A LOCAL MONOTONICITY FORMULA FOR AN INHOMOGENOUS SINGULAR PERTURBATION PROBLEM AND APPLICATIONS

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ABSTRACT. In this paper we prove a local monotonicity formula for solutions to an inhomogeneous singularly perturbed diffusion problem of interest in combustion. This type of monotonicity formula has proved to be very useful for the study of the regularity of limits u of solutions of the singular perturbation problem and of $\partial\{u > 0\}$, in the global homogeneous case.

As a consequence of this formula we prove that u has an asymptotic development at every point in $\partial\{u > 0\}$ where there is a nonhorizontal tangent ball. These type of developments have been essential for the proof of the regularity of $\partial\{u > 0\}$ for Bernoulli and Stefan free boundary problems.

We also present applications of our results to the study of $\partial\{u > 0\}$ in the stationary case including, in particular, its regularity in the case of energy minimizers. We present as well a regularity result for travelling waves of a combustion model that relies on our monotonicity formula and its consequences.

The fact that our results hold for the inhomogeneous problem allows a very wide applicability. In fact, they may be applied to problems with nonlocal diffusion and/or transport.

1. INTRODUCTION

In this paper we develop a local monotonicity formula for solutions to an inhomogeneous singular perturbation problem of interest in combustion theory. It is based on a global monotonicity formula that G. S. Weiss developed for solutions of the global homogeneous problem (see [18]). The problem under consideration here is the following: for $\varepsilon > 0$ we let u^ε be a family of solutions to

$$(P_\varepsilon(f_\varepsilon)) \qquad\qquad \Delta u^\varepsilon - u_t^\varepsilon = \beta_\varepsilon(u^\varepsilon) + f_\varepsilon \quad \text{in } \mathcal{D}$$

where $\varepsilon > 0$, $\mathcal{D}$ is a domain in $\mathbb{R}^{N+1}$, f_ε are uniformly bounded in $L^\infty(\mathcal{D})$, $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon})$ with β a Lipschitz continuous function, $\beta(s) > 0$ for $0 < s < 1$ and $\beta(s) = 0$ otherwise. This type of reaction term appears in the study of the propagation of deflagration flames. In this context ε represents the inverse of the activation energy (see for instance, [2], [3], [15] and the references therein).

We prove that u^ε satisfies a local monotonicity formula. This formula allows us to state our problem in bounded domains and not only in $\mathbb{R}^N \times (0, T)$ as considered in [18]. Moreover, we show that the formula holds for the inhomogeneous equation, this is, $f_\varepsilon \not\equiv 0$.

As an application of this formula, we prove that any limit u of solutions u^ε of this singular perturbation problem (as $\varepsilon \to 0$) has an asymptotic development at each point on $\partial\{u > 0\}$

where there is a nonhorizontal tangent ball contained either in $\{u > 0\}$ or in $\{u \leq 0\}^\circ$, and we determine all the slopes that it is possible to encounter in this way. Moreover, we present applications of the results in this paper to the study of the regularity of the boundary of $\{u > 0\}$ (the flame front in combustion models).

The fact that we are looking at the inhomogeneous equation makes the applicability of these results much wider. In particular, the results in this paper apply to more general equations that include nonlocal diffusion and/or transport. In fact, let $u^\varepsilon = u^\varepsilon(x, t)$ be a family of solutions to:

$$(G_\varepsilon(g_\varepsilon)) \qquad (1-\theta)\int J(x-y)\big(u^\varepsilon(y,t) - u^\varepsilon(x,t)\big)\,dy + \theta\Delta u^\varepsilon +$$
$$+ V_\varepsilon(x,t)\cdot\nabla u^\varepsilon + c_\varepsilon(x,t)u^\varepsilon - u_t^\varepsilon = \beta_\varepsilon(u^\varepsilon) + g_\varepsilon$$

where β_ε is as above, $0 < \theta \leq 1$, $J \in L^1(\mathbb{R}^N)$ with $J(x) = J(-x)$, $J \geq 0$, and with $V_\varepsilon \in (L^\infty)^N$, $c_\varepsilon \in L^\infty$, $g_\varepsilon \in L^\infty$ all with norms uniformly bounded with respect to ε (the term involving the integral in $G_\varepsilon(g_\varepsilon)$ is used to model nonlocal diffusion, see [12]). Then, u^ε satisfy

$$\theta\Delta u^\varepsilon - u_t^\varepsilon = \beta_\varepsilon(u^\varepsilon) + f_\varepsilon$$

with

$$(1.1) \qquad f_\varepsilon = g_\varepsilon - V_\varepsilon(x,t)\cdot\nabla u^\varepsilon - c_\varepsilon(x,t)u^\varepsilon - (1-\theta)\int J(x-y)\big(u^\varepsilon(y,t)-u^\varepsilon(x,t)\big)\,dy.$$

In this paper we prove the monotonicity formula for a family of solutions to $P_\varepsilon(f_\varepsilon)$ under the assumption that the solutions u^ε are uniformly bounded in $Lip(1, 1/2)$ norm and that $\|f_\varepsilon\|_{L^\infty}$ are uniformly bounded.

In case u^ε are solutions to $G_\varepsilon(g_\varepsilon)$ which are uniformly bounded in $Lip\,(1, 1/2)$ norm, the functions f_ε given by (1.1) are uniformly bounded in L^∞ norm. So u^ε are in the situation considered in this paper.

Local bounds for the gradients of solutions u^ε to $G_\varepsilon(g_\varepsilon)$ in terms of bounds of $\|u^\varepsilon\|_{L^\infty}$ were proved in different contexts. For instance, for $V_\varepsilon = 0$ they were proved in [13] in the one phase case (i.e. $u^\varepsilon \geq 0$) and in [14] in the stationary two phase case. For $\theta = 1$ they were proved in the stationary one phase case in [2], when $g_\varepsilon = 0$; in the evolutionary two phase case in [6], when $V_\varepsilon = 0$, $c_\varepsilon = 0$ and $g_\varepsilon = 0$; and in [8], when $g_\varepsilon = 0$. Moreover, in global situations for initial data uniformly bounded in Lipschitz norm, it is possible to obtain these bounds by applying Bernstein type techniques, as done in [11] and [18] when $\theta = 1$, $V_\varepsilon = 0$, $c_\varepsilon = 0$, $g_\varepsilon = 0$ and $u^\varepsilon \geq 0$. Once the bound on the gradients is known, the bound on the Hölder $1/2$ seminorm in the t variable is easily obtained (see, for instance, [9] for the proof).

The main results in the paper are two local monotonicity formulae: one for solutions u^ε of problem $P_\varepsilon(f_\varepsilon)$ (Theorem 2.1) and one for the limits (as $\varepsilon \to 0$) u of u^ε (Theorem 2.2).

The second one, in particular, says that if $u = \lim u^\varepsilon$, $\chi = \lim B_\varepsilon(u^\varepsilon)$, $(x_0, t_0) \in \partial\{u > 0\}$ and $\psi = \psi(x) \in C_0^\infty$, $0 \leq \psi \leq 1$, $\psi \equiv 1$ in a neighborhood of x_0, then there exists

$$(1.2) \qquad \delta(x_0, t_0) = \lim_{r\to 0} \frac{1}{r^2}\int_{t_0-4r^2}^{t_0-r^2}\int_{\mathbb{R}^N}\Big(|\nabla(u\psi)|^2 + 2\psi^2\chi + \frac{1}{2}\frac{(u\psi)^2}{t-t_0}\Big)G(x-x_0, t_0-t)\,dx\,dt,$$

where $G(x,t) = \frac{1}{(4\pi t)^{N/2}}\exp\big(-\frac{|x|^2}{4t}\big)$.

As a corollary to these results we obtain that any blow up limit u_0 of $u = \lim u^\varepsilon$ at a point $(x_0, t_0) \in \partial\{u > 0\}$ $(u_0 = \lim_{\lambda_n \to 0}\frac{1}{\lambda_n}u(x_0 + \lambda_n x, t_0 + \lambda_n^2 t))$ is homogeneous for $t \leq 0$. This is, $u_0(rx, r^2 t) = ru_0(x, t)$, for $t \leq 0$ and any $r > 0$ (Corollary 2.1).

The knowledge of homogeneity properties of blow up limits is a main step in the classification of those limits, which is a key tool towards the understanding of the behavior of u near $\partial\{u > 0\}$ in free boundary problems. See, for instance, [10, 17] for the obstacle problem, [7, 16] for Bernoulli's problem, and Sections 3 and 4 below for the problem under consideration in the present paper.

