VISCOSITY SOLUTIONS AND REGULARITY OF THE FREE BOUNDARY FOR THE LIMIT OF AN ELLIPTIC TWO PHASE SINGULAR PERTURBATION PROBLEM

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1. Introduction.

In this paper we are concerned with the following problem: Study the limit as $\varepsilon \to 0$, of solutions $u^\varepsilon(x)$ to the equation:

$$(E_\varepsilon) \qquad \Delta u^\varepsilon = \beta_\varepsilon(u^\varepsilon),$$

where $\varepsilon > 0$ and $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon})$. Here β is a Lipschitz continuous function with $\beta > 0$ in (0,1), $\beta \equiv 0$ outside (0,1) and $\int \beta(s)ds = M$, where M is a positive constant. The functions $u^\varepsilon(x)$ are defined in $\mathbb{R}^N$, or in a subset of it.

This problem is of interest in the theory of flame propagation. It appears in combustion, in the description of laminar flames as an asymptotic limit for high activation energy (see for instance [3], [5], [18]).

We consider a family u^ε of uniformly bounded solutions to E_ε in a domain $\Omega \subset \mathbb{R}^N$ and we prove that the limit function u is a solution, in an appropriate sense, to the free boundary problem:

$$(E) \qquad \begin{aligned} &\Delta u = 0 && \text{in } \Omega \setminus \partial\{u > 0\}, \\ &u = 0,\ (u_\nu^+)^2 - (u_\nu^-)^2 = 2M && \text{on } \Omega \cap \partial\{u > 0\}, \end{aligned}$$

where M is as above, $u^+ = \max(u, 0)$, $u^- = \max(-u, 0)$ and ν is the inward unit normal to the free boundary $\Omega \cap \partial\{u > 0\}$. In particular, under suitable assumptions, we prove that the free boundary is smooth and therefore, the free boundary condition is satisfied in the classical sense.

The approach in our paper is local, since we do not force the solutions u^ε to be globally defined nor to take on prescribed boundary values. On the other hand, we are concerned with the two phase version of problem E_ε, that is, our solutions are allowed to change sign and become negative.

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First, we prove that u is a solution to E in a pointwise sense. That is, u is harmonic in $\{u > 0\} \cup \{u \leq 0\}^\circ$ and satisfies the free boundary condition at every "regular" free boundary point (Theorem 3.1). Namely, we prove that u has an asymptotic development and that $(u_\nu^+)^2 - (u_\nu^-)^2 = 2M$ holds at a point $x_0 \in \partial\{u > 0\}$, by making only assumptions on u at x_0 (here ν is the inward unit normal to $\partial\{u > 0\}$ at x_0 in the measure theoretic sense).

Next, we prove that under suitable assumptions, our limit function u is a solution to E in a viscosity sense (Theorems 4.1 and 4.2). By a viscosity solution we mean a weak solution to the free boundary problem in the sense introduced in [6] and [7]. In particular, we prove that u is a viscosity solution if $\{u \equiv 0\}^\circ = \emptyset$ (Corollary 4.1) or if u^+ is nondegenerate on $\partial\{u > 0\}$ (Corollary 4.2).

Finally, we study the regularity of the free boundary. We want to remark here that there are limit functions u which do not satisfy the free boundary condition in the classical sense on any portion of $\partial\{u > 0\}$ (for instance, $u = \alpha x_1^+ + \alpha x_1^-$ with $0 < \alpha < \sqrt{2M}$, see [11], Remark 5.1). Thus, extra hypotheses have to be made in order to get regularity results.

On one hand, we prove in Theorem 5.1 that in the absence of zero phase, if u^- is nondegenerate at $x_0 \in \partial\{u > 0\}$, then the free boundary is a $C^{1,\alpha}$ surface in a neighborhood of x_0. Therefore u is a classical solution to E in that neighborhood. We point out that in the strictly two phase case, if the free boundary is smooth, then u^- is nondegerate on the free boundary (see Remark 5.1). One of the corollaries to Theorem 5.1 states that in the absence of zero phase, there is a subset of the free boundary which is locally a $C^{1,\alpha}$ surface. This subset is open and dense in $\partial\{u < 0\}$ (Corollary 5.1).

On the other hand, we prove for the general two phase case that, under nondegeneracy assumptions on u^+, if $x_0 \in \Omega \cap \partial\{u > 0\}$ is such that the free boundary has at x_0 an inward unit normal in the measure theoretic sense, then the free boundary is a $C^{1,\alpha}$ surface in a neighborhood of x_0 (Theorem 5.2). Then, in Corollary 5.3 we prove that, under suitable assumptions on our limit function u, there is a subset of the free boundary which is locally a $C^{1,\alpha}$ surface. Moreover, this smooth subset is open and dense in $\partial\{u > 0\}$ and the remainder of the free boundary has $(N-1)$-dimensional Hausdorff measure zero. We end our work with further results for the one phase case (Theorems 5.3 and 5.4).

The parabolic version of this two phase problem was first studied in [9], [10] and [11], where uniform estimates for uniformly bounded solutions were obtained.

These estimates allow the passage to the limit, as $\varepsilon \to 0$, uniformly. Then, in [12] it was proved that, in the strictly two phase case, the limit function u is a solution in $\mathcal{D} \subset \mathbb{R}^{N+1}$ to the free boundary problem

$$\begin{aligned} &\Delta u - u_t = 0 \qquad \text{in } \mathcal{D} \setminus \partial\{u > 0\}, \\ &u = 0,\ (u_\nu^+)^2 - (u_\nu^-)^2 = 2M \qquad \text{on } \mathcal{D} \cap \partial\{u > 0\}, \end{aligned}$$

in a pointwise sense at regular free boundary points, and in a parabolic viscosity sense (ν is the inward unit spatial normal to the free boundary). All these results apply, in particular, to the present elliptic situation.

A parabolic viscosity solution is a continuous function which satisfies local parabolic comparison principles with classical supersolutions and subsolutions to the evolutionary free boundary problem.

If both the parabolic viscosity solution and the classical subsolution or supersolution are time independent these comparison principles give no information.

Therefore, the notion of parabolic viscosity solution is not as appropriate for an elliptic problem as the one we use in this paper. This is the reason why we prove in Section 4, that limit functions are viscosity solutions in the elliptic sense of [6] and [7]. The results of Section 4 apply, in particular, to the one phase case.

On the other hand, we here improve, in Theorem 3.1, the pointwise result in [12], since it now applies to the one phase case.

In addition to the intrinsic interest of the results of Sections 3 and 4, they are used to prove the regularity results in Section 5.

A mathematical idea introduced in this paper is the use of the local parabolic monotonicity formula ([9]) together with convexity results for eigenvalues ([4]) to derive regularity results of interfaces (see Proposition 5.1 and Theorem 5.1). This new idea replaces more usual geometric measure theoretic arguments.

The use of the local parabolic monotonicity formula, instead of an elliptic one, allows us to prove Proposition 5.1 —which is of independent interest. An analogous result was proven in [2] for dimension $N = 2$ by using an elliptic monotonicity formula (see [2], Lemma 6.6 and Remark 6.1).

In Section 5 we make a substantial use of the regularity results of free boundaries of [6] and [7].

The problem considered in this paper was first studied in the pioneer work [3] in the one phase case and for a more general elliptic equation. The authors proved uniform estimates and found that the limit function u satisfies the free boundary condition $u = 0$, $u_\nu = \sqrt{2M}$ when $\partial\{u > 0\}$ is smooth. For other related works see, for instance, [9], [10], [11], [12], [13], [16] and [18].

Our paper is organized in the following way. We consider a family u^ε of solutions to E_ε which are uniformly bounded in L^∞ norm. In Section 2 we state some preliminary results known for such a family, and for the limit function u (as $\varepsilon \to 0$). In Section 3 we prove that the free boundary condition is satisfied at every "regular" free boundary point. We also show that, under extra hypotheses, a representation formula holds. Next, we prove in Section 4 that the limit function u is a viscosity solution to the free boundary problem E. Finally, in Section 5 we study the regularity of the free boundary.

Notation

Throughout the paper N will denote the spatial dimension and $M = \int_0^1 \beta(s)\,ds$. In addition, the following notation will be used:

- $|S|$ N-dimensional Lebesgue measure of the set S
- $\mathcal{H}^{N-1}$ $(N-1)$-dimensional Hausdorff measure
- $B_r(x_0)$ open ball of radius r and center x_0
- $\fint_{B_r(x_0)} u = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u\,dx$
- $\fint_{\partial B_r(x_0)} u = \frac{1}{\mathcal{H}^{N-1}(\partial B_r(x_0))} \int_{\partial B_r(x_0)} u\,d\mathcal{H}^{N-1}$
- χ_S characteristic function of the set S
- $u^+ = \max(u,0)$, $u^- = \max(-u,0)$
- $\langle \cdot\,,\cdot \rangle$ scalar product in $\mathbb{R}^N$

2. Preliminary results.

In this section we consider a family u^ε of solutions to E_ε in a domain $\Omega \subset \mathbb{R}^N$ which are uniformly bounded in L^∞ norm in Ω and we state —for the sake of completeness— some results known for such a family, on uniform estimates and passage to the limit as $\varepsilon \to 0$, which will be used throughout the paper (Props. 2.1, 2.2 and 2.3). These results were proven in [9], [10] and [11], for the parabolic version of this problem.

We also state some results from [11] on the behavior of certain limit functions.

We remark here that if $u^\varepsilon \in L^\infty_{\rm loc}(\Omega)$ is a solution to E_ε in the distributional sense, then u^ε is a classical solution to E_ε. Therefore, when referring to a solution to E_ε, we will always mean a classical solution.

The following result was proven in [3] for the one phase case, i.e., under the assumption that $u^\varepsilon \geq 0$. For the two phase case, the result is due to Caffarelli; he proved it for the parabolic version of the problem.

Proposition 2.1. *([9], Theorem 3 or [10], Corollary 2) Let u^ε be a family of solutions to E_ε in a domain $\Omega \subset \mathbb{R}^N$ such that $||u^\varepsilon||_{L^\infty(\Omega)} \leq \mathcal{A}$ for some $\mathcal{A} > 0$. Let $K \subset \Omega$ be a compact set and let $\tau > 0$ be such that $B_\tau(x_0) \subset \Omega$, for every $x_0 \in K$. Then, there exists a constant $L = L(\tau, \mathcal{A})$, such that*

$$|\nabla u^\varepsilon(x)| \leq L \qquad \text{for } x \in K.$$

The following result was proven in [3] for the one phase elliptic problem and in [11] for the two phase parabolic problem by using ideas which are similar to those in [3].

Proposition 2.2. *([11], Lemma 3.1 and [11], Proposition 3.1) Let u^ε be a family of solutions to E_ε in a domain $\Omega \subset \mathbb{R}^N$. Let us assume that $||u^\varepsilon||_{L^\infty(\Omega)} \leq \mathcal{A}$ for some $\mathcal{A} > 0$. For every $\varepsilon_n \to 0$ there exist a subsequence $\varepsilon_{n'} \to 0$ and a function u, which is locally Lipschitz continuous in Ω, such that*

i) $u^{\varepsilon_{n'}} \to u$ uniformly on compact subsets of Ω,
ii) $\nabla u^{\varepsilon_{n'}} \to \nabla u$ in $L^2_{\mathrm{loc}}(\Omega)$,
iii) $\Delta u^{\varepsilon_{n'}} \to \Delta u$ locally as measures in Ω,
iv) $\Delta u \geq 0$ in Ω,
v) $\Delta u = 0$ in $\Omega \setminus \partial\{u > 0\}$.

Next, we state a result proven in [11] in the parabolic case.

Proposition 2.3. *([11], Lemma 3.2) Let u^{ε_j} be a family of solutions to E_{ε_j} in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Let $\lambda_n > 0$ and $x_n \in \Omega \cap \partial\{u > 0\}$ be sequences such that $\lambda_n \to 0$ and $x_n \to x_0 \in \Omega \cap \partial\{u > 0\}$ as $n \to \infty$. Let $u_{\lambda_n}(x) = \frac{1}{\lambda_n} u(x_n + \lambda_n x)$ and $(u^{\varepsilon_j})_{\lambda_n}(x) = \frac{1}{\lambda_n} u^{\varepsilon_j}(x_n + \lambda_n x)$. Assume that $u_{\lambda_n} \to U$ as $n \to \infty$ uniformly on compact sets of $\mathbb{R}^N$. Then, there exists $j(n) \to +\infty$ such that for every $j_n \geq j(n)$, there holds that $\frac{\varepsilon_{j_n}}{\lambda_n} \to 0$ and*

1) $(u^{\varepsilon_{j_n}})_{\lambda_n} \to U$ uniformly on compact sets of $\mathbb{R}^N$,

2) $\nabla (u^{\varepsilon_{j_n}})_{\lambda_n} \to \nabla U$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$.

Also, there holds that

3) $\nabla u_{\lambda_n} \to \nabla U$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$.

Finally, we state two results on the behavior of certain limit functions, that were proven in [11] in the parabolic case.

