitrary free boundary point $(x_0, t_0) \in \mathcal{D} \cap \partial\{u > 0\}$. We prove that if $\limsup_{(x,t)\to(x_0,t_0)} |\nabla u^-| \le \gamma$, then we have $\limsup_{(x,t)\to(x_0,t_0)} |\nabla u^+| \le \sqrt{2M + \gamma^2}$.

Next, we introduce the concepts of pointwise and viscosity solutions to problem P. On one hand, in Theorem A.1 we prove that, in the two phase case, the free boundary condition (1) is satisfied in a pointwise sense at any "regular" free boundary point (x_0, t_0). We point out that we only make assumptions on u at (x_0, t_0).

On the other hand we prove, when $\{u \equiv 0\}^\circ = \emptyset$, that the limit function u is a viscosity solution to the free boundary problem P (Theorem A.2). By a viscosity solution, we mean a weak solution of the free boundary problem in the sense that it is a continuous function for which comparison principles with classical supersolutions and subsolutions hold.

We point out that any limit function u is a viscosity supersolution (Theorem 4.1 in [CLW2]) (this is, we do not need the assumption $\{u \equiv 0\}^\circ = \emptyset$).

To conclude this section we state here the main results in [CLW1] and [CLW2]:

Theorem A.1. *(Theorem 3.1 in [CLW2]) Let u^{ε_j} be solutions to P_{ε_j} in a domain $\mathcal{D} \subset \mathbb{R}^{N+1}$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of $\mathcal{D}$ and $\varepsilon_j \to 0$. Let $(x_0, t_0) \in \mathcal{D} \cap \partial\{u > 0\}$ be such that there exists an inward unit spatial normal to the free boundary $\mathcal{D} \cap \partial\{u > 0\}$ in a parabolic measure theoretic sense (see Def. A.1) and such that $\{u \equiv 0\}$ has zero "parabolic density" (see Def. A.2) at (x_0, t_0). Then, there exist $\alpha > 0, \gamma \ge 0$ such that*

$$u(x,t) = \alpha\langle x - x_0, \nu\rangle^+ - \gamma\langle x - x_0, \nu\rangle^- + o(|x - x_0| + |t - t_0|^{\frac{1}{2}})$$

with

$$\alpha^2 - \gamma^2 = 2M.$$

Here ν is the inward unit spatial normal to the free boundary at (x_0, t_0).

Theorem A.2. *(Theorems 4.1, 4.2 in [CLW2]) Let* u^{ε_j} *be a family of solutions to* P_{ε_j} *in a domain* $\mathcal{D} \subset \mathbb{R}^{N+1}$ *such that* $u^{\varepsilon_j} \to u$ *uniformly on compact subsets of* $\mathcal{D}$ *and* $\varepsilon_j \to 0$. *If* $\{u \equiv 0\}^{\circ} \cap \mathcal{D} = \emptyset$, *then* u *is a viscosity solution of the free boundary problem* P.

Definition A.1. We say that ν is the inward unit spatial normal to the free boundary $\partial\{u > 0\}$ at a point $(x_0, t_0) \in \partial\{u > 0\}$ in the parabolic measure theoretic sense, if $\nu \in \mathbb{R}^N$, $|\nu| = 1$ and

$$\lim_{r \to 0} \frac{1}{r^{N+2}} \iint_{Q_r(x_0,t_0)} |\chi_{\{u>0\}} - \chi_{\{(x,t)\,/\,\langle x-x_0,\nu\rangle>0\}}|\, dx\, dt = 0.$$

(We denote $Q_r(x_0, t_0) = \{(x,t)\,/\,|x - x_0| < r\,,\ |t - t_0| < r^2\}$).

Definition A.2. We say that $\{u \equiv 0\}$ has zero "parabolic density" at $(x_0, t_0) \in \partial\{u > 0\}$ if there holds that

$$\lim_{r \to 0} \frac{|\{u \equiv 0\} \cap Q_r(x_0, t_0)|}{|Q_r(x_0, t_0)|} = 0.$$

B. Viscosity solutions and regularity of the free boundary for the limit of an elliptic two phase singular perturbation problem.