A second corollary is that, given $u = \lim u^\varepsilon$, $\chi = \lim B_\varepsilon(u^\varepsilon)$, $(x_0, t_0) \in \partial\{u > 0\}$ and $\delta(x_0, t_0)$ as in (1.2) then, for any sequence $\lambda_n \to 0$, and limits $u_0 = \lim \frac{1}{\lambda_n} u(x_0 + \lambda_n x, t_0 + \lambda_n^2 t)$ and $\chi_0 = \lim \chi(x_0 + \lambda_n x, t_0 + \lambda_n^2 t)$, there holds that, for any $r > 0$,

$$
\begin{aligned}
(1.3) \qquad \delta(x_0, t_0) &= \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} \left(|\nabla u_0|^2 + 2\chi_0 + \frac{1}{2} \frac{u_0^2}{t} \right) G(x, -t) \, dx \, dt \\
&= \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} 2\chi_0 \, G(x, -t) \, dx \, dt
\end{aligned}
$$

and $0 \leq \delta(x_0, t_0) \leq 6M$ where $M = \int \beta$ (Corollary 2.2).

From (1.3) (see Remark 2.3) it is possible to see that the value of $\delta(x_0, t_0)$ is intimately related to the regularity of $\partial\{u > 0\}$ in a neighborhood of (x_0, t_0). Moreover, we prove in Theorem 9.7 in [14], which holds in the stationary case, that $\delta(x_0) = 3M$ if and only if $\partial\{u > 0\}$ is smooth in a neighborhood of x_0 (see also Section 4 in this paper).

Concerning the applications of our monotonicity formula and its corollaries, we present in this paper two type results.

The result on asymptotic developments described above (Theorem 3.1) concerns limits u of solutions u^ε to the singular perturbation problem $P_\varepsilon(f_\varepsilon)$. These limits are solutions to a differential equation in $\{u > 0\}$ and $\{u \leq 0\}^\circ$ (see for instance, [14]). Thus, a thorough understanding of the limits u comes from the understanding of the boundaries $\partial\{u > 0\}$. The asymptotic developments we obtain in Theorem 3.1 are a step towards this end. In fact, asymptotic developments like the ones in this paper have proved to be very useful to obtain regularity results of $\partial\{u > 0\}$ for free boundary problems (see for instance, [1, 4, 5]). In these papers the knowledge of the asymptotic behavior of u at points of $\partial\{u > 0\}$ with a nonhorizontal tangent ball has proved to be essential.

Moreover, in this paper we characterize all the possible slopes at which u may leave $\partial\{u > 0\}$. All these cases can happen as was shown with examples in [13]. So that our results are really sharp.

One of the main tools in the proof of Theorem 3.1 is a result on the asymptotic development at nonhorizontal points (x_0, t_0) in the boundary of a ball B at which $u(x_0, t_0) = 0$, for nonnegative functions satisfying either $\Delta u - u_t \leq C$ in B with $u > 0$ in B, or $\Delta u - u_t \geq -C$ in B^c with $u \equiv 0$ in B. These asymptotic developments were proved by the authors in a previous paper (see [13]).

We point out that the result in Theorem 3.1 is new even for the global problem and/or with $f_\varepsilon \equiv 0$.

The second type of application we present here deals with the regularity of the boundary of $\{u > 0\}$ in the stationary case including, in particular, regularity results for energy minimizers and also with regularity results for travelling waves of a combustion model.

In particular, in many situations we conclude that if u^{ε_j} are stationary solutions to $P_{\varepsilon_j}(f_{\varepsilon_j})$ and $\varepsilon_j \to 0$ then, $u = \lim u^{\varepsilon_j}$ is a classical solution to the following free boundary problem

$$\Delta u = f\chi_{\{u \neq 0\}} \qquad \text{in } \{u > 0\} \cup \{u \leq 0\}^\circ,$$
$$|\nabla u^+|^2 - |\nabla u^-|^2 = 2M \quad \text{on } \partial\{u > 0\}.$$

Concerning the monotonicity formula proved in this paper, it builds up from a formula introduced by G. S. Weiss in [18] for the global homogeneous version of the problem considered here. The formula was a crucial tool in [18] to obtain rectificability results for $\partial\{u > 0\}$.

The paper is organized as follows. In Section 2 we prove the local monotonicity formulae for the singular perturbation problem $P_\varepsilon(f_\varepsilon)$ and its limits. Then, we derive some consequences of them. In particular, we prove some corollaries that imply that for blow up limits u_0 of limit functions u, the global functional of Weiss (cf. [18]) is constant and u_0 is homogeneous. In Section 3 we use these results to obtain asymptotic developments at points on $\partial\{u > 0\}$. In Section 4 we present applications to regularity results of $\partial\{u > 0\}$ that are proved by the authors in [14] using the results in this paper.

Notation. Throughout the paper $\mathcal{D}$ will be a domain in $\mathbb{R}^{N+1}$. We will be considering rescalings of functions and we will denote

$$v_r(x,t) = \frac{1}{r}v(rx, r^2t) \quad \text{and} \quad v^r(x,t) = v(rx, r^2t).$$

We will say that a function v is in the class $Lip\,(1, 1/2)$ in a domain $\mathcal{D}$, if v is bounded and there exists a constant $L = L(\mathcal{D})$ such that

$$|v(x,t) - v(y,\tau)| \leq L(|x - y| + |t - \tau|^{\frac{1}{2}})$$

for every (x,t), (y,τ) in $\mathcal{D}$. The norm in $Lip\,(1, 1/2)$ in $\mathcal{D}$ is

$$\|v\|_{Lip\,(1,1/2)} = \|v\|_{L^\infty(\mathcal{D})} + \sup_{(x,t),(y,\tau)\in\mathcal{D}} \frac{|v(x,t) - v(y,\tau)|}{|x - y| + |t - \tau|^{1/2}}.$$

We denote by

$$|v|_{Lip\,(1,1/2)} = \sup_{(x,t),(y,\tau)\in\mathcal{D}} \frac{|v(x,t) - v(y,\tau)|}{|x - y| + |t - \tau|^{1/2}}$$

the $Lip\,(1, 1/2)$ seminorm in $\mathcal{D}$.

Finally, we will denote $B_\varepsilon(r) = \int_0^r \beta_\varepsilon(s)\,ds$, $M = \int_0^1 \beta(s)\,ds$ and $G(x,t) = \frac{1}{(4\pi t)^{N/2}}\exp(-\frac{|x|^2}{4t})$.

2. Monotonicity formula

In this section we prove the local monotonicity formulae for the singular perturbation problem $P_\varepsilon(f_\varepsilon)$ and its limits (Theorems 2.1 and 2.2). Then we derive some consequences of them.

We first obtain

Theorem 2.1 (ε -**Monotonicity Formula**). *Let u^ε be a family of solutions to $P_\varepsilon(f_\varepsilon)$ in $\mathcal{D}$, uniformly bounded in $Lip\,(1, 1/2)$ with f_ε uniformly bounded in L^∞ norm in $\mathcal{D}$ and $B_R(0) \times (-4, 0) \subset\subset \mathcal{D}$. For $\psi = \psi(x) \in C_0^\infty(B_R(0))$, $0 \leq \psi \leq 1$, $\psi \equiv 1$ in $B_{R/2}(0)$ and $0 < r \leq 1$, let*

$$\mathcal{W}^\varepsilon(r) = \mathcal{W}^\varepsilon(r, u^\varepsilon, \psi) =$$

(2.1)
$$= \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} \left(|\nabla(u^\varepsilon \psi)|^2 + 2\psi^2 B_\varepsilon(u^\varepsilon) + \frac{1}{2} \frac{(u^\varepsilon \psi)^2}{t} \right) G(x, -t) \, dx \, dt,$$

where $G(x,t) = \frac{1}{(4\pi t)^{N/2}} \exp(-\frac{|x|^2}{4t})$.

Then,

(2.2)
$$\frac{\partial \mathcal{W}^\varepsilon}{\partial r}(r) \geq \int_{-4}^{-1} \int_{\mathbb{R}^N} \left(\partial_r w_r^\varepsilon \right)^2 \frac{r G(x, -t)}{-t} \, dx \, dt -$$
$$- C_1 \left(||f_\varepsilon||_{L^\infty(B_R(0) \times (-4,0))} + ||f_\varepsilon||_{L^\infty(B_R(0) \times (-4,0))}^{3/2} \right) -$$
$$- C_2 ||f_\varepsilon||_{L^\infty(B_R(0) \times (-4,0))} \frac{|u^\varepsilon(0,0)|}{r} - C_3 C(R) e^{-C_4 \frac{R^2}{r^2}}.$$

Here $w^\varepsilon(x,t) = \psi(x) u^\varepsilon(x,t)$ and $C(R) = \max\{\frac{1}{R}, \frac{1}{R^{N/2}}\}$.

Let A_i, $i = 1, \cdots, 4$ be such that

$$||u^\varepsilon||_{L^\infty(B_R(0) \times (-4,0))} \leq A_1 \quad, \quad |u^\varepsilon|_{Lip_{(1,1/2)}(B_R(0) \times (-4,0))} \leq A_2$$
$$||u_t^\varepsilon||_{L^2(B_R(0) \times (-4,0))} \leq A_3 \quad, \quad ||\psi||_{C^2(B_R(0))} \leq A_4$$

The constants C_i in (2.2) depend only on A_i, the dimension N, $M_1 = \max_{0 \leq s \leq 1} s\beta(s)$ and $M = \int_0^1 \beta(s) \, ds$.