Proposition 2.4. *([11], Proposition 5.3). Let u^{ε_j} be solutions to E_{ε_j} in a domain $\Omega \subset \mathbb{R}^N$. Let $x_0 \in \Omega$ and assume that u^{ε_j} converge to $\alpha(x-x_0)_1^+ + \overline{\alpha}(x-x_0)_1^-$ uniformly on compact subsets of Ω, with α and $\overline{\alpha}$ positive and $\varepsilon_j \to 0$. Then*

$$\alpha = \overline{\alpha} \leq \sqrt{2M}.$$

Proposition 2.5. *([11], Theorem 6.2). Let u^{ε_j} be a solution to E_{ε_j} in a domain $\Omega_j \subset \mathbb{R}^N$ such that $\Omega_j \subset \Omega_{j+1}$ and $\cup\Omega_j = \mathbb{R}^N$. Let us assume that u^{ε_j} converge to a function U uniformly on compact sets of $\mathbb{R}^N$ and $\varepsilon_j \to 0$. Assume, in addition, that $|\nabla U| \leq L$ in $\mathbb{R}^N$ for some $L > 0$ and $\partial\{U > 0\} \neq \emptyset$. If $\gamma \geq 0$ is such that $|\nabla U^-| \leq \gamma$ in $\mathbb{R}^N$ then,*

$$|\nabla U^+| \leq \sqrt{2M + \gamma^2} \quad \text{in } \mathbb{R}^N.$$

3. Asymptotic development at regular free boundary points.

In this section we consider $u = \lim u^\varepsilon$ (as $\varepsilon \to 0$) and we show that the free boundary condition $(u_\nu^+)^2 - (u_\nu^-)^2 = 2M$ is satisfied in a pointwise sense at every "regular" free boundary point, this is, at every free boundary point x_0 where there is an inward unit normal ν in the measure theoretic sense. (If $\liminf_{r\to 0} \frac{|\{u<0\}\cap B_r(x_0)|}{|B_r(x_0)|} = 0$ we make, in addition, a nondegeneracy assumption on u^+ at the point). In fact, we show that u has an asymptotic development at any such point, which implies that there exist both u_ν^+ and u_ν^- and that the free boundary condition is satisfied (Theorem 3.1).

This asymptotic development, on the other hand, will be frequently used in the next sections.

In Theorem 3.2 we find (via the application of results in [1]) a representation formula for u, which holds when u^+ is locally uniformly nondegenerate on $\partial\{u > 0\}$. In particular, in this case the free boundary has locally finite $(N-1)$-dimensional Hausdorff measure.

We start this section with some definitions.

Definition 3.1. Let $v \geq 0$ be a continuous function in a domain $\Omega \subset \mathbb{R}^N$. We say that v is nondegenerate at a point $x_0 \in \Omega \cap \{v = 0\}$ if there exist $c > 0$ and $r_0 > 0$

such that one of the following conditions holds:

$$\fint_{\partial B_r(x_0)} v \geq c\,r \qquad \text{for } 0 < r \leq r_0, \tag{3.1}$$

$$\fint_{B_r(x_0)} v \geq c\,r \qquad \text{for } 0 < r \leq r_0. \tag{3.2}$$

We say that v is uniformly nondegenerate on $\Gamma \subset \Omega \cap \{v = 0\}$ in the sense of (3.1) (resp. (3.2)), if there exist $c > 0$ and $r_0 > 0$ such that (3.1) (resp. (3.2)) holds for every $x_0 \in \Gamma$.

Remark 3.1. It is easy to see that (3.1) implies (3.2). On the other hand, if $v \geq 0$ is locally Lipschitz continuous in a domain $\Omega \subset \mathbb{R}^N$ and harmonic in $\{v > 0\}$ (which will be our case), there holds that if v is nondegenerate at $x_0 \in \Omega \cap \{v = 0\}$ in the sense of (3.2) then, v is nondegenerate at x_0 in the sense of (3.1). Analogously, under the assumptions above, if $\Omega' \subset\subset \Omega$ and v is uniformly nondegenerate on $\Gamma \subset \Omega' \cap \{v = 0\}$ in the sense of (3.2) then, v is uniformly nondegenerate on Γ in the sense of (3.1).

Let us prove this last assertion (the pointwise result follows similarly). In fact, if the result were not true, there would exist a sequence $x_n \in \Gamma$ such that

$$\fint_{\partial B_{r_n}(x_n)} v \leq \frac{1}{n} r_n \qquad \text{with } r_n \to 0.$$

Let $v_{r_n}(x) := \frac{1}{r_n} v(x_n + r_n x)$. Then, there exists a subsequence, that we still call v_{r_n}, such that $v_{r_n} \to v_0$ uniformly on compact sets of $\mathbb{R}^N$, where $v_0 \geq 0$ is Lipschitz continuous in $\mathbb{R}^N$ and harmonic in $\{v_0 > 0\}$.

By rescaling, we deduce that $\fint_{\partial B_1(0)} v_{r_n} \leq \frac{1}{n}$ and therefore,

$$\fint_{\partial B_1(0)} v_0 = 0.$$

Since v_0 is globally subharmonic, it follows that $v_0 \equiv 0$ in $B_1(0)$. On the other hand, by hypothesis we have $0 < c \leq \fint_{B_1(0)} v_{r_n}$ and thus,

$$c \leq \fint_{B_1(0)} v_0,$$

which is a contradiction and proves the result.

Definition 3.2. Let u be a continuous function in a domain $\Omega \subset \mathbb{R}^N$ and let $x_0 \in \Omega \cap \partial\{u > 0\}$. We say that a unit vector ν is the inward unit normal to $\partial\{u > 0\}$ at x_0 in the measure theoretic sense if (see for instance [14])

$$\lim_{r\to 0} \fint_{B_r(x_0)} |\chi_{\{u>0\}} - \chi_{\{x\,/\,\langle x-x_0,\nu\rangle>0\}}|\,dx = 0. \tag{3.3}$$

First we prove the following lemma

Lemma 3.1. *Let u^{ε_j} be solutions to E_{ε_j} in a domain $\Omega \subset \mathbb{R}^N$ such that u^{ε_j} converge to a function u uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ and, for $\lambda > 0$, let $u_\lambda(x) = \frac{1}{\lambda}u(x_0 + \lambda x)$. Let $\lambda_n \to 0$ and $\tilde\lambda_n \to 0$ be such that*

$$u_{\lambda_n} \to U = \alpha x_1^+ - \gamma x_1^- + o(|x|),$$
$$u_{\tilde\lambda_n} \to \widetilde{U} = \tilde\alpha x_1^+ - \tilde\gamma x_1^- + o(|x|),$$

uniformly on compact sets of $\mathbb{R}^N$, with $\alpha, \tilde\alpha, \gamma, \tilde\gamma \geq 0$.

Then $\alpha\gamma = \tilde\alpha\tilde\gamma$.

Proof. By Lemma 3.1 in [12] there exists a constant δ (independent of the sequence λ_n) such that

$$\delta \equiv \frac{1}{t^2}\left(\int_{-t}^0\int_{\mathbb{R}^N} |\nabla U^+|^2 G(x,-s)dx\,ds\right)\left(\int_{-t}^0\int_{\mathbb{R}^N} |\nabla U^-|^2 G(x,-s)dx\,ds\right) \tag{3.4}$$

for every $t > 0$, where $G(x,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}}e^{-\frac{|x|^2}{4t}}$.

Let $U_\lambda(x) = \frac{1}{\lambda}U(\lambda x)$. Then $U_\lambda \to U_0 = \alpha x_1^+ - \gamma x_1^-$ uniformly on compact sets of $\mathbb{R}^N$, as $\lambda \to 0$. Let us take any sequence converging to 0, for instance λ_n, and let us rescale (3.4). We get, for $t > 0$,

$$\delta \equiv \frac{1}{t^2}\left(\int_{-t}^0\int_{\mathbb{R}^N} |\nabla U_{\lambda_n}^+|^2 G(x,-s)dx\,ds\right)\left(\int_{-t}^0\int_{\mathbb{R}^N} |\nabla U_{\lambda_n}^-|^2 G(x,-s)dx\,ds\right). \tag{3.5}$$

We want to pass to the limit in (3.5) as $n \to \infty$. To this end we first see that, by Proposition 2.3, there exists a sequence $j_n \to \infty$ with $\delta_n = \frac{\varepsilon_{j_n}}{\lambda_n} \to 0$ such that $(u^{\varepsilon_{j_n}})_{\lambda_n} \to U$ uniformly on compact sets of $\mathbb{R}^N$. It is easy to see that $(u^{\varepsilon_{j_n}})_{\lambda_n}$ is a solution to $E_{\frac{\varepsilon_{j_n}}{\lambda_n}}$. Therefore, there exists a sequence u^{δ_n} of solutions to E_{δ_n} such that $u^{\delta_n} \to U$ uniformly on compact sets of $\mathbb{R}^N$ and $\delta_n \to 0$.

Thus, we may apply Proposition 2.3 again, now to the functions U and U_0, to conclude $\nabla U_{\lambda_n} \to \nabla U_0$ in $L^2_{\rm loc}(\mathbb{R}^N)$. So that by taking a subsequence, that we still call λ_n, we may assume that the convergence takes place almost everywhere in $\mathbb{R}^N$. Since U is Lipschitz in $\mathbb{R}^N$, $|\nabla U_{\lambda_n}|$ are bounded in $\mathbb{R}^N$ uniformly in n. Therefore we can pass to the limit in (3.5) to conclude that

$$\delta = \frac{\alpha^2\gamma^2}{4}.$$

Repeating the argument with the sequence $\tilde{\lambda}_n$, we see that

$$\delta = \frac{\tilde{\alpha}^2\tilde{\gamma}^2}{4}.$$

Therefore, $\alpha\gamma = \tilde{\alpha}\tilde{\gamma}$.

Let us prove our first theorem.

Theorem 3.1. *Let u^{ε_j} be solutions to E_{ε_j} in a domain $\Omega \subset \mathbb{R}^N$ such that u^{ε_j} converge to a function u uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ be such that $\partial\{u > 0\}$ has at x_0 an inward unit normal ν in the measure theoretic sense (this is, (3.3) above holds). If $\liminf_{r\to 0} \frac{|\{u<0\}\cap B_r(x_0)|}{|B_r(x_0)|} = 0$, we assume, in addition, that u^+ is nondegenerate at x_0 in the sense of (3.2).*

Then, there exist $\alpha > 0$ and $\gamma \geq 0$, such that

$$u(x) = \alpha\langle x - x_0, \nu\rangle^+ - \gamma\langle x - x_0, \nu\rangle^- + o(|x - x_0|)$$

with

$$\alpha^2 - \gamma^2 = 2M.$$

Proof. Let x_0 be as in the statement of the theorem. We may assume without loss of generality that $x_0 = 0$ and $\nu = e_1$.

We will consider two cases.

<u>Case I</u>. $\limsup_{r\to 0} \frac{|\{u\equiv 0\}\cap B_r(0)|}{|B_r(0)|} = 0$.

In this case, the limit actually exists and it is equal to 0. This is the case considered in Theorem 3.1 in [12] for the parabolic version of this problem. Since [12], Thm. 3.1 applies to our present situation, the theorem is proved in this case.

<u>Case II</u>. $\limsup_{r\to 0} \frac{|\{u\equiv 0\}\cap B_r(0)|}{|B_r(0)|} > 0$.

Let us define for each $\lambda > 0$ the function $u_\lambda(x) = \frac{1}{\lambda}u(\lambda x)$. Since $u(0) = 0$ and u is locally Lipschitz continuous, given a sequence $\lambda_n \to 0$, there exist a subsequence, that we will still call λ_n, and a function U, which is Lipschitz continuous in $\mathbb{R}^N$, such that $u_{\lambda_n} \to U$ uniformly on compact sets of $\mathbb{R}^N$.

Step I. Let us see that there exist $\alpha, \gamma \geq 0$ such that

$$
\begin{aligned}
&U(x) = \alpha x_1^+ + o(|x|) \quad \text{in } \{x_1 > 0\}, \\
&U(x) = -\gamma x_1^- \quad \text{in } \{x_1 < 0\}.
\end{aligned} \tag{3.6}
$$

In fact, by our assumption that e_1 is the inward unit normal to $\partial\{u > 0\}$ at 0 in the measure theoretic sense, we have, as $\lambda \to 0$,

$$
\begin{aligned}
&\chi_{\{u_\lambda > 0\}} \to \chi_{\{x_1 > 0\}} \qquad \text{in } L^1_{\text{loc}}(\mathbb{R}^N), \\
&\chi_{\{u_\lambda \leq 0\}} \to \chi_{\{x_1 < 0\}} \qquad \text{in } L^1_{\text{loc}}(\mathbb{R}^N).
\end{aligned} \tag{3.7}
$$

By (3.7), U is nonnegative in $\{x_1 > 0\}$ and harmonic where positive; and harmonic and nonpositive in $\{x_1 < 0\}$. In fact, if $U^- \not\equiv 0$, then it is harmonic in the set where it is positive and therefore, it is subharmonic in $\mathbb{R}^N$. On the other hand, since u is globally subharmonic, U^- is superharmonic in $\{x_1 < 0\}$. Therefore $U^-(x) = \gamma x_1^-$ in $\mathbb{R}^N$ with $\gamma \geq 0$.

On the other hand, by Lemma A1 in [7], there exists $\alpha \geq 0$ such that $U(x) = \alpha x_1^+ + o(|x|)$ in $\{x_1 > 0\}$. So that (3.6) is verified.

Step II. Let us see that $\alpha > 0$.