In [LW] we pursue further the study in [CLW1] and [CLW2] in the case where the functions u^{ε} do not depend on t; this is, u^{ε} are solutions to

$$(E_\varepsilon) \qquad \Delta u^{\varepsilon} = \beta_\varepsilon(u^{\varepsilon}).$$

In fact, we consider a family u^{ε} of uniformly bounded solutions to E_ε in a domain $\Omega \subset \mathbb{R}^N$ and we prove that the limit function u is a solution —in a more appropriate elliptic sense— to the free boundary problem:

$$(E) \qquad \begin{array}{ll} \Delta u = 0 & \text{in } \Omega \setminus \partial\{u > 0\}, \\ u = 0,\ (u_\nu^+)^2 - (u_\nu^-)^2 = 2M & \text{on } \Omega \cap \partial\{u > 0\}. \end{array}$$

Here ν is the inward unit normal to the free boundary $\Omega \cap \partial\{u > 0\}$. Moreover, under suitable assumptions, we prove that the free boundary is smooth and therefore, the free boundary condition is satisfied in the classical sense.

As in [CLW1] and [CLW2], our study here is of a local nature and we consider the two phase case.

Our first result is of a pointwise nature. Although the pointwise result described above (Theorem A.1) applies to the present elliptic situation, we get here —under weaker assumptions at the point $x_0 \in \partial\{u > 0\}$— the same conclusion: there holds that $(u_\nu^+)^2 - (u_\nu^-)^2 = 2M$ at x_0, where ν is the inward unit normal to $\partial\{u > 0\}$ at

x_0 in the measure theoretic sense (Theorem B.1). This result applies, in particular, to the one phase case.

Next, we prove that under suitable assumptions, our limit function u is a solution to E in a viscosity (elliptic) sense (Theorem B.2). A viscosity (elliptic) solution is a continuous function, harmonic away from the free boundary, which satisfies the free boundary condition in terms of proper asymptotic developments. In particular, we prove that u is a viscosity solution if $\{u \equiv 0\}^\circ = \emptyset$ (Corollary 4.1 in [LW]) or if u^+ is nondegenerate on $\partial\{u > 0\}$ (Corollary 4.2 in [LW]).

Finally, we study the regularity of the free boundary. We use results from [C3] —where it was developed a regularity theory for viscosity solutions to a class of elliptic free boundary problems which includes E. We want to remark here that there are limit functions u which do not satisfy the free boundary condition in the classical sense on any portion of $\partial\{u > 0\}$ (for instance, $u = \alpha x_1^+ + \alpha x_1^-$ with $0 < \alpha < \sqrt{2M}$, see [CLW1], Remark 5.1). Thus, extra hypotheses have to be made in order to get regularity results.

On one hand, we prove that, under suitable assumptions on our limit function u, there is a subset of the free boundary $\partial\{u > 0\}$ which is locally a $C^{1,\alpha}$ surface. As a consequence, on this portion of $\partial\{u > 0\}$, the free boundary condition is satisfied in the classical sense. Moreover, this smooth subset is open and dense in $\partial\{u > 0\}$ and the remainder of the free boundary has $(N-1)$-dimensional Hausdorff measure zero (Theorem B.3).

On the other hand, we prove in Theorem B.4 that if u^- is locally uniformly nondegenerate on $\partial\{u > 0\}$, then the free boundary is locally a $C^{1,\alpha}$ surface. Hence, there are no singularities. We point out that the local uniform nondegeneracy of u^- on $\partial\{u > 0\}$ is a necessary condition for the regularity of the free boundary in the strictly two phase case, even if we only require the free boundary to be locally Lipschitz continuous (see Remark 5.2 in [LW]).

To conclude this section, we state here the main results in [LW]:

Theorem B.1. *(Theorem 3.1 in [LW]) Let u^{ε_j} be solutions to E_{ε_j} in a domain $\Omega \subset \mathbb{R}^N$ such that u^{ε_j} converge to a function u uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ be such that $\partial\{u > 0\}$ has at x_0 an inward unit normal ν in the measure theoretic sense. If $\liminf_{r\to 0} \frac{|\{u<0\}\cap B_r(x_0)|}{|B_r(x_0)|} = 0$, we assume, in addition, that u^+ is nondegenerate at x_0 (see Def. B.1).*

Then, there exist $\alpha > 0$ and $\gamma \geq 0$, such that

$$u(x) = \alpha\langle x - x_0, \nu\rangle^+ - \gamma\langle x - x_0, \nu\rangle^- + o(|x - x_0|)$$

with

$$\alpha^2 - \gamma^2 = 2M.$$

Theorem B.2. *(Theorems 4.1, 4.2 in [LW]) Let u^{ε_j} be solutions to E_{ε_j} in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Assume, in addition, that at every regular point from the nonpositive side $x_0 \in \Omega \cap \partial\{u > 0\}$ (this is, such that there exists a ball $B_\rho(y) \subset \{u \le 0\}$ with $x_0 \in \partial B_\rho(y)$), u satisfies one of the following hypotheses:*

H1) *u^+ is nondegenerate at x_0 (see Def B.1).*

or else

H2) *$u < 0$ in $B_\rho(y)$.*

Then u is a viscosity solution in Ω.