Remark 2.1. Observe that in [13] it was proved in the one phase case that solutions u^ε to $P_\varepsilon(f_\varepsilon)$ with uniformly bounded $Lip(1, 1/2)$ norm and $||f_\varepsilon||_{L^\infty}$ uniformly bounded, in a domain $\mathcal{D}' \supset\supset \mathcal{D}$, have uniformly bounded $||u_t^\varepsilon||_{L^2(\mathcal{D})}$ and satisfy that subsequences of ∇u^ε converge strongly in $L^2(\mathcal{D}')$. The same proof holds for the two phase case.

Proof of the theorem. By rescaling we get

(2.3)
$$\mathcal{W}^\varepsilon(r) =$$
$$= \int_{-4}^{-1} \int_{\mathbb{R}^N} \left(|\nabla w_r^\varepsilon|^2 + 2(\psi^r)^2 B_\varepsilon(r u_r^\varepsilon) + \frac{1}{2} \frac{(w_r^\varepsilon)^2}{t} \right) G(x, -t) \, dx \, dt.$$

Then,

(2.4)
$$\frac{\partial \mathcal{W}^\varepsilon}{\partial r}(r) = \int_{-4}^{-1} \int_{\mathbb{R}^N} \left(2\nabla w_r^\varepsilon \nabla(\partial_r w_r^\varepsilon) + 2(\psi^r)^2 \beta_\varepsilon(r u_r^\varepsilon)(u_r^\varepsilon + r\partial_r u_r^\varepsilon) + \right.$$
$$\left. + 4\psi^r \partial_r \psi^r B_\varepsilon(r u_r^\varepsilon) + \frac{1}{t} w_r^\varepsilon \partial_r w_r^\varepsilon \right) G(x, -t) \, dx \, dt \geq$$
$$\geq \int_{-4}^{-1} \int_{\mathbb{R}^N} \left(-2\Delta w_r^\varepsilon \partial_r w_r^\varepsilon + 2(\psi^r)^2 \beta_\varepsilon(r u_r^\varepsilon) r\partial_r u_r^\varepsilon + \right.$$
$$\left. + 4\psi^r \partial_r \psi^r B_\varepsilon(r u_r^\varepsilon) + \frac{1}{t} w_r^\varepsilon \partial_r w_r^\varepsilon - 2\partial_r w_r^\varepsilon \frac{x}{2t} \cdot \nabla w_r^\varepsilon \right) G(x, -t) \, dx \, dt$$

since $u_r^\varepsilon \beta_\varepsilon(r u_r^\varepsilon) \geq 0$ and $\nabla G(x, -t) = \frac{x}{2t} G(x, -t)$. Thus,

$$\frac{\partial \mathcal{W}^\varepsilon}{\partial r}(r) \geq \int_{-4}^{-1} \int_{\mathbb{R}^N} \Big(\partial_r w_r^\varepsilon \big(-2\Delta w_r^\varepsilon + \frac{1}{t} w_r^\varepsilon - 2\frac{x}{2t} \cdot \nabla w_r^\varepsilon \big) + 2(\psi^r)^2 \beta_\varepsilon(ru_r^\varepsilon)\, r\partial_r u_r^\varepsilon +$$
$$+ 4\psi^r \partial_r \psi^r B_\varepsilon(ru_r^\varepsilon) \Big) G(x,-t)\, dx\, dt$$

$$(2.5) \quad = \int_{-4}^{-1} \int_{\mathbb{R}^N} \Big(\partial_r w_r^\varepsilon \big(-2\psi^r (\Delta - \partial_t) u_r^\varepsilon - 2u_r^\varepsilon \Delta \psi^r - 4\nabla\psi^r \nabla u_r^\varepsilon - \frac{r}{t}\partial_r w_r^\varepsilon \big)$$
$$+ 2\psi^r r\beta_\varepsilon(ru_r^\varepsilon)(\partial_r w_r^\varepsilon - u_r^\varepsilon \partial_r \psi^r) + 4\psi^r \partial_r \psi^r B_\varepsilon(ru_r^\varepsilon) \Big) G(x,-t)\, dx\, dt$$

$$= \int_{-4}^{-1} \int_{\mathbb{R}^N} \Big(\partial_r w_r^\varepsilon \big(-2\psi^r r f_\varepsilon{}^r - 2u_r^\varepsilon \Delta \psi^r - 4\nabla\psi^r \nabla u_r^\varepsilon - \frac{r}{t}\partial_r w_r^\varepsilon \big)$$
$$- 2\psi^r \beta_\varepsilon(ru_r^\varepsilon)\, ru_r^\varepsilon \partial_r \psi^r + 4\psi^r \partial_r \psi^r B_\varepsilon(ru_r^\varepsilon) \Big) G(x,-t)\, dx\, dt.$$

We have used that

$$(2.6) \quad \begin{aligned} -2\Delta w_r^\varepsilon - 2\frac{x}{2t} \cdot \nabla w_r^\varepsilon + \frac{w_r^\varepsilon}{t} &= -2(\Delta - \partial_t) w_r^\varepsilon - 2\partial_t w_r^\varepsilon - \frac{x}{t} \cdot \nabla w_r^\varepsilon + \frac{w_r^\varepsilon}{t} = \\ &= -2(\Delta - \partial_t) w_r^\varepsilon - \frac{r}{t}\partial_r w_r^\varepsilon = -2\psi^r(\Delta - \partial_t) u_r^\varepsilon - 2u_r^\varepsilon \Delta \psi^r - 4\nabla\psi^r \nabla u_r^\varepsilon - \frac{r}{t}\partial_r w_r^\varepsilon, \end{aligned}$$

$$\psi^r \partial_r u_r^\varepsilon = \partial_r w_r^\varepsilon - u_r^\varepsilon \partial_r \psi^r, \quad \text{and} \quad (\Delta - \partial_t) u_r^\varepsilon = r\beta_\varepsilon(ru_r^\varepsilon) + r f_\varepsilon{}^r.$$

Therefore,

$$\frac{\partial \mathcal{W}^\varepsilon}{\partial r}(r) \geq \int_{-4}^{-1} \int_{\mathbb{R}^N} (\partial_r w_r^\varepsilon)^2 \frac{rG(x,-t)}{-t}\, dx\, dt +$$

$$+ \int_{-4}^{-1} \int_{\mathbb{R}^N} (\partial_r w_r^\varepsilon)(-2\psi^r r f_\varepsilon{}^r) G(x,-t)\, dx\, dt +$$

$$(2.7) \quad + \int_{-4}^{-1} \int_{\mathbb{R}^N} (\partial_r w_r^\varepsilon)(-2u_r^\varepsilon \Delta\psi^r - 4\nabla\psi^r \nabla u_r^\varepsilon) G(x,-t)\, dx\, dt +$$

$$+ \int_{-4}^{-1} \int_{\mathbb{R}^N} \big(-2\psi^r \beta_\varepsilon(ru_r^\varepsilon)\, ru_r^\varepsilon \partial_r \psi^r + 4\psi^r \partial_r \psi^r B_\varepsilon(ru_r^\varepsilon) \big) G(x,-t)\, dx\, dt =$$

$$= \int_{-4}^{-1} \int_{\mathbb{R}^N} (\partial_r w_r^\varepsilon)^2 \frac{rG(x,-t)}{-t}\, dx\, dt + I + II + III.$$

Now,

$$\partial_r w_r^\varepsilon = -\frac{w^\varepsilon(rx, r^2 t)}{r^2} + \frac{\nabla w^\varepsilon(rx, r^2 t)}{r} \cdot x + 2t w_t^\varepsilon(rx, r^2 t).$$

There holds,

$$|w^\varepsilon(rx, r^2 t)| \leq |w^\varepsilon(0,0)| + (A_1 + A_2)\, A_4\, (|x| + |t|^{1/2})\, r,$$

and for $0 < r < 1$,

$$\Big| \big(-\frac{w^\varepsilon(rx, r^2 t)}{r^2} + \frac{\nabla w^\varepsilon(rx, r^2 t)}{r} \cdot x \big) \big(-2\psi^r r f_\varepsilon{}^r \big) G(x,-t) \Big| \leq$$
$$\leq \Big(2\frac{|u^\varepsilon(0,0)|}{r} A_4 + 4(A_1 + A_2)\, A_4\, (|x| + |t|^{1/2}) \Big) \|f_\varepsilon\|_{L^\infty}\, G(x,-t).$$

So that,

$$\int_{-4}^{-1} \int_{\mathbb{R}^N} \left(-\frac{w^\varepsilon(rx, r^2 t)}{r^2} + \frac{\nabla w^\varepsilon(rx, r^2 t)}{r} \cdot x \right)(-2\psi^r r f_\varepsilon{}^r)) G(x, -t) \, dx \, dt \geq$$

$$\geq -C_2 \|f_\varepsilon\|_{L^\infty} \frac{|u^\varepsilon(0,0)|}{r} - \bar{C}_1 \|f_\varepsilon\|_{L^\infty}.$$