Indeed, if there holds that

$$
\liminf_{r \to 0} \frac{|\{u < 0\} \cap B_r(0)|}{|B_r(0)|} = 0,
$$

then, u^+ satisfies (3.2) at 0 by hypothesis. Thus, rescaling (3.2) we get, for every $r > 0$ and n large,

$$
⨍_{B_r(0)} u^+_{\lambda_n} = \frac{1}{\lambda_n} ⨍_{B_{\lambda_n r}(0)} u^+ \geq c\, r,
$$

so that

$$
⨍_{B_r(0)} U^+ \geq c\, r,
$$

for every $r > 0$. This would not be possible if $U^+(x) = o(|x|)$. Therefore, $\alpha > 0$.

If, instead, there holds that

$$
\liminf_{r \to 0} \frac{|\{u < 0\} \cap B_r(0)|}{|B_r(0)|} > 0,
$$

we may apply Theorem 6.3 in [11], since there is an inward unit normal at $0 \in \Omega \cap \partial\{u > 0\}$ in the measure theoretic sense. Therefore, there exists $C > 0$ such that

$$
\sup_{B_r(0)} u \geq C\, r,
$$

for every $r > 0$ small. Rescaling and passing to the limit, we get, for $r > 0$,

$$\sup_{B_r(0)} U \geq C\, r,$$

which implies that $\alpha > 0$.

Step III. Let us now see that $\gamma = 0$. To this end we will apply Lemma 3.1. In fact, since in this case we have $\limsup_{r\to 0} \frac{|\{u\equiv 0\}\cap B_r(0)|}{|B_r(0)|} > 0$, there exists a sequence $\tilde\lambda_n \to 0$ such that

$$\lim_{n\to\infty} \frac{|\{u \equiv 0\} \cap B_{\tilde\lambda_n}(0)|}{|B_{\tilde\lambda_n}(0)|} = 2c > 0.$$

We then have

$$\frac{|\{u_{\tilde\lambda_n} \equiv 0\} \cap B_1(0)|}{|B_1(0)|} \geq c \tag{3.8}$$

for n sufficiently large.

Reasoning as in Step I, we get for a subsequence, that we still call $\tilde\lambda_n$, that $u_{\tilde\lambda_n} \to \widetilde U$ where

$$\widetilde U(x) = \tilde\alpha x_1^+ + o(|x|) \qquad \text{in } \{x_1 > 0\},$$
$$\widetilde U(x) = -\tilde\gamma x_1^- \qquad \text{in } \{x_1 < 0\},$$

uniformly on compact sets of $\mathbb{R}^N$, with $\tilde\alpha, \tilde\gamma \geq 0$.

Let us see that $\tilde\gamma = 0$. Since e_1 is the inward unit normal to $\partial\{u > 0\}$ at the origin in the measure theoretic sense we have that

$$\lim_{r\to 0} \frac{|\{u \leq 0\} \cap \{x_1 > 0\} \cap B_r(0)|}{|B_r(0)|} = 0.$$

So that

$$\lim_{n\to\infty} \frac{|\{u_{\tilde\lambda_n} \equiv 0\} \cap \{x_1 > 0\} \cap B_1(0)|}{|B_1(0)|} = 0.$$

Therefore, by (3.8)

$$\frac{|\{\widetilde U \equiv 0\} \cap \{x_1 < 0\} \cap B_1(0)|}{|B_1(0)|} \geq c.$$

Since $\widetilde U(x) = -\tilde\gamma x_1^-$ in $\{x_1 < 0\}$ with $\tilde\gamma \geq 0$, we deduce that necessarily $\tilde\gamma = 0$.

We can now apply Lemma 3.1 to conclude that $\gamma = 0$, since $\alpha > 0$ and $\alpha\gamma = \tilde\alpha\tilde\gamma = 0$.

Step IV. Now we will prove that $\alpha = \sqrt{2M}$.

By Proposition 2.3, there exists a sequence $j_n \to \infty$ with $\delta_n = \frac{\varepsilon_{j_n}}{\lambda_n} \to 0$ such that $(u^{\varepsilon_{j_n}})_{\lambda_n} \to U$ uniformly on compact sets of $\mathbb{R}^N$. It is easy to see that $(u^{\varepsilon_{j_n}})_{\lambda_n}$

is a solution to $E_{\frac{\varepsilon_{j_n}}{\lambda_n}}$. Therefore, there exists a sequence u^{δ_n} of solutions to E_{δ_n} such that $u^{\delta_n} \to U$ uniformly on compact sets of $\mathbb{R}^N$ and $\delta_n \to 0$.

Let $U_\lambda(x) = \frac{1}{\lambda}U(\lambda x)$. Then $U_\lambda \to U_0 = \alpha x_1^+$ uniformly on compact sets of $\mathbb{R}^N$, as $\lambda \to 0$. Therefore, we may apply Proposition 2.3 again, now to the functions U and U_0 to conclude that there exist a sequence $\hat{\delta}_n \to 0$ and solutions $u^{\hat{\delta}_n}$ to $E_{\hat{\delta}_n}$ such that $u^{\hat{\delta}_n} \to U_0$ uniformly on compact sets of $\mathbb{R}^N$ and $\nabla u^{\hat{\delta}_n} \to \nabla U_0$ in $L^2_{\rm loc}(\mathbb{R}^N)$.

Let $\psi \in C_0^\infty(\mathbb{R}^N)$. Let us multiply equation $E_{\hat{\delta}_n}$ by $(u^{\hat{\delta}_n})_{x_1}\psi$ and integrate by parts. We have

$$\frac{1}{2}\int \psi \frac{\partial}{\partial x_1}|\nabla u^{\hat{\delta}_n}|^2 + \int (u^{\hat{\delta}_n})_{x_1}\nabla u^{\hat{\delta}_n}\nabla\psi = \int B_{\hat{\delta}_n}(u^{\hat{\delta}_n})\psi_{x_1}$$

where $B_{\hat{\delta}_n}(s) = \int_0^s \beta_{\hat{\delta}_n}(\tau)\,d\tau$. Therefore,

$$\frac{1}{2}\int |\nabla u^{\hat{\delta}_n}|^2\psi_{x_1} + \int B_{\hat{\delta}_n}(u^{\hat{\delta}_n})\psi_{x_1} = \int (u^{\hat{\delta}_n})_{x_1}\nabla u^{\hat{\delta}_n}\nabla\psi. \tag{3.9}$$

We want to pass to the limit in (3.9) for $n \to \infty$.

Let us see that

$$B_{\hat{\delta}_n}(u^{\hat{\delta}_n}) \to M\chi_{\{x_1>0\}} + \overline{M}\chi_{\{x_1<0\}} \quad \text{in } L^1_{\rm loc}(\mathbb{R}^N),$$

for some constant $0 \le \overline{M} \le M$. In fact,

$$B_{\hat{\delta}_n}(u^{\hat{\delta}_n}) = \int_0^{\frac{u^{\hat{\delta}_n}}{\hat{\delta}_n}} \beta(s)ds.$$

Therefore, $B_{\hat{\delta}_n}(u^{\hat{\delta}_n}) \equiv M$ in every $\mathcal{U} \subset\subset \{x_1 > 0\}$, if n is large.

Let us see that there exists a function $\overline{M}(x)$ such that $B_{\hat{\delta}_n}(u^{\hat{\delta}_n}) \to \overline{M}(x)$ in $L^1_{\rm loc}(\{x_1 \le 0\})$.

To this end, we will first show that $B_{\hat{\delta}_n}(u^{\hat{\delta}_n}) \to \overline{M}(x)$ in $L^1_{\rm loc}(\{x_1 < 0\})$. In fact, let $\mathcal{U} \subset\subset \{x_1 < 0\}$, then

$$\int_{\mathcal{U}} |\nabla B_{\hat{\delta}_n}(u^{\hat{\delta}_n})| = \int_{\mathcal{U}} \beta_{\hat{\delta}_n}(u^{\hat{\delta}_n})|\nabla u^{\hat{\delta}_n}| \le C\int_{\mathcal{U}} \beta_{\hat{\delta}_n}(u^{\hat{\delta}_n}) \to 0 \qquad (n \to \infty)$$

since, by Prop. 2.2, $\beta_{\hat{\delta}_n}(u^{\hat{\delta}_n}) = \Delta u^{\hat{\delta}_n} \to \Delta U_0$ locally as measures.

Therefore, $\int_{\mathcal{U}} |\nabla B_{\hat{\delta}_n}(u^{\hat{\delta}_n})| \le C(\mathcal{U})$ for every $\mathcal{U} \subset\subset \{x_1 < 0\}$. So that for a subsequence and a function $\overline{M}(x)$ we have $B_{\hat{\delta}_n}(u^{\hat{\delta}_n}) \to \overline{M}(x)$ in $L^1_{\rm loc}(\{x_1 < 0\})$.

In particular, we may assume that the convergence takes place almost everywhere in $\{x_1 < 0\}$ and, since $0 \le B_{\hat\delta_n}(u^{\hat\delta_n}) \le M$, we conclude that $B_{\hat\delta_n}(u^{\hat\delta_n}) \to \overline{M}(x)$ in $L^1_{\rm loc}(\{x_1 \le 0\})$.

Thus, we actually have

$$B_{\hat\delta_n}(u^{\hat\delta_n}) \to M\chi_{\{x_1>0\}} + \overline{M}(x)\chi_{\{x_1<0\}} \quad \text{in } L^1_{\rm loc}(\mathbb{R}^N).$$

On the other hand, since $\int_{\mathcal{U}} |\nabla B_{\hat\delta_n}(u^{\hat\delta_n})| \to 0$ as $n\to\infty$, for every $\mathcal{U} \subset\subset \{x_1 < 0\}$, we deduce that $\overline{M}(x)$ is constant, i.e. $\overline{M}(x) \equiv \overline{M}$, with $0 \le \overline{M} \le M$.

Let us next see that necessarily $\overline{M} = 0$ or $\overline{M} = M$.

In fact, let $\varepsilon_1, \varepsilon_2 > 0$ and $K \subset\subset \{x_1 < 0\}$. There exists $0 < \eta < 1$ such that (with $\beta_\eta = \inf_{[\eta,1-\eta]} \beta > 0$)

$$\begin{aligned}|\{x \in K/\, \varepsilon_1 < B_{\hat\delta_n}(u^{\hat\delta_n}) < M - \varepsilon_2\}| &\le |\{x \in K/\, \eta < \frac{u^{\hat\delta_n}}{\hat\delta_n} < 1-\eta\}| \\ \le& |\{x \in K/\, \beta_{\hat\delta_n}(u^{\hat\delta_n}) \ge \frac{\beta_\eta}{\hat\delta_n}(> \beta_\eta > 0)\}| \to 0 \quad (\text{as } n\to\infty)\end{aligned}$$

since $\beta_{\hat\delta_n}(u^{\hat\delta_n}) \to 0$ in $L^1(K)$. Therefore, the fact that $B_{\hat\delta_n}(u^{\hat\delta_n}) \to \overline{M}$ in $L^1(K)$ implies that

$$|\{x \in K/\, \varepsilon_1 < \overline{M} < M - \varepsilon_2\}| = 0 \quad \text{for every } \varepsilon_1, \varepsilon_2 > 0, K \subset\subset \{x_1 < 0\},$$

so that $\overline{M} = 0$ or $\overline{M} = M$.

Let us now pass to the limit in (3.9). We find

$$\frac{\alpha^2}{2}\int_{\{x_1>0\}} \psi_{x_1} = M\int_{\{x_1>0\}} \psi_{x_1} + \overline{M}\int_{\{x_1<0\}} \psi_{x_1}.$$

Integrating, we obtain

$$\frac{\alpha^2}{2}\int_{\{x_1=0\}} \psi = M\int_{\{x_1=0\}} \psi - \overline{M}\int_{\{x_1=0\}} \psi.$$

Since ψ is arbitrary

$$\alpha = \sqrt{2(M-\overline{M})}.$$

From the fact that $\overline{M} = M$ or $\overline{M} = 0$, and $\alpha > 0$ we find

$$\alpha = \sqrt{2M}.$$

Step V. Conclusion.

We have proved that

$$U(x) = \sqrt{2M}x_1^+ + o(|x|) \qquad \text{in } \{x_1 > 0\} \tag{3.10}$$

and

$$U \equiv 0 \qquad \text{in } \{x_1 < 0\}. \tag{3.11}$$

Since $U \geq 0$ then, by Proposition 2.5, $|\nabla U| \leq \sqrt{2M}$. Therefore,

$$U(x) \leq \sqrt{2M}x_1 \ \text{ in } \{x_1 > 0\}.$$

By (3.10) and Hopf's Principle we deduce that

$$U(x) = \sqrt{2M}x_1 \ \text{ in } \{x_1 > 0\}. \tag{3.12}$$

By (3.11) and (3.12), there holds that the limit U does not depend on the sequence λ_n and

$$u(x) = \sqrt{2M}x_1^+ + o(|x|) \ \text{ in } \Omega.$$

So that the theorem is proved also in Case II.

Remark 3.2. Let u be a continuous function in a domain $\Omega \subset \mathbb{R}^N$. If we have $\mathcal{H}^{N-1}(\Omega \cap \partial\{u > 0\}) < \infty$, then $\{u > 0\}$ is a set of finite perimeter in Ω (see [14]). In this situation we will call, as usual, reduced boundary (and denote $\partial_{\text{red}}\{u > 0\}$), the subset of points in $\partial\{u > 0\}$ which have an inward unit normal in the measure theoretic sense (see Def. 3.2).