Theorem B.3. *(Theorem 5.1 in [LW]) Let u^{ε_j} be solutions to E_{ε_j} in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Assume, in addition, that:*

i) *u^+ is locally uniformly nondegenerate on $\Omega \cap \partial\{u > 0\}$ (see Def. B.1)*

ii) *The set $\{u \le 0\}$ has locally uniform positive density on $\Omega \cap \partial\{u > 0\}$ (see Def.B.2).*

Then, there is a subset $\mathcal{R}$ of the free boundary $\Omega \cap \partial\{u > 0\}$ ($\mathcal{R} = \partial_{\rm red}\{u > 0\}$) which is locally a $C^{1,\alpha}$ surface. Moreover, $\mathcal{R}$ is open and dense in $\Omega \cap \partial\{u > 0\}$ and the remainder of the free boundary has $(N-1)$-dimensional Hausdorff measure zero.

Theorem B.4. *(Theorem 5.2 in [LW]) Let u^{ε_j} be solutions to E_{ε_j} in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Assume, in addition, that there is $\mathcal{U} \subset\subset \Omega$ such that u^- is uniformly nondegenerate on $\mathcal{U} \cap \partial\{u > 0\}$ (see Def. B.1).*

Then $\mathcal{U} \cap \partial\{u > 0\}$ is locally a $C^{1,\alpha}$ surface.

Definition B.1. Let $v \ge 0$ be a continuous function in a domain $\Omega \subset \mathbb{R}^N$. We say that v is nondegenerate at a point $x_0 \in \Omega \cap \{v = 0\}$ if there exist $c > 0$ and $r_0 > 0$ such that

$$-\!\!\!\!\!\!\int_{B_r(x_0)} v \ge c\, r \qquad \text{for } 0 < r \le r_0. \tag{2}$$

We say that v is uniformly nondegenerate on $\Gamma \subset \Omega \cap \{v = 0\}$, if there exist $c > 0$ and $r_0 > 0$ such that (2) holds for every $x_0 \in \Gamma$.

Definition B.2. Let v be a continuous function in a domain $\Omega \subset \mathbb{R}^N$. We say that the set $\{v \le 0\}$ has positive density at a point $x_0 \in \Omega \cap \partial\{v > 0\}$, if there exist $c > 0$ and $r_0 > 0$ such that

$$\frac{|\{v \le 0\} \cap B_r(x_0)|}{|B_r(x_0)|} \ge c \qquad \text{for } 0 < r \le r_0. \tag{3}$$

We say that the set $\{v \le 0\}$ has uniform positive density on $\Gamma \subset \Omega \cap \partial\{v > 0\}$, if there exist $c > 0$ and $r_0 > 0$ such that (3) holds for every $x_0 \in \Gamma$.

C. Uniqueness of solution to a free boundary problem from combustion.

In [LVW] we consider the initial-boundary value problem associated to the free boundary problem (P) described in Part A, in the one phase case (i.e., assuming $u \geq 0$). We will call this problem (IVP). The purpose of [LVW] is to contribute to the questions of unique characterization of the solution to the free boundary problem (IVP) and the consistency of the different solution concepts.

This free boundary problem admits classical solutions only for good data in the small. Several generalized concepts of solution have been proposed, among them the concepts of limit solution and viscosity solution. We investigate conditions under which the three concepts agree and produce a unique solution.

A limit solution is a function which is obtained as the limit (as $\varepsilon \to 0$) of solutions u^ε to

$$(P_\varepsilon) \qquad \Delta u^\varepsilon - u^\varepsilon_t = \beta_\varepsilon(u^\varepsilon)$$

where $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon})$, with β a Lipschitz continuous function such that $\beta > 0$ in $(0,1)$ and zero otherwise, and $\int \beta(s)\, ds = M$.

A viscosity solution is a continuous function which satisfies comparison principles with classical supersolutions and subsolutions to the free boundary problem.