Now,

$$\left| \int_{-4}^{-1} \int_{\mathbb{R}^N} \left(2t w_t^\varepsilon(rx, r^2 t)(-2\psi^r r f_\varepsilon{}^r) \right) G(x, -t) \, dx \, dt \right| \leq$$

$$\leq 16 \|f_\varepsilon\|_{L^\infty(B_R(0) \times (-4,0))} \int_{-4}^{-1} \int_{\mathbb{R}^N} |w_t^\varepsilon(rx, r^2 t)| \, \psi^r \, r \, G(x, -t) \, dx \, dt =$$

$$= 16 \|f_\varepsilon\|_{L^\infty(B_R(0) \times (-4,0))} \int_{-4}^{-1} \int_{\mathbb{R}^N} |\partial_t(u_r^\varepsilon)| \, (\psi^r)^2 G(x, -t) \, dx \, dt \leq$$

$$\leq 32 \|f_\varepsilon\|_{L^\infty(B_R(0) \times (-4,0))} \left(\int_{-4}^{-1} \int_{\mathbb{R}^N} |\partial_t(u_r^\varepsilon)|^2 (\psi^r)^2 G(x, -t) \, dx \, dt \right)^{1/2}.$$

Now, let $v = u_r^\varepsilon$. Then v is a solution to $\Delta v - v_t = \beta_{\varepsilon/r}(v) + r f_\varepsilon{}^r$. Thus,

$$\int_{-4}^{-1} \int_{\mathbb{R}^N} v_t^2 (\psi^r)^2 G(x, -t) \, dx \, dt = \int_{-4}^{-1} \int_{\mathbb{R}^N} v_t \Delta v (\psi^r)^2 G(x, -t) \, dx \, dt -$$

$$- \int_{-4}^{-1} \int_{\mathbb{R}^N} \beta_{\varepsilon/r}(v) \, v_t \, (\psi^r)^2 G(x, -t) \, dx \, dt - \int_{-4}^{-1} \int_{\mathbb{R}^N} r f_\varepsilon{}^r v_t (\psi^r)^2 G(x, -t) \, dx \, dt =$$

$$= i) + ii) + iii).$$

There holds,

$$i) = -\int_{-4}^{-1} \int_{\mathbb{R}^N} \nabla v_t \nabla v \, (\psi^r)^2 G(x, -t) \, dx \, dt - 2 \int_{-4}^{-1} \int_{\mathbb{R}^N} v_t \nabla v \, \psi^r \nabla \psi^r G(x, -t) \, dx \, dt -$$

$$- \int_{-4}^{-1} \int_{\mathbb{R}^N} v_t \nabla v \, (\psi^r)^2 \nabla G \, dx \, dt.$$

Since for $x \in \mathbb{R}^N$, $-4 \leq t \leq -1$, $|\nabla G(x, -t)| = |\frac{x}{2t} G(x, -t)| \leq C \, G(x, -t)^{1/2} G(x, -2t)^{1/2}$, $|G_t(x, -t)| \leq C \, G(x, -2t)$ and $|\nabla v(x, t)| = |\nabla u^\varepsilon(rx, r^2 t)| \leq A_2$ in the support of ψ^r, there holds

$$i) \leq \int_{\mathbb{R}^N} |\nabla v|^2 (\psi^r)^2 G(x, -t) \, dx \Big|_{t=-4}^{t=-1} + \int_{-4}^{-1} \int_{\mathbb{R}^N} |\nabla v|^2 (\psi^r)^2 |G_t| \, dx \, dt$$

$$+ 2 \left(\int_{-4}^{-1} \int_{\mathbb{R}^N} v_t^2 (\psi^r)^2 G(x, -t) \, dx \, dt \right)^{1/2} \left(\int_{-4}^{-1} \int_{\mathbb{R}^N} |\nabla v|^2 |\nabla \psi^r|^2 G(x, -t) \, dx \, dt \right)^{1/2} +$$

$$+ C \left(\int_{-4}^{-1} \int_{\mathbb{R}^N} v_t^2 (\psi^r)^2 G(x, -t) \, dx \, dt \right)^{1/2} \left(\int_{-4}^{-1} \int_{\mathbb{R}^N} |\nabla v|^2 (\psi^r)^2 G(x, -2t) \, dx \, dt \right)^{1/2} \leq$$

$$\leq \eta \int_{-4}^{-1} \int_{\mathbb{R}^N} v_t^2 (\psi^r)^2 G(x, -t) \, dx \, dt + C_\eta(A_2, A_4)$$

for $r \leq 1$.

Proceeding in a similar way,

$$iii) \leq \eta \int_{-4}^{-1} \int_{\mathbb{R}^N} v_t^2 (\psi^r)^2 G(x, -t) \, dx \, dt + C_\eta \|f_\varepsilon\|_{L^\infty(B_R(0) \times (-4,0))}$$

for $r \leq 1$.

Using that $\beta_{\varepsilon/r}(v) \, v_t = \partial_t B_{\varepsilon/r}(v)$ and $0 \leq B_{\varepsilon/r}(s) \leq M$ we get for ii)

$$ii) \leq C(M).$$

Thus,

$$(2.8) \qquad \left| \int_{-4}^{-1} \int_{\mathbb{R}^N} (\partial_t u_r^\varepsilon)^2 (\psi^r)^2 G(x, -t) \, dx \, dt \right| \leq C(A_2, A_4, M) + C\|f_\varepsilon\|_{L^\infty(B_R(0) \times (-4,0))}.$$

Summing up,

$$I \geq -C_2 \|f_\varepsilon\|_{L^\infty(B_R(0) \times (-4,0))} \frac{|u^\varepsilon(0,0)|}{r} - C_1 \left(\|f_\varepsilon\|_{L^\infty(B_R(0) \times (-4,0))} + \|f_\varepsilon\|_{L^\infty(B_R(0) \times (-4,0))}^{3/2} \right).$$

Since $\psi \equiv 1$ in $B_{R/2}(0)$,

$$II \geq -C(A_1, A_2, A_4) \Big\{ \frac{1}{r} \|G\|_{L^1((|x| > R/2r) \times (1,4))} + \||x| G\|_{L^1((|x| > R/2r) \times (1,4))}$$

$$+ \frac{1}{r^{\frac{N}{2}}} \|u_t^\varepsilon\|_{L^2(B_R(0) \times (-4,0))} \|G\|_{L^2((|x| > R/2r) \times (1,4))} \Big\} \geq -\bar{C}_3 C(R) e^{-\bar{C}_4 \frac{R^2}{r^2}}$$

where $C(R) = \max\{R^{-1}, R^{-N/2}\}$.

Since $0 \leq s\, \beta_\varepsilon(s) \leq M_1$, $0 \leq B_\varepsilon(s) \leq M$ and $\psi \equiv 1$ in $B_{R/2}(0)$,

$$II + III \geq -C_3 C(R) e^{-C_4 \frac{R^2}{r^2}}.$$

The theorem is proved. $\square$

By passing to the limit as ε goes to 0 we get a monotonicity formula for limits u of solutions u^ε to $P_\varepsilon(f_\varepsilon)$. Moreover, we prove that —when $u(0,0) = 0$— the functional $\mathcal{W}(r)$ has a limit as $r \to 0^+$.

Theorem 2.2 (Monotonicity Formula). *Let u^ε, f_ε and ψ as in Theorem 2.1. Let $u = \lim_{\varepsilon \to 0} u^\varepsilon$ uniformly on compact subsets of $\mathcal{D}$, $\chi = \lim_{\varepsilon \to 0} B_\varepsilon(u^\varepsilon)$ $*$-weakly in $L^\infty(\mathcal{D})$. Let $A \geq \|f_\varepsilon\|_{L^\infty(B_R(0) \times (-4,0))}$ and, for $0 < r \leq 1$,*

$$(2.9) \qquad \begin{aligned} \mathcal{W}(r) &= \mathcal{W}(r, u, \psi, \chi) \\ &= \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} \left(|\nabla(u\psi)|^2 + 2\psi^2 \chi + \frac{1}{2} \frac{(u\psi)^2}{t} \right) G(x, -t) \, dx \, dt. \end{aligned}$$

Then, for $1 \geq \rho_1 > \rho_2 > 0$

$$(2.10) \qquad \begin{aligned} \mathcal{W}(\rho_1) - \mathcal{W}(\rho_2) &\geq \int_{\rho_2}^{\rho_1} \int_{-4}^{-1} \int_{\mathbb{R}^N} \left(\partial_r w_r \right)^2 \frac{r G(x, -t)}{-t} \, dx \, dt \, dr - \\ &\quad - C_1 (A + A^{3/2}) (\rho_1 - \rho_2) - C_2 A |u(0,0)| \log\left(\frac{\rho_1}{\rho_2} \right) - C_3 C(R) \int_{\rho_2}^{\rho_1} e^{-C_4 \frac{R^2}{r^2}} \, dr, \end{aligned}$$

where C_i and $C(R)$ are as in Theorem 2.1 and $w(x,t) = \psi(x) u(x,t)$. In particular, if $u(0,0) = 0$, there exists

$$\delta := \lim_{r \to 0^+} \mathcal{W}(r) > -\infty.$$

Proof. Integration of (2.2) gives

$$\mathcal{W}^\varepsilon(\rho_1, u^\varepsilon, \psi) - \mathcal{W}^\varepsilon(\rho_2, u^\varepsilon, \psi) \geq \int_{\rho_2}^{\rho_1} \int_{-4}^{-1} \int_{\mathbb{R}^N} \left(\partial_r w_r^\varepsilon\right)^2 \frac{rG(x,-t)}{-t} \, dx \, dt \, dr -$$

$$- C_1 \left(A + A^{3/2}\right) (\rho_1 - \rho_2) - C_2 \, A \, |u^\varepsilon(0,0)| \log\left(\frac{\rho_1}{\rho_2}\right) - C_3 C(R) \int_{\rho_2}^{\rho_1} e^{-C_4 \frac{R^2}{r^2}} \, dr$$

and the result follows by passing to the limit. In fact,

$$\mathcal{W}^\varepsilon(r, u^\varepsilon, \psi) \to \mathcal{W}(r, u, \psi, \chi)$$

since $u^\varepsilon \to u$ and $\nabla u^\varepsilon \to \nabla u$ almost everywhere and $|u^\varepsilon| \leq A_1$, $|\nabla u^\varepsilon| \leq A_2$ on the support of ψ. We also use that $B_\varepsilon(u^\varepsilon) \to \chi$ $*$-weakly in L^∞_{loc}.