We will next prove a representation formula for u which holds when u^+ is locally uniformly nondegenerate. We will denote by $\mathcal{H}^{N-1}\lfloor\,\partial\{u > 0\}$ the measure $\mathcal{H}^{N-1}$ restricted to the set $\partial\{u > 0\}$.

Theorem 3.2. *Let u^{ε_j} be solutions to E_{ε_j} in a domain $\Omega \subset \mathbb{R}^N$ such that u^{ε_j} converge to a function u uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Let us assume that u^+ is locally uniformly nondegenerate on $\Omega \cap \partial\{u > 0\}$ in the sense of (3.1). Then,*
3.2.1) $\mathcal{H}^{N-1}(\Omega' \cap \partial\{u > 0\}) < \infty$, for every $\Omega' \subset\subset \Omega$.
3.2.2) There exist borelian functions q_u^+ and q_u^- defined on $\Omega \cap \partial\{u > 0\}$ such that

$$\Delta u^+ = q_u^+\, \mathcal{H}^{N-1}\lfloor\,\partial\{u > 0\},$$
$$\Delta u^- = q_u^-\, \mathcal{H}^{N-1}\lfloor\,\partial\{u > 0\}.$$

3.2.3) For every $\Omega' \subset\subset \Omega$ there exist $C > 0, c > 0$ and $r_1 > 0$ such that

$$cr^{N-1} \leq \mathcal{H}^{N-1}(B_r(x_0) \cap \partial\{u > 0\}) \leq Cr^{N-1}$$

for every $x_0 \in \Omega' \cap \partial\{u > 0\}, 0 < r < r_1$ and, in addition,
3.2.4) $0 < c \leq q_u^+ \leq C$ and $0 \leq q_u^- \leq C$ in $\Omega' \cap \partial\{u > 0\}$, $q_u^- = 0$ in $\partial\{u > 0\} \setminus \partial\{u < 0\}$.
3.2.5) u has the following asymptotic development at $\mathcal{H}^{N-1}$-almost every point x_0 in $\partial_{\mathrm{red}}\{u > 0\}$ (this is, at $\mathcal{H}^{N-1}$-almost every point x_0 such that $\partial\{u > 0\}$ has an inward unit normal ν in the measure theoretic sense)

$$u(x) = q_u^+(x_0)\langle x - x_0, \nu\rangle^+ - q_u^-(x_0)\langle x - x_0, \nu\rangle^- + o(|x - x_0|).$$

Remark 3.3. Under the assumptions of Thm. 3.2, we have that Thm. 3.1 applies at every point x_0 in the reduced boundary. Therefore, the constants α and γ in Thm. 3.1 verify that $\alpha = q_u^+(x_0)$ and $\gamma = q_u^-(x_0)$ where q_u^+ and q_u^- are the borelian functions in 3.2.2).

Proof of Thm. 3.2. In order to prove the theorem we will make use of some results in [1]. In fact, u^+ is harmonic in $\{u > 0\}$ and locally uniformly nondegenerate on $\Omega \cap \partial\{u > 0\}$ in the sense of (3.1). On the other hand, since u is locally Lipschitz, for every $\Omega' \subset\subset \Omega$ there exist $C > 0$ and $r_1 > 0$ such that

$$\fint_{\partial B_r(x_0)} u^+ \leq Cr$$

for $x_0 \in \Omega' \cap \partial\{u > 0\}$, $0 < r \leq r_1$.

Under these conditions it was proved in [1], 4.5 that 3.2.1) holds and there exists a borelian function q_u^+ defined on $\Omega \cap \partial\{u > 0\}$ such that

$$\Delta u^+ = q_u^+ \mathcal{H}^{N-1} \lfloor \partial\{u > 0\}.$$

Also 3.2.3) holds by the results of [1].

Moreover, it follows from [1], 4.5 that the local uniform nondegeneracy of u^+ on $\Omega \cap \partial\{u > 0\}$ in the sense of (3.1) implies that $q_u^+ \geq c > 0$.

On the other hand, u^- is a nonnegative harmonic function in $\Omega \cap \{u < 0\}$. Since u is Lipschitz,

$$\fint_{\partial B_r(x_0)} u^- \leq Cr$$

for every $x_0 \in \Omega' \cap \partial\{u<0\}$, $0<r\le r_1$. On the other hand, since u is harmonic in $\Omega \cap \{u \le 0\}^\circ$ then $\partial\{u<0\} \subset \partial\{u>0\}$. Therefore, by 3.2.1) and [1], Remark 4.6, there exists a borelian function q_u^- defined on $\Omega \cap \partial\{u<0\}$ such that

$$\Delta u^- = q_u^- \mathcal{H}^{N-1} \lfloor \partial\{u<0\}.$$

Thus, if we define $q_u^- = 0$ in $\partial\{u>0\} \setminus \partial\{u<0\}$ there holds that $q_u^- \ge 0$ and

$$\Delta u^- = q_u^- \mathcal{H}^{N-1} \lfloor \partial\{u>0\}.$$

Therefore 3.2.2) is proved.

It also follows that

$$q_u^+ \le C \ \text{ in } \Omega' \cap \partial\{u>0\} \ , \ q_u^- \le C \ \text{ in } \Omega' \cap \partial\{u>0\}$$

so 3.2.4) is true.

Finally, by [1], 4.8 and 4.9

$$u^+(x) = q_u^+(x_0)\langle x-x_0,\nu\rangle^+ + o(|x-x_0|)$$

for $\mathcal{H}^{N-1}$-almost every x_0 in $\partial_{\mathrm{red}}\{u>0\}$. Proceeding as in [1], 4.8 we can also deduce that

$$u^-(x) = q_u^-(x_0)\langle x-x_0,\nu\rangle^- + o(|x-x_0|)$$

for $\mathcal{H}^{N-1}$-almost every x_0 in $\partial_{\mathrm{red}}\{u>0\}$ and 3.2.5) follows.

So that the theorem is proved.

4. Viscosity solutions to the free boundary problem E.

In this section we prove that the limit function u is a viscosity solution to the free boundary problem

$$(E) \qquad \begin{aligned} &\Delta u = 0 \qquad \text{in } \Omega \setminus \partial\{u>0\},\\ &u=0,\ (u_\nu^+)^2-(u_\nu^-)^2 = 2M \qquad \text{on } \Omega\cap\partial\{u>0\},\end{aligned}$$

where $u^+ = \max(u,0)$, $u^- = \max(-u,0)$, and ν is the inward unit normal to the free boundary $\Omega \cap \partial\{u>0\}$. This notion of weak solution was introduced by Caffarelli in [6], [7].

We start the section with some definitions and we then prove our main results in Theorems 4.1 and 4.2. We also include some corollaries at the end of the section (Corollaries 4.1, 4.2 and Proposition 4.1).

Definition 4.1. Let Ω be a domain in $\mathbb{R}^N$. Let u be a continuous function in Ω. Then u is called a viscosity supersolution in Ω if

(i) $\Delta u \leq 0$ in $\Omega^+ := \Omega \cap \{u > 0\}$.

(ii) $\Delta u \leq 0$ in $\Omega^- := \Omega \cap \{u \leq 0\}^\circ$.

(iii) Along $F = \Omega \cap \partial\{u > 0\}$, u satisfies the condition

$$(u_\nu^+)^2 - (u_\nu^-)^2 \leq 2M$$

in the following weak sense. If $x_0 \in F$, F has a tangent ball at x_0 from Ω^+ (i.e, there is a $B_\rho(y) \subset \Omega^+$, such that $x_0 \in \partial B_\rho(y)$), and, in $B_\rho(y)$

$$u(x) \geq \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|),$$

for $\alpha \geq 0$ (ν given by the inward unit radial direction of the ball at x_0), then

$$u(x) < -\gamma\langle x - x_0, \nu\rangle^- + o(|x - x_0|)$$

in $B_\rho(y)^c$ for any $\gamma \geq 0$ such that $\alpha^2 - \gamma^2 > 2M$.

Definition 4.2. Let u be a continuous function in Ω. Then u is called a viscosity subsolution in Ω if

(i) $\Delta u \geq 0$ in $\Omega^+ := \Omega \cap \{u > 0\}$.

(ii) $\Delta u \geq 0$ in $\Omega^- := \Omega \cap \{u \leq 0\}^\circ$.

(iii) Along $F = \Omega \cap \partial\{u > 0\}$, u satisfies the condition

$$(u_\nu^+)^2 - (u_\nu^-)^2 \geq 2M$$

in the following weak sense. If $x_0 \in F$, F has a tangent ball at x_0 from Ω^- (i.e, there is a $B_\rho(y) \subset \Omega^-$, such that $x_0 \in \partial B_\rho(y)$), and, in $B_\rho(y)$

$$u(x) \leq -\gamma\langle x - x_0, \nu\rangle^- + o(|x - x_0|),$$

for $\gamma \geq 0$ (ν given by the outward unit radial direction of the ball at x_0), then

$$u(x) > \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|)$$

in $B_\rho(y)^c$ for any $\alpha \geq 0$ such that $\alpha^2 - \gamma^2 < 2M$.

Definition 4.3. We say that u is a viscosity solution in Ω if it is both a viscosity subsolution and a viscosity supersolution.

Definition 4.4. Let u be a continuous function in a domain $\Omega \subset \mathbb{R}^N$. We say that a point $x_0 \in \Omega \cap \partial\{u > 0\}$ is a regular point from the positive side if there is a tangent ball at x_0 from $\{u > 0\}$ (i.e, there is a $B_\rho(y) \subset \{u > 0\}$, such that $x_0 \in \partial B_\rho(y)$).

Analogously, we say that a point $x_0 \in \Omega \cap \partial\{u > 0\}$ is a regular point from the nonpositive side if there is a tangent ball at x_0 from $\{u \leq 0\}^\circ$ (i.e, there is a $B_\rho(y) \subset \{u \leq 0\}^\circ$, such that $x_0 \in \partial B_\rho(y)$).

We will use throughout this section the following auxiliary lemma on the asymptotic behavior of nonnegative subharmonic functions at boundary points which are regular from the nonpositive side.

Lemma 4.1. *Let U be a Lipschitz function in some ball B centered at the origin. Assume that U is nonnegative and subharmonic in B, $U(0) = 0$. Assume, in addition, that $U \equiv 0$ in some ball $B_\rho(y) \subset \{x_1 < 0\}$, $B_\rho(y) \subset\subset B$, $0 \in \partial B_\rho(y)$.*

Then, near the origin, U has the asymptotic development

$$U(x) = \alpha x_1^+ + o(|x|) \quad \text{in } \{x_1 > 0\},$$

with $\alpha \geq 0$.

Proof. Let $\rho' > \rho$ be such that $B_{\rho'}(y) \subset\subset B$. Since U is Lipschitz continuous, there exists a constant $\Lambda > 0$ such that $U < \Lambda$ on $\partial B_{\rho'}(y)$. Let v be the harmonic function in $\mathcal{B} = B_{\rho'}(y) \setminus \overline{B}_\rho(y)$ such that $v = 0$ on $\partial B_\rho(y)$ and $v = \Lambda$ on $\partial B_{\rho'}(y)$. Clearly, v is positive in $\mathcal{B}$ and smooth in $\overline{\mathcal{B}}$ and it vanishes at $0 \in \partial\mathcal{B}$. Therefore,

$$v(x) = \delta x_1^+ + o(|x|) \quad \text{in } \{x_1 > 0\}, \tag{4.1}$$

for some $\delta > 0$. On the other hand, the function $w = v - U$ is superharmonic and positive in $\mathcal{B}$ and, in particular, it is so in the subset $\mathcal{B} \cap \{x_1 > 0\}$. Therefore, we can use Lemma A1 in [7] (which applies to superharmonic functions even though it is stated for harmonic functions) to deduce that

$$w(x) = \bar{\delta} x_1^+ + o(|x|) \quad \text{in } \{x_1 > 0\},$$

for some $\bar{\delta} > 0$, which implies, by (4.1),

$$U(x) = \alpha x_1^+ + o(|x|) \quad \text{in } \{x_1 > 0\},$$

with $\alpha \geq 0$.

Now we will prove our main results in the section.

Theorem 4.1. *Let u^{ε_j} be solutions to E_{ε_j} in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Then u is a viscosity supersolution in Ω.*

Proof. Clearly, u satisfies (i) and (ii) in Definition 4.1 (see for instance Prop. 2.2). We will show that (iii) also holds. Let $B_\rho(y) \subset \{u > 0\}$ be a ball touching $\partial\{u > 0\}$ at a point x_0, and assume that, for some $\alpha \geq 0$,

$$u(x) \geq \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \tag{4.2}$$

in $B_\rho(y)$ (ν given by the inward unit radial direction of the ball at x_0).

If $\alpha^2 \leq 2M$ there is nothing to prove. Otherwise, let $\gamma \geq 0$ be such that $\alpha^2 - \gamma^2 > 2M$. We will show that

$$u(x) < -\gamma\langle x - x_0, \nu\rangle^- + o(|x - x_0|)$$

in $B_\rho(y)^c$.

We will assume without loss of generality that $\nu = e_1$ and $x_0 = 0$.

We claim that $u(x) = \overline{\alpha}x_1^+ - \overline{\gamma}x_1^- + o(|x|)$ with $\overline{\alpha}, \overline{\gamma} > 0$.