Theorem C.1. *(Theorem 5.1 in [LVW]) Let us assume that $\Omega = (0,+\infty) \times \Sigma$, with $\Sigma \subset \mathbb{R}^{N-1}$, a smooth bounded domain. Let u be a classical solution to IVP in $\overline{\Omega} \times [0,T]$ such that $\frac{\partial u}{\partial \eta} = 0$ on $(0,+\infty) \times \partial\Sigma \times (0,T)$. Assume, in addition, that $\partial\{u>0\}$ is bounded and that $u_0(x) = u(x,0)$ is a strictly decreasing function on $\overline{\{u_0 > 0\}}$ in the direction e_1. We also require that $u(0,x',t) > c_1 > 0$ and $u_{x_1}(0,x',t) \leq -c_2 < 0$ for $(x',t) \in \Sigma \times (0,T)$.*

Let $\widehat{u} = \lim u^\varepsilon$, where u^ε is a family of bounded solutions to P_ε with $\frac{\partial u^\varepsilon}{\partial \eta} = 0$ on $(0,+\infty) \times \partial\Sigma \times (0,T)$, such that $u^\varepsilon \to \widehat{u}$ uniformly on compact subsets of $\overline{\Omega} \times [0,T]$.

Assume, that $\widehat{u} = u$ on $\partial_p(\Omega \times (0,T)) \cap (\{x_1 = 0\} \cup \{t = 0\})$ and, in addition, $\{x \,/\, u^\varepsilon(x,0) > 0\} \to \{x \,/\, u_0(x) > 0\}$.

Then $\widehat{u} = u$.

Remark C.1. In order for P_ε to approximate problem IVP properly, we need to impose some condition on how the initial datum $u_0(x)$ is approximated by $u_0^\varepsilon(x)$ so that we get a limit solution with a free boundary starting from the initial free boundary $\partial\{x \,/\, u_0(x) > 0\}$. In fact, if for example we take u_0^ε satisfying $u_0^\varepsilon \geq \varepsilon$, we obtain that the limit function $\widehat{u}$ is the solution to the heat equation with initial datum u_0, and therefore, the uniqueness result stated above does not hold. (However, we can relax the assumption that $\{x \,/\, u_0^\varepsilon(x) > 0\} \to \{x \,/\, u_0(x) > 0\}$, imposing instead a suitable control on the growth of u_0^ε away from $\{x \,/\, u_0(x) > 0\}$).

Theorem C.2. *(Theorem 6.1 in [LVW]) Let $\mathcal{D} = \Omega \times (0,T)$, where $\Omega = \mathbb{R} \times \Sigma$, with $\Sigma \subset \mathbb{R}^{N-1}$ a smooth bounded domain. Let u be a classical solution to IVP in $\overline{\mathcal{D}}$ and assume that, on $\overline{\{u>0\}} \cap \partial_p \mathcal{D}$, u is a strictly decreasing function in the direction e_1. Let v be a viscosity solution to IVP in $\overline{\mathcal{D}}$ such that*

(i) $v = u$ *on* $\partial_p \mathcal{D}$,

(ii) $\overline{\{v>0\}} \cap \partial_p \mathcal{D} = \overline{\{u>0\}} \cap \partial_p \mathcal{D}$.

Then $v = u$ in $\overline{\mathcal{D}}$.

In addition, we can prove the same result if we let $\Omega = (0,+\infty) \times \Sigma$. In this case, we require that $u(0,x',t) > c_1 > 0$ and $u_{x_1}(0,x',t) \le -c_2 < 0$ for $(x',t) \in \Sigma \times (0,T)$.

Moreover, the result also holds if we replace on $(0,+\infty) \times \partial\Sigma \times (0,T)$ (or $\mathbb{R} \times \partial\Sigma \times (0,T)$) the Dirichlet condition by the homogeneous Neumann condition.

Remark C.2. Conditions (i) and (ii) imply that the free boundaries of u and v coincide on $\partial_p \mathcal{D}$ and both start from $\partial\{u_0 > 0\}$. A condition like (ii) is necessary in order to get the uniqueness result since, for instance, the solution of the heat equation with initial datum u_0 is a viscosity solution to IVP.

Proposition C.1. *If u is a classical solution to IVP in $\overline{\mathcal{D}}$ then u is a viscosity solution to IVP in $\overline{\mathcal{D}}$.*

Final Comments on Part C. In conclusion, under suitable assumptions on the domain and on the initial and boundary data, *if a classical solution to the free boundary problem exists in a certain time interval, then it is at the same time the unique classical solution, the unique limit solution and also the unique viscosity solution in that time interval. In particular, there is a unique limit solution independently of the choice of the function β.*

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Claudia Lederman
Departamento de Matematica
Facultad de Ciencias Exactas
Universidad de Buenos Aires
(1428) Buenos Aires - Argentina

E-mail address: `clederma@mate.dm.uba.ar`

Noemí Wolanski
Departamento de Matematica
Facultad de Ciencias Exactas
Universidad de Buenos Aires
(1428) Buenos Aires - Argentina

E-mail address: `wolanski@mate.dm.uba.ar`