On the other hand, for every $0 < \rho_2 < r < \rho_1 \leq 1$,

$$\liminf_{\varepsilon \to 0} \int_{-4}^{-1} \int_{\mathbb{R}^N} \left(\partial_r w_r^\varepsilon\right)^2 \frac{rG(x,-t)}{-t} \, dx \, dt \geq \int_{-4}^{-1} \int_{\mathbb{R}^N} \left(\partial_r w_r\right)^2 \frac{rG(x,-t)}{-t} \, dx \, dt.$$

In fact,

$$\partial_r w_r^\varepsilon = -\frac{u^\varepsilon(rx, r^2 t)\, \psi(rx)}{r^2} + \frac{\psi(rx)\, \nabla u^\varepsilon(rx, r^2 t) \cdot x}{r} +$$

$$+ \frac{u^\varepsilon(rx, r^2 t)\, \nabla \psi(rx) \cdot x}{r} + 2t\psi(rx)\, \partial_t u^\varepsilon(rx, r^2 t).$$

We use the convergence and boundedness of u^ε and ∇u^ε stated above and the weak convergence of $\partial_t u^\varepsilon$ to $\partial_t u$ in L^2 on compact sets together with estimate (2.8) that implies that

$$\psi(rx)\, \partial_t u^\varepsilon(rx, r^2 t) \rightharpoonup \psi(rx)\, \partial_t u(rx, r^2 t)$$

weakly in $L^2\left(\mathbb{R}^N \times (-4, -1), (-t)G(x,-t)\, dx\, dt\right)$. Also, (2.8) gives uniform bounds that allow to pass to the limit in the triple integral to get

$$\liminf_{\varepsilon \to 0} \int_{\rho_2}^{\rho_1} \int_{-4}^{-1} \int_{\mathbb{R}^N} \left(\partial_r w_r^\varepsilon\right)^2 \frac{rG(x,-t)}{-t} \, dx \, dt \, dr \geq$$

$$\geq \int_{\rho_2}^{\rho_1} \int_{-4}^{-1} \int_{\mathbb{R}^N} \left(\partial_r w_r\right)^2 \frac{rG(x,-t)}{-t} \, dx \, dt \, dr$$

and this proves the theorem. $\qquad\square$

Now, we prove that any blow up limit of a function $u = \lim u^\varepsilon$ at a point where $u(0,0) = 0$ verifies that the global functional of Weiss (cf. [18]) is constant with a constant independent of the blow up sequence and we prove that the blow up limit is homogeneous.

Corollary 2.1. *Let u and χ as in Theorem 2.2. Assume $(0,0) \in \mathcal{D} \cap \partial\{u > 0\}$. Let $u_\lambda(x,t) = \frac{1}{\lambda} u(\lambda x, \lambda^2 t)$, $\chi^\lambda(x,t) = \chi(\lambda x, \lambda^2 t)$. Let $u_0 = \lim_{j \to \infty} u_{\lambda_j}$ uniformly on compact sets of $\mathbb{R}^N \times (-\infty, 0]$, $\chi_0 = \lim_{j \to \infty} \chi^{\lambda_j}$ $*$-weakly in L^∞ on compact sets of $\mathbb{R}^N \times (-\infty, 0]$, with $\lambda_j \to 0^+$.*
Then, for $r > 0$

$$\mathcal{W}_0(r) = \mathcal{W}(r, u_0, 1, \chi_0)$$

(2.11)
$$= \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} \left(|\nabla u_0|^2 + 2\chi_0 + \frac{1}{2}\frac{u_0^2}{t}\right) G(x,-t) \, dx \, dt \equiv \delta$$

where δ is the constant in Theorem 2.2. In particular, δ is independent of the sequence λ_j and of the cut off function ψ. Moreover, for $\rho_1 \geq \rho_2 > 0$

$$(2.12) \qquad 0 = \mathcal{W}_0(\rho_1) - \mathcal{W}_0(\rho_2) \geq \int_{\rho_2}^{\rho_1} \int_{-4}^{-1} \int_{\mathbb{R}^N} \left(\partial_r(u_0)_r\right)^2 \frac{rG(x,-t)}{-t} \, dx \, dt \, dr.$$

Therefore, u_0 is homogeneous in $\mathbb{R}^N \times (-\infty, 0)$. This is, for every $x \in \mathbb{R}^N$, $t < 0$,

$$u_0(rx, r^2 t) = r u_0(x, t) \quad for \quad r > 0.$$

Proof. Let $r > 0$. By Theorem 2.2 if $\psi \in C_0^\infty(B_R(0))$, $\psi \equiv 1$ in $B_{R/2}(0)$, $0 \leq \psi \leq 1$,

$$\mathcal{W}(\lambda\, r, u, \psi, \chi) \to \delta \quad \text{as} \quad \lambda \to 0.$$

On the other hand,

$$\mathcal{W}(\lambda_j r, u, \psi, \chi) = \int_{-4}^{-1} \int_{\mathbb{R}^N} \left(|\nabla(u\psi)_{\lambda_j r}|^2 + 2(\psi^2\chi)^{\lambda_j r} + \frac{1}{2} \frac{(u\psi)^2_{\lambda_j r}}{t}\right) G(x,-t) \, dx \, dt =$$

$$= \int_{-4}^{-1} \int_{\mathbb{R}^N} \left(|\nabla(u_{\lambda_j}\psi^{\lambda_j})_r|^2 + 2((\psi^{\lambda_j})^2\chi^{\lambda_j})^r + \frac{1}{2} \frac{(u_{\lambda_j}\psi^{\lambda_j})^2_r}{t}\right) G(x,-t) \, dx \, dt =$$

$$= \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} \left(|\nabla(u_{\lambda_j}\psi^{\lambda_j})|^2 + 2(\psi^{\lambda_j})^2\chi^{\lambda_j} + \frac{1}{2} \frac{(u_{\lambda_j}\psi^{\lambda_j})^2}{t}\right) G(x,-t) \, dx \, dt =$$

$$= \mathcal{W}(r, u_{\lambda_j}, \psi^{\lambda_j}, \chi^{\lambda_j}).$$

Now, there holds that $u_{\lambda_j} \to u_0$ as $j \to \infty$ uniformly on compact sets of $\mathbb{R}^N \times (-\infty, 0]$, $\nabla u_{\lambda_j} \to \nabla u_0$ almost everywhere in $\mathbb{R}^N \times (-\infty, 0]$ and $|u_{\lambda_j}| \leq A_1$, $|\nabla u_{\lambda_j}| \leq A_2$ on the support of ψ^{λ_j} where A_i are defined in Theorem 2.1.