In fact, by Lemma 4.1,

$$u^-(x) = \overline{\gamma}x_1^- + o(|x|) \qquad \text{in } \{x_1 < 0\}, \tag{4.3}$$

for some $\overline{\gamma} \geq 0$. Let us consider, for $\lambda > 0$,

$$u_\lambda(x) = \frac{1}{\lambda}u(\lambda x).$$

Since u is locally Lipschitz continuous and $u(0) = 0$ then, for every sequence $\lambda_n \to 0$, there exists a subsequence, that we still call λ_n, such that $u_{\lambda_n} \to u_0$ uniformly on compact sets of $\mathbb{R}^N$, where u_0 is Lipschitz in $\mathbb{R}^N$. By (4.2) and (4.3) we know that

$$u_0^-(x) = \overline{\gamma}x_1^- \qquad \text{in } \mathbb{R}^N$$

and

$$u_0 > 0 \;\text{ and harmonic in } \{x_1 > 0\}.$$

(Let us recall at this moment our assumption that $\alpha > \sqrt{2M} > 0$).

We will consider two cases.

Case I: $\overline{\gamma} > 0$.

In this case, $u_0 < 0$ in $\{x_1 < 0\}$. Therefore $u_0 = 0$ on $\{x_1 = 0\}$ and since u_0 is Lipschitz, we have

$$u_0^+(x) = \overline{\alpha} x_1^+ \qquad \text{in } \mathbb{R}^N,$$

for some $\overline{\alpha} > 0$. Thus, we deduce that

$$u_0(x) = \overline{\alpha} x_1^+ - \overline{\gamma} x_1^-, \qquad \overline{\alpha}, \overline{\gamma} > 0. \tag{4.4}$$

Case II: $\overline{\gamma} = 0$.

In this case, $u_0 \geq 0$ in $\mathbb{R}^N$. Since $u_0 > 0$ and harmonic in $\{x_1 > 0\}$, then by Lemma A1 in [7], there exists $\overline{\alpha} > 0$ such that

$$u_0(x) = \overline{\alpha} x_1^+ + o(|x|) \qquad \text{in } \{x_1 > 0\}. \tag{4.5}$$

Necessarily

$$\overline{\alpha} \geq \alpha > \sqrt{2M}. \tag{4.6}$$

Let us consider, for $\lambda > 0$,

$$(u_0)_\lambda(x) = \frac{1}{\lambda} u_0(\lambda x).$$

Since u_0 is Lipschitz continuous and $u_0(0) = 0$, we deduce that there exists a sequence $\overline{\lambda}_n \to 0$ such that $(u_0)_{\overline{\lambda}_n} \to u_{00}$ uniformly on compact sets of $\mathbb{R}^N$, where u_{00} is a Lipschitz continuous function in $\mathbb{R}^N$. By (4.5),

$$u_{00}(x) = \overline{\alpha} x_1^+ \qquad \text{in } \{x_1 > 0\}.$$

Also, $u_{00} \geq 0$ in $\mathbb{R}^N$ and harmonic in the set where it is positive. Since $u_{00} = 0$ in $\{x_1 = 0\}$, then the application of Lemma A1 in [7] yields

$$u_{00}(x) = \tilde{\alpha} x_1^- + o(|x|) \qquad \text{in } \{x_1 < 0\},$$

for some $\tilde{\alpha} \geq 0$. Let us finally consider for $\lambda > 0$

$$(u_{00})_\lambda(x) = \frac{1}{\lambda} u_{00}(\lambda x).$$

As before, we see that there exist a sequence $\tilde{\lambda}_n \to 0$ and a Lipschitz function u_{000} in $\mathbb{R}^N$, such that $u_{\tilde{\lambda}_n} \to u_{000}$ uniformly on compact sets of $\mathbb{R}^N$. From our computations above we deduce that

$$u_{000}(x) = \overline{\alpha} x_1^+ + \tilde{\alpha} x_1^-, \qquad \overline{\alpha} > 0, \tilde{\alpha} \geq 0.$$

Applying Prop. 2.3 we see that there exist a sequence $\delta_n \to 0$ and solutions u^{δ_n} to E_{δ_n} such that

$$u^{\delta_n} \to u_0 \tag{4.7}$$

uniformly on compact sets of $\mathbb{R}^N$. Applying Prop. 2.3 two more times, we see that there exist a sequence $\tilde{\delta}_n \to 0$ and solutions $u^{\tilde{\delta}_n}$ to $E_{\tilde{\delta}_n}$ such that $u^{\tilde{\delta}_n} \to u_{000}$ uniformly on compact sets of $\mathbb{R}^N$.

It follows that u_{000} is under the hypotheses of Thm. 3.1 or Prop. 2.4. In either case,

$$\overline{\alpha} \leq \sqrt{2M} < \alpha$$

which contradicts (4.6).

Thus we get a contradiction. So that Case II is not possible and (4.4) holds.

By (4.7), we see that we may apply Thm. 3.1 to the function u_0 to conclude that

$$\overline{\alpha}^2 - \overline{\gamma}^2 = 2M. \tag{4.8}$$

By Lemma 3.1, there exists δ independent of the sequence λ_n such that

$$\overline{\alpha}\,\overline{\gamma} = \delta, \tag{4.9}$$

By (4.8) and (4.9) we see that the limit function u_0 does not depend on the sequence λ_n. So that we have

$$u_\lambda \to u_0$$

uniformly on compact sets of $\mathbb{R}^N$ (as $\lambda \to 0$). Therefore,

$$u(x) = \overline{\alpha} x_1^+ - \overline{\gamma} x_1^- + o(|x|).$$

In particular,

$$u(x) = -\overline{\gamma} x_1^- + o(|x|) \qquad \text{in } B_\rho(y)^c. \tag{4.10}$$

By (4.8), we have

$$\overline{\gamma} = \sqrt{\overline{\alpha}^2 - 2M} \geq \sqrt{\alpha^2 - 2M} > \gamma. \tag{4.11}$$

So that by (4.10) and (4.11), we deduce that

$$u(x) < -\gamma x_1^- + o(|x|) \qquad \text{in } B_\rho(y)^c,$$

and the theorem is proved.

Theorem 4.2. *Let u^{ε_j} be solutions to E_{ε_j} in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Assume, in addition, that at every regular point from the nonpositive side $x_0 \in \Omega \cap \partial\{u > 0\}$ (this is, such that there exists a ball $B_\rho(y) \subset \{u \leq 0\}$ with $x_0 \in \partial B_\rho(y)$), u satisfies one of the following hypotheses:*

H1) *u^+ is nondegenerate at x_0 in the sense of (3.2).*

or else

H2) *$u < 0$ in $B_\rho(y)$.*

Then u is a viscosity subsolution in Ω.

Proof. Clearly, u satisfies (i) and (ii) in Definition 4.2 (see for instance Prop. 2.2). We will show that (iii) also holds. Let $B_\rho(y) \subset \{u \leq 0\}^\circ$ be a ball touching $\partial\{u > 0\}$ at a point x_0, and assume that for some $\gamma \geq 0$

$$u(x) \leq -\gamma \langle x - x_0, \nu \rangle^- + o(|x - x_0|) \tag{4.12}$$

in $B_\rho(y)$, (ν given by the outward unit radial direction of the ball at x_0).

Let $\alpha \geq 0$ be such that $\alpha^2 - \gamma^2 < 2M$. We will show that

$$u(x) > \alpha \langle x - x_0, \nu \rangle^+ + o(|x - x_0|)$$

in $B_\rho(y)^c$.

We will assume without loss of generality that $\nu = e_1$ and $x_0 = 0$.

By Lemma 4.1 we know that there exists $\overline{\alpha} \geq 0$ such that

$$u^+(x) = \overline{\alpha} x_1^+ + o(|x|) \qquad \text{in } \{x_1 > 0\}. \tag{4.13}$$

Let us consider, for $\lambda > 0$, u_λ as in Theorem 4.1. For every sequence $\lambda_n \to 0$ there exist a subsequence, that we still call λ_n, and u_0 Lipschitz continuous in $\mathbb{R}^N$ such that $u_{\lambda_n} \to u_0$ uniformly on compact sets of $\mathbb{R}^N$.

Then, by (4.13) we have

$$u_0^+(x) = \overline{\alpha} x_1^+ \qquad \text{in } \{x_1 > 0\}.$$

On the other hand, since $B_\rho(y) \subset \{u \le 0\}$, $0 \in \partial B_\rho(y)$ and e_1 is the outward unit radial direction to $B_\rho(y)$ at the origin, then $u_0 \le 0$ in $\{x_1 < 0\}$. Since u is harmonic in the set $\{u \le 0\}^\circ$, we can actually have only one of the following situations:
i) $u \equiv 0$ in $B_\rho(y)$, in which case

$$u_0 \equiv 0 \qquad \text{in } \{x_1 < 0\}. \tag{4.14}$$

ii) $u < 0$ in $B_\rho(y)$. In this case, by Hopf's Principle there exists $\theta > 0$ such that $u(x) \le -\theta x_1^- + o(|x|)$ in $B_\rho(y)$ and therefore,

$$u_0 < 0 \qquad \text{in } \{x_1 < 0\}. \tag{4.15}$$

We will consider two cases (independently on whether (4.14) or (4.15) holds).
Case I: $\overline{\alpha} > 0$. Then we have $u_0^+ = u_0$ in $\{x_1 > 0\}$. Since $u_0 \le 0$ in $\{x_1 < 0\}$, we deduce that $u_0 = 0$ in $\{x_1 = 0\}$. Thus, u_0 is Lipschitz in $\mathbb{R}^N$, nonnegative and harmonic in $\{x_1 < 0\}$ and vanishes on $\{x_1 = 0\}$. We conclude that there exists $\overline{\gamma} \ge 0$ such that

$$u_0(x) = -\overline{\gamma} x_1^- \qquad \text{in } \{x_1 < 0\}.$$

Therefore,

$$u_0(x) = \overline{\alpha} x_1^+ - \overline{\gamma} x_1^- \qquad \text{in } \mathbb{R}^N. \tag{4.16}$$

Case II: $\overline{\alpha} = 0$. In this case we must have $u < 0$ in $B_\rho(y)$. In fact, if not, by H1) u^+ must satisfy (3.2) at the origin. Rescaling (3.2) we get that

$$⨍_{B_r(0)} u_{\lambda_n}^+ \ge c\, r$$

for every small $r > 0$. So that

$$⨍_{B_r(0)} u_0^+ \ge c\, r$$

which is not possible if $\overline{\alpha} = 0$.

Therefore $u < 0$ in $B_\rho(y)$ and thus, $u_0 < 0$ in $\{x_1 < 0\}$ (see ii) above).

On the other hand, since $u_0^+ = 0$, we have that $u_0 \le 0$ in $\mathbb{R}^N$. By the arguments in Thm. 4.1 we know that there exist a sequence $\delta_n \searrow 0$ and solutions u^{δ_n} of E_{δ_n} such that $u^{\delta_n} \to u_0$ uniformly on compact sets of $\mathbb{R}^N$. Therefore, u_0 is harmonic in $\mathbb{R}^N$ since any such limit is harmonic in the interior of the set where it is nonnegative (see Prop. 2.2).

From the fact that $u_0(0) = 0$ we deduce that we must have $u_0 \equiv 0$ in $\mathbb{R}^N$ which we have just seen that is not possible.

So that Case II is not possible and we have (4.16) with $\overline{\alpha} > 0$ and $\overline{\gamma} \ge 0$.

As in Thm. 4.1 we see that we may apply Thm. 3.1 to the function u_0 to deduce that $\overline{\alpha}$, $\overline{\gamma}$ satisfy the free boundary condition (4.8).

As in Thm. 4.1, by the application of Lemma 3.1, we see that the limit function u_0 does not depend on the sequence $\lambda_n \to 0$ and therefore,

$$u(x) = \overline{\alpha} x_1^+ - \overline{\gamma} x_1^- + o(|x|).$$

In particular,

$$u(x) = \overline{\alpha} x_1^+ + o(|x|) \qquad \text{in } B_\rho(y)^c.$$

By (4.12) we see that necessarily $\overline{\gamma} \ge \gamma$. Thus,

$$\overline{\alpha} = \sqrt{2M + \overline{\gamma}^2} \ge \sqrt{2M + \gamma^2} > \alpha.$$

So that

$$u(x) > \alpha x_1^+ + o(|x|) \qquad \text{in } B_\rho(y)^c$$

and the theorem is proved.

In particular we obtain the following corollaries.

Corollary 4.1. *Let u^{ε_j} be solutions to E_{ε_j} in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Assume, in addition, that $\{u \equiv 0\}^\circ = \emptyset$. Then u is a viscosity solution in Ω.*

Corollary 4.2. *Let u^{ε_j} be solutions to E_{ε_j} in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Assume, in addition, that u^+ is nondegenerate at every point $x_0 \in \Omega \cap \partial\{u > 0\}$ in the sense of (3.2). Then u is a viscosity solution in Ω.*

As a corollary to the proof of Theorem 4.2 we have the following proposition which is of the type of Theorem 3.1. This is a result of a pointwise nature.

Proposition 4.1. *Let u^{ε_j} be solutions to E_{ε_j} in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ be a regular point from the nonpositive side (this is, such that there exists a ball $B_\rho(y) \subset \{u \leq 0\}$ with $x_0 \in \partial B_\rho(y)$). If $u \equiv 0$ in $B_\rho(y)$ we assume in addition that u^+ is nondegenerate at x_0 in the sense of (3.2). Then u has the following asymptotic development*

$$u(x) = \alpha\langle x - x_0, \nu\rangle^+ - \gamma\langle x - x_0, \nu\rangle^- + o(|x - x_0|)$$

with $\alpha > 0$, $\gamma \geq 0$ and $\alpha^2 - \gamma^2 = 2M$.

5. Regularity of the free boundary.

In this section we study the regularity of the free boundary. We first want to remark that there are limit functions u which do not satisfy the free boundary condition in the classical sense on any portion of $\partial\{u > 0\}$ (for instance, $u = \alpha x_1^+ + \alpha x_1^-$ with $0 < \alpha < \sqrt{2M}$, see [11], Remark 5.1). Thus, extra hypotheses have to be made in order to get regularity results.

On one hand, we prove in Theorem 5.1 that in the absence of zero phase, if u^- is nondegenerate at $x_0 \in \partial\{u > 0\}$, then the free boundary is a $C^{1,\alpha}$ surface in a neighborhood of x_0. Therefore u is a classical solution to E in that neighborhood. We point out that in the strictly two phase case, if the free boundary is smooth, then u^- is nondegerate on the free boundary (see Remark 5.1). As a corollary to Theorem 5.1 we deduce that in the absence of zero phase, there is a subset of the free boundary which is locally a $C^{1,\alpha}$ surface. This subset is open and dense in $\partial\{u < 0\}$ (Corollary 5.1). See also Corollary 5.2.

On the other hand, we prove for the general two phase case that, under nondegeneracy assumptions on u^+, if $x_0 \in \Omega \cap \partial\{u > 0\}$ is such that the free boundary has at x_0 an inward unit normal in the measure theoretic sense, then the free boundary is a $C^{1,\alpha}$ surface in a neighborhood of x_0 (Theorem 5.2). Then, in Corollary 5.3 we prove that, under suitable assumptions on our limit function u, there is a subset of the free boundary which is locally a $C^{1,\alpha}$ surface. Moreover, this smooth subset is open and dense in $\partial\{u > 0\}$ and the remainder of the free boundary has $(N-1)$-dimensional Hausdorff measure zero. We end our work with further results for the one phase case (Theorems 5.3 and 5.4).

In order to get our regularity results we use, on one hand, the results we have obtained in the previous sections for the limit function u and the regularity theory developed in [6], [7]. On the other hand, we use the result of Proposition 5.1 —which replaces, in Theorem 5.1, more usual geometric measure theoretic arguments.

We start the section with a definition and some auxiliary lemmas:

Definition 5.1. Let v be a continuous function in a domain $\Omega \subset \mathbb{R}^N$. We say that the set $\{v \leq 0\}$ has positive density at a point $x_0 \in \Omega \cap \partial\{v > 0\}$, if there exist $c > 0$ and $r_0 > 0$ such that

$$\frac{|\{v \leq 0\} \cap B_r(x_0)|}{|B_r(x_0)|} \geq c \quad \text{for } 0 < r \leq r_0. \tag{5.1}$$

We say that the set $\{v \leq 0\}$ has uniform positive density on $\Gamma \subset \Omega \cap \partial\{v > 0\}$, if there exist $c > 0$ and $r_0 > 0$ such that (5.1) holds for every $x_0 \in \Gamma$.

Lemma 5.1. *Let u be a locally Lipschitz continuous function in a domain $\Omega \subset \mathbb{R}^N$, harmonic in $\{u > 0\} \cup \{u \leq 0\}^\circ$ and globally subharmonic. Assume that u^+ is locally uniformly nondegenerate on $\Omega \cap \partial\{u > 0\}$ in the sense of (3.2) and that the set $\{u \leq 0\}$ has locally uniform positive density on $\Omega \cap \partial\{u > 0\}$. Then, for every $\Omega' \subset\subset \Omega$ there exist $C > 0$ and $\rho_0 > 0$ such that, for $x \in \Omega'$,*

$$u^+(x) \geq C d(x, \{u \leq 0\}) \quad \textit{if } d(x, \{u \leq 0\}) \leq \rho_0. \tag{5.2}$$

Here $d(x, A)$ is the distance from the point x to the set A.

Proof. If the result does not hold, then there exists a sequence x_n in $\Omega' \subset\subset \Omega$ such that $d(x_n, \{u \leq 0\}) < \frac{1}{n}$ and $u^+(x_n) < \frac{1}{n} d(x_n, \{u \leq 0\})$.

Let $z_n \in \partial\{u > 0\}$ be such that $\lambda_n := d(x_n, \{u \leq 0\}) = |x_n - z_n|$, and let $u_{\lambda_n}(x) = \frac{1}{\lambda_n} u(z_n + \lambda_n x)$. Then, $\lambda_n \to 0$ and there is a function u_0 such that, for a subsequence, $u_{\lambda_n} \to u_0$ uniformly on compact sets of $\mathbb{R}^N$. We can choose the subsequence so that there exists

$$v := \lim_{n \to \infty} v_n, \quad \text{where } v_n := \frac{x_n - z_n}{\lambda_n},$$

and without loss of generality we will assume that $v = e_1$.

Then, u_0 is Lipschitz continuous and harmonic in $\{u_0 > 0\} \cup \{u_0 < 0\}$, and from the fact that

$$B_1(v_n) \subset \{u_{\lambda_n} > 0\},$$

we deduce that u_0 is harmonic and nonnegative in $B_1(e_1)$. Since $u_0(e_1) = 0$, it follows that $u_0 \equiv 0$ in $B_1(e_1)$. Now, we can apply Lemma 4.1 to u_0^+ and u_0^- to deduce that

$$u_0(x) = \alpha x_1^- + o(|x|) \quad \text{in } \{x_1 < 0\},$$

with $\alpha \in \mathbb{R}$. Next, we consider for $\lambda > 0$, $(u_0)_\lambda(x) = \frac{1}{\lambda} u_0(\lambda x)$. There exists a sequence $\bar\lambda_n \to 0$ such that $(u_0)_{\bar\lambda_n} \to u_{00}$ uniformly on compact sets of $\mathbb{R}^N$. It follows that

$$u_{00}(x) = \alpha x_1^- \quad \text{ in } \mathbb{R}^N.$$

On the other hand, since u^+ is locally uniformly nondegenerate on $\Omega \cap \partial\{u > 0\}$ in the sense of (3.2), it follows that, for $r > 0$,

$$\frac{1}{\lambda_n r} \fint_{B_{\lambda_n r}(z_n)} u^+ \geq c > 0, \quad \text{if } n \text{ large}$$

and therefore,

$$\frac{1}{r} \fint_{B_r(0)} u_0^+ \geq c \quad \text{for } r > 0,$$

which gives

$$\fint_{B_1(0)} u_{00}^+ \geq c.$$

Hence $\alpha > 0$.

Using now that the set $\{u \leq 0\}$ has locally uniform positive density on $\Omega \cap \partial\{u > 0\}$, we see that, for $r > 0$,

$$0 < c \leq \frac{|\{u \leq 0\} \cap B_{\lambda_n r}(z_n)|}{|B_{\lambda_n r}(z_n)|} \quad \text{if } n \text{ large},$$

and therefore,

$$c \leq \frac{|\{u_{\lambda_n} \leq 0\} \cap B_r(0)|}{|B_r(0)|} = \frac{|\{u_{\lambda_n} \leq 0\} \cap B_1(v_n)^c \cap B_r(0)|}{|B_r(0)|}.$$

Letting $n \to \infty$, we get

$$c \leq \frac{|\{u_0 \leq 0\} \cap B_1(e_1)^c \cap B_r(0)|}{|B_r(0)|},$$

and then, rescaling and letting $n \to \infty$ again,

$$\begin{aligned} c &\leq \limsup_{n\to\infty} \frac{|\{(u_0)_{\bar\lambda_n} \leq 0\} \cap B_{1/\bar\lambda_n}(e_1/\bar\lambda_n)^c \cap B_1(0)|}{|B_1(0)|} \\ &\leq \frac{|\{u_{00} \leq 0\} \cap \{x_1 < 0\} \cap B_1(0)|}{|B_1(0)|}. \end{aligned}$$

But this contradicts the fact that $u_{00} > 0$ in $\{x_1 < 0\}$ and the lemma follows.

Lemma 5.2. *Let u be a locally Lipschitz continuous function in a domain $\Omega \subset \mathbb{R}^N$, harmonic in $\{u > 0\} \cup \{u \le 0\}^\circ$ and globally subharmonic. Assume that u^- is nondegenerate at $x_0 \in \Omega \cap \partial\{u > 0\}$ in the sense of (3.2). Then u^+ is nondegenerate at x_0 in the same sense.*

Proof. Let us proceed by contradiction. Then, there exist a sequence $\lambda_n \to 0$ such that

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \fint_{B_{\lambda_n}(x_0)} u^+ = 0.$$

Now consider the sequence $u_{\lambda_n}(x) = \frac{1}{\lambda_n} u(x_0 + \lambda_n x)$. Since u is locally Lipschitz continuous and $u(x_0) = 0$ there exist a subsequence, that we still call λ_n, and a Lipschitz continuous function u_0 such that $u_{\lambda_n} \to u_0$ uniformly on compact subsets of $\mathbb{R}^N$. We have

$$\fint_{B_1(0)} u_0^+ = \lim_{n\to\infty} \fint_{B_1(0)} u_{\lambda_n}^+ = \lim_{n\to\infty} \frac{1}{\lambda_n} \fint_{B_{\lambda_n}(x_0)} u^+ = 0.$$

Thus, $u_0^+ \equiv 0$ in $B_1(0)$ and then, $u_0 \le 0$ in $B_1(0)$. So that u_0 is a nonpositive harmonic function in $B_1(0)$ which vanishes at the origin. We deduce that $u_0 \equiv 0$ in $B_1(0)$. But this contradicts the fact that u^- is nondegenerate at x_0 in the sense of (3.2). Indeed,

$$0 = \lim_{n\to\infty} \fint_{B_1(0)} u_{\lambda_n}^- = \lim_{n\to\infty} \frac{1}{\lambda_n} \fint_{B_{\lambda_n}(x_0)} u^- \ge c > 0.$$

Therefore, u^+ is nondegenerate at x_0 in the sense of (3.2) and the proof is complete.

Proposition 5.1. *Let u^{ε_j} be solutions to E_{ε_j} in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ and let $\lambda_n > 0$ be a sequence such that $\lambda_n \to 0$. Consider the functions $u_{\lambda_n}(x) = \frac{1}{\lambda_n} u(x_0 + \lambda_n x)$ and assume that $u_{\lambda_n} \to U$ as $n \to \infty$ uniformly on compact sets of $\mathbb{R}^N$. If u^- is nondegenerate at x_0 in the sense of (3.2), then*

$$U(x) = \alpha\langle x, \nu\rangle^+ - \gamma\langle x, \nu\rangle^- \qquad \text{in } \mathbb{R}^N,$$

where ν is a unit vector, and α, γ are positive constants satisfying $\alpha^2 - \gamma^2 = 2M$.

Proof. Let us consider, for $t > 0$,

$$J_U(t) := \frac{1}{t^2} \left(\int_{-t}^0 \int_{\mathbb{R}^N} |\nabla U^+|^2 G(x, -s)\,dx\,ds \right) \left(\int_{-t}^0 \int_{\mathbb{R}^N} |\nabla U^-|^2 G(x, -s)\,dx\,ds \right),$$

where $G(x,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{|x|^2}{4t}}$.

From the local parabolic monotonicity formula in [9] it is possible to deduce that there exists $\delta \geq 0$ independent of the sequence λ_n such that

$$J_U(t) \equiv \delta \qquad \text{for } t > 0. \tag{5.3}$$

In fact, this was done by Caffarelli and the authors, for the parabolic version of this problem (Lemma 3.1 in [12]).

Let us see that we necessarily have $\delta > 0$. In fact, assume that

$$\int_{-t}^{0} \int_{\mathbb{R}^N} |\nabla U^-|^2 G(x,-s)\,dxds = 0$$

for some $t > 0$. Then, $U^- \equiv 0$ in $\mathbb{R}^N$ and therefore, for any $r > 0$,

$$0 = \lim_{n\to\infty} \frac{1}{r} \fint_{B_r(0)} u^-_{\lambda_n} = \lim_{n\to\infty} \frac{1}{\lambda_n r} \fint_{B_{\lambda_n r}(x_0)} u^-,$$

which contradicts the nondegeneracy of u^- at x_0 in the sense of (3.2). Since also u^+ is nondegenerate at x_0 in the same sense (recall Lemma 5.2), we proceed analogously with U^+.

That is, we have shown that (5.3) holds with $\delta > 0$.

We will now conclude that

$$U(x) = \alpha\langle x,\nu\rangle^+ - \gamma\langle x,\nu\rangle^- \qquad \text{in } \mathbb{R}^N,$$

with $\alpha > 0$, $\gamma > 0$ and ν a unit vector.

We will show that this can be deduced from the proof of the global parabolic monotonicity formula in Theorem 1 in [9], and from the convexity results for the first eigenvalue of the Dirichlet problem for Gaussian measure in [4], since (5.3) holds with $\delta > 0$.