On the other hand, $\psi^{\lambda_j} \to 1$ almost everywhere, $\nabla\psi^{\lambda_j} \to 0$ almost everywhere with $|\nabla\psi^{\lambda_j}| \leq A_4$ for $\lambda_j \leq 1$. Therefore,

$$\frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} \left(|\nabla(u_{\lambda_j}\psi^{\lambda_j})|^2 + \frac{1}{2} \frac{(u_{\lambda_j}\psi^{\lambda_j})^2}{t}\right) G(x,-t) \, dx \, dt \to$$

$$\to \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} \left(|\nabla u_0|^2 + \frac{1}{2} \frac{u_0^2}{t}\right) G(x,-t) \, dx \, dt \qquad \text{as} \quad j \to \infty.$$

Moreover, $\chi^{\lambda_j} \to \chi_0$ $*$-weakly in L^∞ on compact sets of $\mathbb{R}^N \times (-\infty, 0]$. Thus,

$$(2.13) \qquad \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} (\psi^{\lambda_j})^2 \chi^{\lambda_j} G(x,-t) \, dx \, dt \quad \to \quad \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} \chi_0 G(x,-t) \, dx \, dt.$$

In fact, let $\varepsilon > 0$ and let $K > 0$ be such that $\dfrac{1}{r^2}\displaystyle\int_{-4r^2}^{-r^2}\int_{|x|>K} M\,G(x,-t)\,dx\,dt < \dfrac{\varepsilon}{4}$. Then,

$$\left| \frac{1}{r^2}\int_{-4r^2}^{-r^2}\int_{\mathbb{R}^N} \left\{ (\psi^{\lambda_j})^2\chi^{\lambda_j} - \chi_0 \right\} G(x,-t)\,dx\,dt \right| \le$$

$$\le \left| \frac{1}{r^2}\int_{-4r^2}^{-r^2}\int_{|x|<K} \left\{ (\psi^{\lambda_j})^2\chi^{\lambda_j} - \chi_0 \right\} G(x,-t)\,dx\,dt \right| + \frac{\varepsilon}{2} =$$

$$= \left| \frac{1}{r^2}\int_{-4r^2}^{-r^2}\int_{|x|<K} \left\{ \chi^{\lambda_j} - \chi_0 \right\} G(x,-t)\,dx\,dt \right| + \frac{\varepsilon}{2} < \varepsilon$$

if j is sufficiently large. Thus, (2.13) holds. Therefore, for every $r > 0$,

$$\mathcal{W}(r,u_0,1,\chi_0) = \lim_{j\to\infty}\mathcal{W}(r,u_{\lambda_j},\psi^{\lambda_j},\chi^{\lambda_j}) = \lim_{j\to\infty}\mathcal{W}(\lambda_j r,u,\psi,\chi) \equiv \delta.$$

On the other hand, if $0 < \rho_2 \le \rho_1$, by Theorem 2.2,

$$0 = \mathcal{W}(\rho_1,u_0,1,\chi_0) - \mathcal{W}(\rho_2,u_0,1,\chi_0) = \lim_{j\to\infty}\left\{ \mathcal{W}(\lambda_j\rho_1,u,\psi,\chi) - \mathcal{W}(\lambda_j\rho_2,u,\psi,\chi) \right\} \ge$$

$$\ge \liminf_{j\to\infty}\left[\int_{\rho_2}^{\rho_1}\int_{-4}^{-1}\int_{\mathbb{R}^N} \left(\partial_r(w_{\lambda_j})_r \right)^2 \frac{rG(x,-t)}{-t}\,dx\,dt\,dr - \right.$$

$$\left. - C_1\left(A + A^{3/2}\right)\lambda_j(\rho_1 - \rho_2) - C_3 C(R)\lambda_j \int_{\rho_2}^{\rho_1} e^{-C_4\frac{R^2}{\lambda_j^2 r^2}}\,dr \right] =$$

$$= \liminf_{j\to\infty}\int_{\rho_2}^{\rho_1}\frac{1}{r^4}\int_{-4r^2}^{-r^2}\int_{\mathbb{R}^N} \left(-u_{\lambda_j}\psi^{\lambda_j} + \psi^{\lambda_j}\nabla u_{\lambda_j}\cdot x + u_{\lambda_j}\nabla\psi^{\lambda_j}\cdot x + \right.$$

$$\left. + 2t\psi^{\lambda_j}\partial_t u_{\lambda_j} \right)^2 \frac{rG(x,-t)}{-t}\,dx\,dt\,dr =$$

$$= \liminf_{j\to\infty}\left\{ \int_{\rho_2}^{\rho_1}\frac{1}{r^4}\int_{-4r^2}^{-r^2}\int_{\mathbb{R}^N} (-u_{\lambda_j}\psi^{\lambda_j} + \psi^{\lambda_j}\nabla u_{\lambda_j}\cdot x + u_{\lambda_j}\nabla\psi^{\lambda_j}\cdot x)^2 \frac{rG(x,-t)}{-t}\,dx\,dt\,dr + \right.$$

$$+ \int_{\rho_2}^{\rho_1}\frac{1}{r^4}\int_{-4r^2}^{-r^2}\int_{\mathbb{R}^N} (2t\psi^{\lambda_j}\partial_t u_{\lambda_j})^2 \frac{rG(x,-t)}{-t}\,dx\,dt\,dr +$$

$$+ \int_{\rho_2}^{\rho_1}\frac{2}{r^4}\int_{-4r^2}^{-r^2}\int_{\mathbb{R}^N} (-u_{\lambda_j}\psi^{\lambda_j} + \psi^{\lambda_j}\nabla u_{\lambda_j}\cdot x + u_{\lambda_j}\nabla\psi^{\lambda_j}\cdot x)$$

$$\left. (2t\psi^{\lambda_j}\partial_t u_{\lambda_j})\frac{rG(x,-t)}{-t}\,dx\,dt\,dr \right\} =$$

$$= \liminf_{j\to\infty}\left\{ \int_{\rho_2}^{\rho_1} I\,dr + \int_{\rho_2}^{\rho_1} II\,dr + \int_{\rho_2}^{\rho_1} III\,dr \right\}.$$

It is easy to see that for every $0 < \rho_2 < r < \rho_1$ and j sufficiently large

$$|I| = \left| \frac{1}{r^4}\int_{-4r^2}^{-r^2}\int_{\mathbb{R}^N} (-u_{\lambda_j}\psi^{\lambda_j} + \psi^{\lambda_j}\nabla u_{\lambda_j}\cdot x + u_{\lambda_j}\nabla\psi^{\lambda_j}\cdot x)^2 \frac{rG(x,-t)}{-t}\,dx\,dt \right| \le C$$

and moreover,

(2.14)
$$I \to \frac{1}{r^4}\int_{-4r^2}^{-r^2}\int_{\mathbb{R}^N} (-u_0 + \nabla u_0\cdot x)^2 \frac{rG(x,-t)}{-t}\,dx\,dt.$$

On the other hand, by (2.8) we get for $0 < \rho_2 < r < \rho_1$ and $\varepsilon < \varepsilon_0(\lambda)$,

$$\left| \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} \left(\partial_t u_\lambda^\varepsilon \right)^2 (\psi^\lambda)^2 G(x, -t) \, dx \, dt \right| \leq C(A_2, A_4, M, A, \rho_1, \rho_2).$$

Thus, since $\partial_t u_{\lambda_j}^\varepsilon \rightharpoonup \partial_t u_{\lambda_j}$ as $\varepsilon \to 0$ weakly in L^2 on compact sets,

$$\left| \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} \left(\partial_t u_{\lambda_j} \right)^2 (\psi^{\lambda_j})^2 G(x, -t) \, dx \, dt \right| \leq C(A_2, A_4, M, A, \rho_1, \rho_2).$$

Therefore,

$$(2.15) \qquad\qquad\qquad |II| \leq \tilde{C}(A_2, A_4, M, A, \rho_1, \rho_2).$$

Thus, for a subsequence that we still call λ_j,

$$(2.16) \qquad\qquad \psi^{\lambda_j} \, \partial_t u_{\lambda_j} = \partial_t (\psi^{\lambda_j} \, u_{\lambda_j}) \rightharpoonup \partial_t u_0 \quad \text{as} \quad j \to \infty$$

weakly in $L^2 \big(\mathbb{R}^N \times (-4r^2, -r^2); (-t) G(x, -t) \, dx \, dt \big)$.

In fact, the estimate (2.15) implies that $\psi^{\lambda_j} \, \partial_t u_{\lambda_j}$ converges weakly to a function v_0 in $L^2 \big(\mathbb{R}^N \times (-4r^2, -r^2); (-t) G(x, -t) \, dx \, dt \big)$. Since $\partial_t u_{\lambda_j}$ converges to $\partial_t u_0$ weakly in $L^2_{loc}(\mathbb{R}^N \times (-\infty, 0))$, and $\psi^{\lambda_j} \equiv 1$ in $|x| < \frac{1}{2\lambda_j}$, we get (2.16).

Now, (2.15) and (2.16) imply that

$$(2.17) \qquad \liminf_{j \to \infty} \frac{1}{r^4} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} \left(2t \psi^{\lambda_j} \partial_t u_{\lambda_j} \right)^2 \frac{r G(x, -t)}{-t} \, dx \, dt \geq$$
$$\geq \frac{1}{r^4} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} \left(2t \partial_t u_0 \right)^2 \frac{r G(x, -t)}{-t} \, dx \, dt,$$

and

$$(2.18) \qquad III \to \frac{2}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} \left(-u_0 + \nabla u_0 \cdot x \right) \left(2t \, \partial_t u_0 \right) \frac{r G(x, -t)}{-t} \, dx \, dt.$$

Moreover,

$$|III| \leq C$$

with C independent of λ and r, for λ small enough and $0 < \rho_2 < r < \rho_1$.