In fact, in [4] the authors consider for any domain $\Omega \subset \mathbb{R}^N$

$$\lambda(\Omega) = \inf_{K(\Omega)} \frac{\int_\Omega |\nabla v|^2 e^{-|\xi|^2} d\xi}{\int_\Omega v^2 e^{-|\xi|^2} d\xi}$$

where

$$K(\Omega) = \Big\{ v \,/\, \int_\Omega (v^2 + |\nabla v|^2)\, e^{-|\xi|^2} d\xi < \infty,\ v = 0 \text{ on } \partial\Omega,\ v \not\equiv 0 \Big\}.$$

For every $0 < m < \int_{\mathbb{R}^N} e^{-|\xi|^2} d\xi$ they prove that

$$\inf_{\Omega \,/\, \int_\Omega e^{-|\xi|^2} d\xi = m} \lambda(\Omega) = \lambda(\Omega_a) =: \lambda(a),$$

where $\Omega_a := \{\xi_N > a\}$ and a is such that $\int_{\Omega_a} e^{-|\xi|^2} d\xi = m$.

Also, the authors prove in [4]:

(1) $\lambda(a)$ is a convex function of a.

(2) If $\lambda(\Omega) = \lambda(a)$, with $\int_\Omega e^{-|\xi|^2} d\xi = m$, then $\Omega = \Omega_a$ (for a certain system of coordinates).

Moreover, they prove that

(3) If $\lambda(a) + \lambda(-a) = 2\lambda(0)$, then $a = 0$.

Let us also observe that, from the definition of $\lambda(\Omega)$, it follows that

$$\Omega_1 \subset \Omega_2 \qquad \text{implies that} \qquad \lambda(\Omega_1) \geq \lambda(\Omega_2).$$

On the other hand, since $J_U(t) \equiv \delta$ for $t > 0$, there follows that $J_U'(t) \equiv 0$ which is equivalent to

$$\frac{I_1'(t)}{I_1(t)} + \frac{I_2'(t)}{I_2(t)} = \frac{2}{t}$$

with

$$I_i(t) = \int_{-t}^0 \int_{\mathbb{R}^N} |\nabla u_i(x)|^2 G(x, -s)\, dx ds$$

where $u_1 = U^+$ and $u_2 = U^-$. Here it is used that, since $\delta > 0$, $I_i(t) \neq 0$ for every $t > 0$. Now, since u_1 and u_2 satisfy the hypothesis of Theorem 1 in [9], there holds that (see [9], or [4])

$$\frac{I_i'(t)}{I_i(t)} \geq \frac{\lambda\big(\{u_i(2\sqrt{t}\xi) > 0\}\big)}{2t} \geq \frac{\lambda(a_i)}{2t}$$

where a_i is such that $\int_{\{\xi_N > a_i\}} e^{-|\xi|^2} d\xi = \int_{\{u_i(2\sqrt{t}\xi) > 0\}} e^{-|\xi|^2} d\xi$. Therefore,

$$\begin{aligned} \frac{2}{t} = \frac{I_1'(t)}{I_1(t)} + \frac{I_2'(t)}{I_2(t)} &\geq \frac{\lambda\big(\{u_1(2\sqrt{t}\xi) > 0\}\big) + \lambda\big(\{u_2(2\sqrt{t}\xi) > 0\}\big)}{2t} \\ &\geq \frac{\lambda(a_1) + \lambda(a_2)}{2t} \geq \frac{\lambda(-a_2) + \lambda(a_2)}{2t} \geq \frac{\lambda(0)}{t} = \frac{2}{t} \end{aligned}$$

since $a_1 \geq -a_2$ and $\lambda(a)$ is increasing and convex. In fact, since

$$\{u_1(2\sqrt{t}\xi) > 0\} \cap \{u_2(2\sqrt{t}\xi) > 0\} = \emptyset,$$

there follows that

$$\int_{\{\xi_N>a_1\}} e^{-|\xi|^2}d\xi + \int_{\{\xi_N<-a_2\}} e^{-|\xi|^2}d\xi = \int_{\{\xi_N>a_1\}} e^{-|\xi|^2}d\xi + \int_{\{\xi_N>a_2\}} e^{-|\xi|^2}d\xi$$
$$= \int_{\{u_1(2\sqrt{t}\xi)>0\}} e^{-|\xi|^2}d\xi + \int_{\{u_2(2\sqrt{t}\xi)>0\}} e^{-|\xi|^2}d\xi \leq \int_{\mathbb{R}^N} e^{-|\xi|^2}d\xi.$$

Therefore, $a_1 \geq -a_2$.

So that, there holds that

$$\frac{\lambda(a_2)+\lambda(-a_2)}{2} = \lambda(0)$$

and, by the results of [4] we deduce that $a_2 = 0$. But, there also holds

$$\frac{\lambda(a_1)+\lambda(a_2)}{2} = \frac{\lambda(a_1)+\lambda(0)}{2} = \lambda(0)$$

so that $\lambda(a_1) = \lambda(0)$ and it follows that $a_1 = 0$.

Now, since $\lambda\big(\{u_i(2\sqrt{t}\xi) > 0\}\big) \geq \lambda(a_i)$, it follows that

$$\lambda\big(\{u_i(2\sqrt{t}\xi) > 0\}\big) = \lambda(a_i).$$

Therefore, by the results in [4] (since $a_i = 0$) there exists a unit vector ν_i (for $i = 1, 2$) such that

$$\{u_i(2\sqrt{t}\xi) > 0\} = \{\xi \,/\, \langle \xi, \nu_i\rangle > 0\}$$

and thus,

$$\{U^+(x) > 0\} = \{x \,/\, \langle x, \nu_1\rangle > 0\}.$$

Analogously,

$$\{U^-(x) > 0\} = \{x \,/\, \langle x, \nu_2\rangle > 0\}.$$

Since $\{U^+(x) > 0\} \cap \{U^-(x) > 0\} = \emptyset$, there follows that $\nu_2 = -\nu_1$.

Finally, from the fact that U is Lipschitz continuous in $\mathbb{R}^N$ and harmonic in $\{U > 0\} \cup \{U \leq 0\}^\circ$ it follows that (with $\nu = \nu_1$)

$$U(x) = \alpha\langle x, \nu\rangle^+ - \gamma\langle x, \nu\rangle^-$$

with $\alpha, \gamma > 0$.

Now Prop. 2.3 and Thm. 3.1 give $\alpha^2 - \gamma^2 = 2M$ and the proof is complete.

Let us start with the regularity results.

Theorem 5.1. *Let u^{ε_j} be solutions to E_{ε_j} in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Assume that $\{u \equiv 0\}^\circ = \emptyset$ and that u^- is nondegenerate at $x_0 \in \Omega \cap \partial\{u > 0\}$ in the sense of (3.2). Then the free boundary is a $C^{1,\alpha}$ surface in a neighborhood of x_0.*

The same result holds if we replace the assumption that $\{u \equiv 0\}^\circ = \emptyset$ by the hypothesis of Theorem 4.2.

Proof. Since u falls under the hypotheses of Corollary 4.1 it follows that u is a viscosity solution to the free boundary problem in Ω. (If instead of the fact that $\{u \equiv 0\}^\circ = \emptyset$ there holds that u satisfies the hypothesis of Theorem 4.2, the same conclusion follows).

We will now derive our regularity result.

Let $\lambda_n > 0$ be a sequence such that $\lambda_n \to 0$ and such that $u_{\lambda_n}(x) = \frac{1}{\lambda_n} u(x_0 + \lambda_n x)$ converges to a function U as $n \to \infty$, uniformly on compact sets of $\mathbb{R}^N$.

From Proposition 5.1, it follows that

$$U(x) = \alpha\langle x, \nu\rangle^+ - \gamma\langle x, \nu\rangle^- \qquad \text{in } \mathbb{R}^N,$$

with ν a unit vector, and α, γ positive constants satisfying $\alpha^2 - \gamma^2 = 2M$.

Therefore, given $0 < \mu < 1$ and $\frac{\pi}{4} < \theta_0 < \frac{\pi}{2}$, there exists n_0 such that, for any $n \geq n_0$, u_{λ_n} is μ-monotone in $B_1(0)$ in any direction τ of the cone

$$\Gamma(\theta_0, \nu) = \{\tau : \text{angle}(\tau, \nu) \leq \theta_0\}.$$

That is,

$$u_{\lambda_n}(x + r\tau) \geq u_{\lambda_n}(x) \quad \text{for any } 1 \geq r \geq \mu.$$

Since u_{λ_n} is a viscosity solution to problem E, it follows from Theorem 1 in [7] that u_{λ_n} is fully monotone in $B_{1/2}(0)$ in any direction of a smaller cone. Therefore the free boundary of u is Lipschitz in $B_{\lambda_n/2}(x_0)$.

Now Theorem 1 in [6] and the remarks in Section 1 in [7] imply that $\partial\{u > 0\}$ is a $C^{1,\alpha}$ surface in a neighborhood of x_0.

Remark 5.1. We point out that in the strictly two phase case, if the free boundary is smooth, then u^- is nondegenerate on the free boundary. This is the case even if we only require the free boundary to be locally Lipschitz continuous.

In fact, if $\{u \equiv 0\}^\circ = \emptyset$, then u is a viscosity solution (Corollary 4.1). Therefore, if the free boundary is locally Lipschitz continuous, we may apply the results in [6] and [7] to conclude that it is locally a $C^{1,\alpha}$ surface. Thus, by Hopf's principle u^- is nondegenerate on the free boundary.

When $\{u \equiv 0\}^\circ = \emptyset$, the free boundary cannot be smooth in a neighborhood of a point $x_0 \notin \partial\{u < 0\} \subset \partial\{u > 0\}$. For $\partial\{u < 0\}$ we have the following regularity result:

Corollary 5.1. *Let u^{ε_j} be solutions to E_{ε_j} in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Assume that $\{u \equiv 0\}^\circ = \emptyset$. Then, there is a subset $\mathcal{R}$ of $\Omega \cap \partial\{u > 0\}$ which is locally a $C^{1,\alpha}$ surface. Moreover, $\mathcal{R}$ is open and dense in $\Omega \cap \partial\{u < 0\}$.*

Proof. Let $x_0 \in \Omega \cap \partial\{u < 0\}$ and let $r > 0$ small. Let $y \in \{u < 0\} \cap B_{r/2}(x_0)$ and $z \in \partial\{u < 0\}$ be such that $d = \operatorname{dist}(y, \partial\{u < 0\}) = |y - z|$. Then, $B_d(y)$ is tangent to $\partial\{u < 0\}$ at the point z. Moreover, $u < 0$ in $B_d(y)$ so that u^- is nondegenerate at z in the sense of (3.2). By Theorem 5.1 the free boundary is a $C^{1,\alpha}$ surface in a neighborhood of z. In addition, the distance from z to x_0 is at most r. The corollary is proved.

Corollary 5.2. *Let u^{ε_j} be solutions to E_{ε_j} in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Assume that there exists a domain $\mathcal{U} \subset \Omega$ such that u^- is nondegenerate at every point in $\mathcal{U} \cap \partial\{u > 0\}$ in the sense of (3.2). Then $\mathcal{U} \cap \partial\{u > 0\}$ is locally a $C^{1,\alpha}$ surface.*

Proof. Since u^- is nondegenerate at every point in $\mathcal{U} \cap \partial\{u > 0\}$ in the sense of (3.2), then by Lemma 5.2 u^+ is nondegenerate at every point in $\mathcal{U} \cap \partial\{u > 0\}$ in the same sense. Therefore u satisfies the hypothesis of Theorem 4.2 and we may apply Theorem 5.1 at every point $x_0 \in \mathcal{U} \cap \partial\{u > 0\}$.

For the general two phase case, we have the following result:

Theorem 5.2. *Let u^{ε_j} be solutions to E_{ε_j} in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Assume that u^+ is locally uniformly nondegenerate in the sense that (5.2) holds on every compact subset of Ω. If $x_0 \in \Omega \cap \partial\{u > 0\}$ is such that $\partial\{u > 0\}$ has at x_0 an inward unit normal in the measure theoretic sense then, the free boundary is a $C^{1,\alpha}$ surface in a neighborhood of x_0.*

Proof. We will see that u^+ is locally uniformly nondegenerate on the free boundary in the sense (3.2), so that u falls under the hypotheses of Corollary 4.2 and we deduce that u is a viscosity solution in Ω.

In fact, under our hypotheses we can apply Lemma 7 in [8] to u^+. Therefore, for every $\Omega' \subset\subset \Omega$ there are positive constants c and r_0 such that

$$\sup_{B_r(y_0)} u^+ \geq cr,$$

for every $y_0 \in \Omega' \cap \partial\{u > 0\}$ and $0 < r < r_0$.

Next, let $y_0 \in \Omega' \cap \partial\{u > 0\}$, $0 < \rho < r_0$ and consider the function h which is harmonic in $B_\rho(y_0)$ with boundary values u^+. We clearly have

$$h \geq u^+ \quad \text{in } B_\rho(y_0), \qquad h(y_0) = \fint_{\partial B_\rho(y_0)} u^+,$$

and then, by Harnack's inequality applied to h we get

$$\fint_{\partial B_\rho(y_0)} u^+ = h(y_0) \geq \frac{1}{2^N} \sup_{B_{\rho/2}(y_0)} u^+ \geq C\rho.$$

Therefore,

$$\fint_{B_r(y_0)} u^+ \geq Cr \qquad \text{for every } 0 < r \leq r_0.$$

This is, u^+ is locally uniformly nondegenerate on $\Omega \cap \partial\{u > 0\}$ in the sense of (3.2).