Finally, the uniform estimates and the convergence (2.14), (2.17) and (2.18) imply that

$$0 = \mathcal{W}(\rho_1, u_0, 1, \chi_0) - \mathcal{W}(\rho_2, u_0, 1, \chi_0) = \lim_{j \to \infty} \left\{ \mathcal{W}(\lambda_j \rho_1, u, \psi, \chi) - \mathcal{W}(\lambda_j \rho_2, u, \psi, \chi) \right\} \geq$$
$$\geq \int_{\rho_2}^{\rho_1} \int_{-4}^{-1} \int_{\mathbb{R}^N} \left(\partial_r (u_0)_r \right)^2 \frac{r G(x, -t)}{-t} \, dx \, dt \, dr \geq 0.$$

Thus, for every $r > 0$, $-4 \leq t \leq -1$,

$$u_0(rx, r^2 t) = r u_0(x, t).$$

In particular, for $t = -1$,

$$u_0(rx, -r^2) = r u_0(x, -1) = r H(x).$$

Therefore, for any $t < 0$, $y \in \mathbb{R}^N$,

$$u_0(y, t) = u_0(rx, -r^2) = \sqrt{-t} \, H\left(\frac{y}{\sqrt{-t}} \right),$$

and

$$u_0(ry, r^2t) = ru_0(y, t)$$

for every $r > 0$, $y \in \mathbb{R}^N$, $t < 0$.

The corollary is proved. □

Proposition 2.1. *Let u, χ, ψ as in Theorem 2.2. Assume $(0,0) \in \mathcal{D} \cap \partial\{u > 0\}$ and let δ as in Theorem 2.2. Then,*

$$\delta = \lim_{r \to 0} \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} 2\psi^2 \chi G(x, -t) \, dx \, dt.$$

Proof. In Theorem 2.2 we proved that

$$\delta = \lim_{r \to 0} \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} \left\{ |\nabla(u\psi)|^2 + 2\psi^2 \chi + \frac{(u\psi)^2}{2t} \right\} G(x, -t) \, dx \, dt.$$

We claim that

$$L := \lim_{r \to 0} \int_{-4r^2}^{-r^2} \frac{1}{r^2} \int_{\mathbb{R}^N} \left\{ |\nabla(u\psi)|^2 + \frac{(u\psi)^2}{2t} \right\} G(x, -t) \, dx \, dt = 0.$$

In fact, let $\lambda_j \to 0$ be a sequence such that u_{λ_j} has a limit u_0. Then, for $r > 0$ fixed

$$L = \lim_{j \to \infty} \int_{-4\lambda_j^2 r^2}^{-\lambda_j r^2} \frac{1}{\lambda_j^2 r^2} \int_{\mathbb{R}^N} \left\{ |\nabla(u\psi)|^2 + \frac{(u\psi)^2}{2t} \right\} G(x, -t) \, dx \, dt =$$

$$= \lim_{j \to \infty} \int_{-4r^2}^{-r^2} \frac{1}{r^2} \int_{\mathbb{R}^N} \left\{ |\nabla(u_{\lambda_j} \psi^{\lambda_j})^2| + \frac{(u_{\lambda_j} \psi^{\lambda_j})^2}{2t} \right\} G(x, -t) \, dx \, dt =$$

$$= \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} \left(|\nabla u_0|^2 + \frac{1}{2} \frac{u_0^2}{t} \right) G(x, -t) \, dx \, dt$$

as proved in Corollary 2.1. Let us see that this last integral is zero. This is a consequence of the fact that u_0 is homogeneous and a solution to the homogeneous heat equation in the set $\{u_0 > 0\} \cup \{u \le 0\}^\circ$. In fact, let $H = H(y)$ be such that $u_0(x, t) = \sqrt{-t} H\left(\frac{x}{\sqrt{-t}}\right)$. Then,

$$\Delta H(y) = \frac{\partial}{\partial t} u_0(y, -1) = -\frac{1}{2} H(y) + \frac{1}{2} y \cdot \nabla H(y).$$

Thus,

$$\frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} \left(|\nabla u_0|^2 + \frac{1}{2} \frac{u_0^2}{t} \right) G(x, -t) \, dx \, dt =$$

$$= \frac{3}{(2\sqrt{\pi})^N} \int_{\mathbb{R}^N} \left\{ |\nabla H(y)|^2 - \frac{1}{2} H^2(y) \right\} e^{-\frac{1}{4}|y|^2} \, dy =$$

$$= \frac{3}{(2\sqrt{\pi})^N} \left\{ -\int_{\mathbb{R}^N} H(y) \operatorname{div} \left(e^{-\frac{1}{4}|y|^2} \nabla H(y) \right) dy - \int_{\mathbb{R}^N} \frac{1}{2} H^2(y) e^{-\frac{1}{4}|y|^2} \, dy \right\} = 0.$$

□

Corollary 2.2. *Let u, χ, u_0 and χ_0 as in Corollary 2.1. Assume $(0,0) \in \mathcal{D} \cap \partial\{u > 0\}$ and let δ as in Theorem 2.2. Then, for any $r > 0$,*

$$\delta \equiv \mathcal{W}(r, u_0, 1, \chi_0)$$

(2.19)
$$= \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} \left(|\nabla u_0|^2 + 2\chi_0 + \frac{1}{2} \frac{u_0^2}{t} \right) G(x, -t) \, dx \, dt$$

$$= \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} 2\chi_0 \, G(x, -t) \, dx \, dt.$$

In particular $0 \leq \delta \leq 6M$.

Proof. The result follows from Corollary 2.1, the proof of Proposition 2.1 and the fact that $0 \leq \chi_0 \leq M$. $\qquad\square$

Remark 2.2. Let $\mathcal{D} \subset \mathbb{R}^{N+1}$ be a domain. Let u^ε be a family of uniformly bounded — in $Lip(1, 1/2)$ norm— solutions to $P_\varepsilon(f_\varepsilon)$ in $\mathcal{D}$ with f_ε uniformly bounded in $L^\infty(\mathcal{D})$. Let $u = \lim u^\varepsilon$ and $\chi = \lim B_\varepsilon(u^\varepsilon)$ in $\mathcal{D}$. Let $(x_0, t_0) \in \mathcal{D} \cap \partial\{u > 0\}$. Then, if λ_0 is small enough, all the results of this section can be applied to the functions

$$u^\sigma(x, t) = \frac{1}{\lambda_0} u^\varepsilon(x_0 + \lambda_0 x, t_0 + \lambda_0^2 t),$$

$$\tilde{u} = \lim u^\sigma,$$

$$\tilde{\chi} = \lim B_\sigma(u^\sigma)$$

where $\sigma = \varepsilon/\lambda_0$.

In fact, u^σ are solutions to $P_\sigma(f_\sigma)$ with $f_\sigma(x, t) = \lambda_0 f_\varepsilon(x_0 + \lambda_0 x, t_0 + \lambda_0^2 t)$. Also, $(0,0) \in \tilde{\mathcal{D}} \cap \partial\{\tilde{u} > 0\}$ and $B_1(0) \times [-4, 0] \subset \tilde{\mathcal{D}}$ where $\tilde{\mathcal{D}} = \frac{1}{\lambda_0}(\mathcal{D} - (x_0, t_0))$.

Remark 2.3. The value of $\delta(x_0, t_0)$ is closely related to the regularity of the boundary of $\{u > 0\}$ in a neighborhood of $(x_0, t_0) \in \partial\{u > 0\}$.

Indeed, let u^ε and f_ε as in Remark 2.2. Let $u = \lim u^\varepsilon$, $\chi = \lim B_\varepsilon(u^\varepsilon)$, $(x_0, t_0) \in \partial\{u > 0\}$ and $\chi_0(x, t) = \lim \chi(x_0 + \lambda_n x, t_0 + \lambda_n^2 t)$, with $\lambda_n \to 0$. Assume that $\partial\{u > 0\}$ is smooth in a neighborhood of (x_0, t_0) with a nonhorizontal normal at (x_0, t_0), and that u^+ is nondegenerate at (x_0, t_0) (i.e, u^+ grows linearly from $\partial\{u > 0\}$). Let ψ be such that $\psi = \psi(x) \in C_0^\infty$, $0 \leq \psi \leq 1$, $\psi \equiv 1$ in a neighborhood of x_0. Then

$$\delta(x_0, t_0) = \lim_{r \to 0} \frac{1}{r^2} \int_{t_0 - 4r^2}^{t_0 - r^2} \int_{\mathbb{R}^N} \left(|\nabla(u\psi)|^2 + 2\psi^2 \chi + \frac{1}{2} \frac{(u\psi)^2}{t - t_0} \right) G(x - x_0, t_0 - t) \, dx \, dt =$$

$$= \int_{-4}^{-1} \int_{\mathbb{R}^N} 2\chi_0 \, G(x, -t) \, dx \, dt = 3M.$$

In fact, let $u_\lambda(x, t) = \frac{1}{\lambda} u(x_0 + \lambda x, t_0 + \lambda^2 t)$. Then, in some system of coordinates, $u_\lambda(x, t) \to u_0(x, t) = \alpha x_1^+ - \gamma x_1^-$ with $\alpha > 0$ and $\gamma \geq 0$.