We will derive our regularity result from Theorem 2' in [7]. To do so, let us see that the free boundary is flat in a neighborhood of x_0. In fact, since the free boundary has at x_0 an inward unit normal ν in the measure theoretic sense we can apply Theorem 3.1 to deduce that

$$u(x) = \alpha\langle x - x_0, \nu\rangle^+ - \gamma\langle x - x_0, \nu\rangle^- + o(|x - x_0|),$$

with $\alpha^2 - \gamma^2 = 2M$, $\alpha > 0$, $\gamma \geq 0$.

Then, given $\lambda_n \to 0$, the sequence $u_{\lambda_n}(x) = \frac{1}{\lambda_n} u(x_0 + \lambda_n x)$ converges uniformly on compact sets of $\mathbb{R}^N$ to $u_0(x) = \alpha\langle x, \nu\rangle^+ - \gamma\langle x, \nu\rangle^-$.

It is not hard to see that for any $\varepsilon > 0$ small, there holds that

$$u_{\lambda_n} > 0 \quad \text{in } B_1(0) \cap \{\langle x, \nu\rangle > \varepsilon\}, \tag{5.4}$$

$$u_{\lambda_n} \leq 0 \quad \text{in } B_1(0) \cap \{\langle x, \nu\rangle < -\varepsilon\}, \tag{5.5}$$

if n is large enough. Indeed, (5.4) follows easily and the same happens with (5.5) in case $\gamma > 0$.

Let us derive (5.5) in case $\gamma = 0$. In fact, suppose that there holds that $B_1(0) \cap \{\langle x, \nu\rangle < -\varepsilon\} \cap \partial\{u_{\lambda_{n_k}} > 0\} \neq \emptyset$, for a subsequence. Then, we arrive at a contradiction by using that u^+ is locally uniformly nondegenerate on

$\Omega \cap \partial\{u > 0\}$ in the sense of (3.2). If, on the other hand, there holds that $u_{\lambda_{n_k}} > 0$ in $B_1(0) \cap \{\langle x, \nu\rangle < -\varepsilon\}$ for a subsequence, then the contradiction follows from the fact that u^+ satisfies (5.2) in a neighborhood of x_0. Thus, (5.5) holds.

Therefore, u falls under the hypotheses of Thm. 2' in [7] for small balls around x_0. This eventually implies that $\partial\{u > 0\}$ is a $C^{1,\alpha}$ surface in a neighborhood of x_0.

Corollary 5.3. *Let u^{ε_j} be solutions to E_{ε_j} in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Assume, in addition, that:*

i) *u^+ is locally uniformly nondegenerate on $\Omega \cap \partial\{u > 0\}$ in the sense of (3.2).*

ii) *The set $\{u \leq 0\}$ has locally uniform positive density on $\Omega \cap \partial\{u > 0\}$.*

Then, there is a subset $\mathcal{R}$ of the free boundary $\Omega \cap \partial\{u > 0\}$ ($\mathcal{R} = \partial_{\rm red}\{u > 0\}$) which is locally a $C^{1,\alpha}$ surface. Moreover, $\mathcal{R}$ is open and dense in $\Omega \cap \partial\{u > 0\}$ and the remainder of the free boundary has $(N-1)$-dimensional Hausdorff measure zero.

Proof. Let us first observe that, since Theorem 3.2 applies (see Remark 3.1), the free boundary $\Omega \cap \partial\{u > 0\}$ has locally finite $(N-1)$-dimensional Hausdorff measure and therefore, $\{u > 0\}$ has locally finite perimeter in Ω.

By Lemma 5.1, u^+ is locally uniformly nondegenerate in the sense that it satisfies (5.2) on every compact set. Therefore u is under the hypothesis of Theorem 5.2 for every point $x_0 \in \Omega \cap \partial\{u > 0\}$ at which the free boundary has an inward unit normal in the measure theoretic sense, this is, at every point $x_0 \in \partial_{\rm red}\{u > 0\}$. Thus, the free boundary is a $C^{1,\alpha}$ surface in a neighborhood of any such point.

To conclude the proof we observe that under our hypotheses, we have for every point $\bar{x} \in \Omega \cap \partial\{u > 0\}$

$$\liminf_{r\to 0} \frac{|B_r(\bar{x}) \cap \{x/u(x) > 0\}|}{|B_r(\bar{x})|} > 0, \quad \liminf_{r\to 0} \frac{|B_r(\bar{x}) \cap \{x/u(x) \leq 0\}|}{|B_r(\bar{x})|} > 0,$$

and therefore, Lemma 1 in [15], Section 5.8, gives that $\mathcal{H}^{N-1}$- almost all $\bar{x} \in \Omega \cap \partial\{u > 0\}$ is in the reduced boundary. Finally, Theorem 3.2 (see 3.2.3)) implies that the reduced boundary is dense in $\Omega \cap \partial\{u > 0\}$. Thus, the corollary is proved.

We obtain the following result for the one phase case:

Theorem 5.3. *Let u and $\mathcal{R}$ be as in Corollary 5.3. Assume in addition that $u \geq 0$ in $\mathcal{U} \subset\subset \Omega$. Then $\mathcal{R} \cap \mathcal{U}$ is locally an analytic surface.*

Proof. The previous results imply that for every free boundary point $x_0 \in \mathcal{R}$ there is a neighborhood $\mathcal{N} = \mathcal{N}_{x_0} \subset \mathcal{U}$ such that $\mathcal{N} \cap \partial\{u > 0\}$ is a $C^{1,\alpha}$ surface. Therefore, u satisfies

$$\Delta u = 0 \qquad \text{in } \mathcal{N} \cap \{u > 0\},$$
$$u = 0,\ u_\nu = \sqrt{2M} \qquad \text{on } \mathcal{N} \cap \partial\{u > 0\}$$

(ν the inward unit normal to the free boundary) in the classical sense and then, Theorem 2 in [17] implies that $\mathcal{N} \cap \partial\{u > 0\}$ is analytic.

We finally study the behavior near (not necessarily regular) free boundary points, in the one phase case:

Theorem 5.4. *Let u^{ε_j} be solutions to E_{ε_j} in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Assume that $u \geq 0$ in $\mathcal{U} \subset\subset \Omega$.*

Let $\Gamma \subset \mathcal{U} \cap \partial\{u > 0\}$ denote the set of free boundary points in $\mathcal{U}$ which are regular from the positive side and assume, in addition, that the set $\{u \leq 0\}$ has uniform positive density on Γ. (We call this property, Property (D)). Then, for every $x_0 \in \mathcal{U} \cap \partial\{u > 0\}$ there holds that

$$\limsup_{\substack{x \to x_0 \\ u(x) > 0}} |\nabla u(x)| = \sqrt{2M}.$$

The same result holds if we replace Property (D) by the following property which we call Property (L): For every point $z \in \Gamma$, and for every ball $B_{\rho(z)}(y_z) \subset \{u > 0\}$ with $z \in \partial B_{\rho(z)}(y_z)$, there exists a unit vector $\tilde{e}_z$, with $\langle \tilde{e}_z, y_z - z\rangle > \theta||y_z - z||$, such that

$$u(z - s\tilde{e}_z) \leq 0 \quad \text{for } 0 < s < s_0.$$

Here θ and s_0 are positive constants not depending on the point z.

Remark 5.2 In [11], Theorem 6.1, it was proved (for the parabolic version of this problem) that if u is a limit function and $\limsup_{x \to x_0} |\nabla u^-(x)| \leq \gamma$, then $\limsup_{x \to x_0} |\nabla u^+(x)| \leq \sqrt{2M + \gamma^2}$.

In particular, any limit function $u \geq 0$ satisfies

$$\limsup_{\substack{x \to x_0 \\ u(x) > 0}} |\nabla u(x)| \leq \sqrt{2M}.$$

On the other hand, it was shown in [11], Remark 5.1, that we cannot expect the equality to hold in general.

In Theorem 5.4 we show, when $u \geq 0$, that under some additional hypothesis at points which are regular from the positive side (Property (D) or Property (L)), the equality holds.

Proof of Theorem 5.4. Let $x_0 \in \mathcal{U} \cap \partial\{u > 0\}$, and set

$$\alpha := \limsup_{\substack{x \to x_0 \\ u(x)>0}} |\nabla u(x)|.$$

Since u is locally Lipschitz continuous, we have $0 \leq \alpha < \infty$.
Case I. Assume that $\alpha > 0$. We will show that in this case $\alpha = \sqrt{2M}$.

We know that there is a sequence $y_n \to x_0$ with

$$u(y_n) > 0, \qquad |\nabla u(y_n)| \to \alpha.$$

Let $d_n := \operatorname{dist}(y_n, \partial\{u > 0\})$. Then $d_n \to 0$. Let $z_n \in \mathcal{U} \cap \partial\{u > 0\}$ be such that $d_n = |y_n - z_n|$. We have

$$B_{d_n}(y_n) \subset \{u > 0\}.$$

Let us consider the sequence $u_{d_n}(x) = \frac{1}{d_n} u(z_n + d_n x)$. There is a function u_0 such that, for a subsequence, $u_{d_n} \to u_0$ uniformly on compact sets of $\mathbb{R}^N$. We can choose the subsequence so that there exists

$$v := \lim_{n\to\infty} \frac{y_n - z_n}{d_n},$$

and without loss of generality we will assume that $v = e_1$.

Then, u_0 is Lipschitz continuous and harmonic in $\{u_0 > 0\}$ and there holds, by construction, that either

$$B_1(e_1) \subset \{u_0 > 0\} \quad \text{or} \quad u_0 \equiv 0 \ \text{ in } B_1(e_1).$$

Also,

$$|\nabla u_0| \leq \alpha \ \text{ in } \{u_0 > 0\} \quad \text{and} \quad |\nabla u_0(e_1)| = \alpha$$

and therefore, $u_0 > 0$ in $B_1(e_1)$.

Let $\nu := \frac{\nabla u_0(e_1)}{|\nabla u_0(e_1)|}$, and consider the function $\frac{\partial u_0}{\partial \nu}$, which is harmonic in $\{u_0 > 0\}$ and satisfies

$$\frac{\partial u_0}{\partial \nu} \leq \alpha \ \text{ in } \ B_1(e_1), \qquad \frac{\partial u_0}{\partial \nu}(e_1) = \alpha.$$

The strong maximum principle hence implies

$$\frac{\partial u_0}{\partial \nu}(x) \equiv \alpha \qquad \text{and} \qquad u_0(x) = \alpha\langle x, \nu\rangle + c \qquad \text{in } \ B_1(e_1),$$

so that necessarily $c = 0$ and $\nu = e_1$. Finally, by analytic continuation we get

$$u_0(x) = \alpha x_1^+ \text{ in } \{x_1 \geq 0\}.$$

Next, the application of Lemma A1 in [7] to u_0 in $\{x_1 < 0\}$ yields

$$u_0(x) = \gamma x_1^- + o(|x|) \quad \text{in } \{x_1 < 0\}, \quad \gamma \geq 0.$$

Let us consider for $\lambda > 0$, $(u_0)_\lambda(x) = \frac{1}{\lambda} u_0(\lambda x)$. There exists a sequence $\lambda_n \to 0$ such that $(u_0)_{\lambda_n} \to u_{00}$ uniformly on compact sets of $\mathbb{R}^N$. It follows that

$$u_{00}(x) = \alpha x_1^+ + \gamma x_1^- \quad \text{ in } \mathbb{R}^N.$$

If Property (D) holds, it is not hard to see that the set $\{u_0 \leq 0\}$ has positive density at the origin and therefore, the same property holds for u_{00}. This implies that $\gamma = 0$.

We get the same conclusion in case Property (L) holds instead of Property (D).

Finally, we apply Theorem 3.1 (arguing as we did in Theorem 4.1) to deduce that $\alpha = \sqrt{2M}$.

Case II. We will now assume that $\alpha = 0$ and arrive at a contradiction.

In fact, given $\delta > 0$ small, we have

$$|\nabla u(x)| < \delta \quad \text{ in } B_\rho(x_0), \tag{5.6}$$

for some $\rho > 0$.

Let us choose a point $z_0 \in B_\rho(x_0) \cap \partial\{u > 0\}$ which is regular from the positive side so that u^+ is nondegenerate at z_0 in the sense of (3.2). Let us consider sequences λ_n, $\overline{\lambda}_n$ and $\tilde{\lambda}_n$ going to zero such that, for some functions u_0, u_{00} and u_{000},

$$u_{\lambda_n}(x) = \frac{1}{\lambda_n} u(\lambda_n x + z_0), \quad u_{\lambda_n} \to u_0,$$

$$(u_0)_{\overline{\lambda}_n}(x) = \frac{1}{\overline{\lambda}_n} u_0(\overline{\lambda}_n x), \quad (u_0)_{\overline{\lambda}_n} \to u_{00},$$

$$(u_{00})_{\tilde{\lambda}_n}(x) = \frac{1}{\tilde{\lambda}_n} u_{00}(\tilde{\lambda}_n x), \quad (u_{00})_{\tilde{\lambda}_n} \to u_{000}.$$

Then, we have from (5.6)

$$|\nabla u_{000}(x)| < \delta \quad \text{in } \mathbb{R}^N. \tag{5.7}$$

On the other hand, an argument similar to the one used in Thm. 4.1, in combination with the fact that u^+ is nondegenerate at z_0 in the sense of (3.2) and with Property (D) or (L), implies (via the application of Theorem 3.1) that

$$u_{000}(x) = \sqrt{2M}\langle x, e\rangle^+,$$

for some unit vector e. This is in contradiction with (5.7) and therefore, $\alpha > 0$. Now the proof is complete.

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