Since for some sequence $\delta_n = \frac{\varepsilon_{j_n}}{\lambda_n} \to 0$ there holds that $\chi_0 = \lim B_{\delta_n}(u^{\delta_n})$ with $u^{\delta_n} \to u_0$ solutions to $P_{\delta_n}(f_{\delta_n})$ uniformly bounded in $Lip(1, 1/2)$, with $f_{\delta_n} \to 0$ (see the proof of Theorem 3.1) and $\alpha > 0$, there holds that $\chi_0 = M$ in $\{x_1 > 0\}$.

If $\gamma > 0$, then $\chi_0 = 0$ in $\{x_1 < 0\}$. Let us see that this is also true if $\gamma = 0$.

In fact, as in the proof of Proposition 5.1 in [13] we deduce that $\chi_0(x,t) = \chi_0(t) \in \{0, M\}$ in $\{x_1 < 0\}$ and $0 < \frac{1}{2}\alpha^2 = M - \chi_0(t)$ (see also Proposition 5.2 in [9]). Thus, $\chi_0(t) \equiv 0$ in $\{x_1 < 0\}$.

Replacing the value of χ_0 in the formula of $\delta(x_0, t_0)$ we conclude that

$$\delta(x_0, t_0) = 3M.$$

Moreover, we prove in Theorem 9.7 in [14] that the regularity of the boundary of $\{u > 0\}$ in a neighborhood of a boundary point is equivalent to δ being $3M$ at that point, for the stationary case (see Section 4).

3. APPLICATION I: ASYMPTOTIC DEVELOPMENTS

In this section we apply the results of Section 2 to prove a result on asymptotic developments (Theorem 3.1). This result concerns points of $\partial\{u > 0\}$ at which there is a one sided tangent ball. Note that these points are dense in $\partial\{u > 0\}$. On the other hand, the knowledge of this type of asymptotic developments at points with a one sided tangent ball have proved to be an essential tool for the proof of the regularity of $\partial\{u > 0\}$ in free boundary problems (see, for instance, [1], [4], [5]).

We point out that all the cases present in Theorem 3.1 appear in different situations, as the examples in [13] show.

We have

Theorem 3.1. *Let u^{ε_j} be a family of solutions to $P_{\varepsilon_j}(f_{\varepsilon_j})$ in $\mathcal{D}$, uniformly bounded in $Lip\,(1, 1/2)$ with f_{ε_j} uniformly bounded in $L^\infty(\mathcal{D})$ and $\varepsilon_j \to 0$. Let $u = \lim u^{\varepsilon_j}$ in $\mathcal{D}$. Let $(x_0, t_0) \in \partial\{u > 0\} \cap \mathcal{D}$. Assume $\partial\{u > 0\}$ has a nonhorizontal tangent ball B at the point (x_0, t_0). This is, there is a ball B such that $(x_0, t_0) \in \partial B$, the normal to B at (x_0, t_0) has a nonzero spatial component and*

$$B \cap \{t \leq t_0\} \subset \{u > 0\} \quad or \quad B \cap \{t \leq t_0\} \subset \{u \leq 0\}.$$

Let us call ν the interior unit spatial normal to B at (x_0, t_0) when $B \cap \{t \leq t_0\} \subset \{u > 0\}$ and the exterior unit spatial normal to B at (x_0, t_0) when $B \cap \{t \leq t_0\} \subset \{u \leq 0\}$. Then, there exist $\alpha \geq 0$ and $\sigma \in \mathbb{R}$ such that

$$(3.1) \qquad u(x, t) = \alpha\langle x - x_0, \nu\rangle^+ + \sigma\langle x - x_0, \nu\rangle^- + o(|x - x_0| + |t - t_0|^{1/2}) \quad in \quad \{t \leq t_0\}.$$

Moreover, when $B \cap \{t \leq t_0\} \subset \{u > 0\}$, there holds that

(1) If $\alpha > \sqrt{2M}$, then $\sigma < 0$ and $\alpha^2 - \sigma^2 = 2M$.
(2) If $\alpha = \sqrt{2M}$, then $\sigma = \sqrt{2M}$ or $\sigma = 0$.
(3) If $0 \leq \alpha < \sqrt{2M}$, then $\sigma = \alpha$.

When $B \cap \{t \leq t_0\} \subset \{u \leq 0\}$, the only possible cases are

(i) $\alpha \geq \sqrt{2M}$ and $\sigma \leq 0$ with $\alpha^2 - \sigma^2 = 2M$.
(ii) $\alpha = \sigma = 0$.

Proof. Without loss of generality we may assume that $(x_0, t_0) = (0, 0)$ and $\nu = e_1$.

CASE I: Assume first that $B \cap \{t \leq 0\} \subset \{u > 0\}$ and e_1 is the interior unit spatial normal to B at $(0, 0)$. By Lemma A.1 in [13] we know that there exists $\alpha \geq 0$ such that

$$u(x, t) = \alpha x_1^+ + o(|x| + |t|^{1/2}) \quad in \quad B \cap \{t \leq 0\}.$$

On the other hand, by Lemma A.2 in [13], there exists $\gamma \geq 0$ such that

$$u^-(x,t) = \gamma x_1^- + o(|x| + |t|^{1/2}) \quad \text{in} \quad \{t \leq 0\}.$$

<u>Step A</u> Assume $\gamma > 0$. Then, there holds that

$$u(x,t) = \alpha x_1^+ - \gamma x_1^- + o(|x| + |t|^{1/2}) \quad \text{in} \quad \{t \leq 0\}$$

and if $u_0 = \lim_{n \to \infty} u_{\lambda_n}$ with $\lambda_n \to 0$ there holds that $u_0 = \alpha x_1^+ - \gamma x_1^-$ in $\{t \leq 0\}$.

Now, we know that $u_0 = \lim_{n \to \infty} u_{\lambda_n}$ satisfies that $u_0 = \lim_{n \to \infty} u^{\delta_n}$ with $\delta_n := \varepsilon_{j_n}/\lambda_n \to 0$ for a certain choice of $j_n \to \infty$. Moreover, $u^{\delta_n} = (u^{\varepsilon_{j_n}})_{\lambda_n}$ and they are uniformly bounded in $Lip\,(1, 1/2)$ and solutions to $P_{\delta_n}(f_{\delta_n})$ with $f_{\delta_n} \to 0$. (This argument was used frequently in previous articles. See for instance [9], [13]).

Thus, since $\gamma > 0$, there holds that $\alpha^2 - \gamma^2 = 2M$. (This follows as in the proof of Proposition 5.1 in [9] with the necessary changes due to the presence of the forcing term f_{δ_n}, with arguments as those used in [13] in the one phase case and in [14] in the two phase stationary case).

<u>Step B</u> Assume now $\gamma = 0$. Then, there holds that $u_0 \geq 0$, $\Delta u_0 = u_{0t}$ in $\{u_0 > 0\}$, u_0 is $Lip\,(1, 1/2)$ and $u_0 = 0$ on $\{x_1 = 0\}$. Thus, (see Corollary A.1 in [9])

$$u_0(x,t) = \bar{\alpha} x_1^- + o(|x| + |t|^{1/2}) \quad \text{in} \quad \{x_1 < 0\} \cap \{t \leq 0\}$$

with $\bar{\alpha} \geq 0$.

Let us see that we actually have

(3.2) $$u_0(x,t) = \bar{\alpha} x_1^- \quad \text{in} \quad \{x_1 < 0\} \cap \{t < 0\}.$$

In fact, as a consequence of Corollary 2.1 we know that u_0 is homogeneous. This is, $u_0(x,t) = \frac{1}{\lambda} u_0(\lambda x, \lambda^2 t)$, for $\lambda > 0$. Then, for $x_1 < 0$, $t < 0$,

$$u_0(x,t) = \frac{1}{\lambda} u_0(\lambda x, \lambda^2 t) \to \bar{\alpha} x_1^- \quad \text{as} \quad \lambda \to 0.$$

So that, (3.2) holds.

Thus, $u_0(x,t) = \alpha x_1^+ + \bar{\alpha} x_1^-$, for $t \leq 0$, with $\alpha \geq 0$ independent of the sequence λ_n and $\bar{\alpha} \geq 0$.

B.1 Assume that $\bar{\alpha} > 0$. Let us see that $\bar{\alpha} = \alpha$ and that

$$u(x,t) = \alpha x_1^+ + \alpha x_1^- + o(|x| + |t|^{1/2}) \quad \text{for} \quad \{t \leq 0\}.$$

Let us first see that we may choose the sequence ε_{j_n} in such a way that we also have

(3.3) $$B_{\delta_n}(u^{\delta_n}) \rightharpoonup \chi_0 \quad * \text{–weakly in } L^\infty_{loc}(\mathbb{R}^{N+1})$$

where $\chi_0 = \lim \chi^{\lambda_n} * $ –weakly in $L^\infty_{loc}(\mathbb{R}^{N+1})$ and $\chi = \lim B_{\varepsilon_j}(u^{\varepsilon_j}) * $ –weakly in $L^\infty(\mathcal{D})$. Here, as above, $\delta_n = \dfrac{\varepsilon_{j_n}}{\lambda_n}$ and $u^{\delta_n} = (u^{\varepsilon_{j_n}})_{\lambda_n}$.

In fact, for each $n \in \mathbb{N}$, let ε_{j_n} be such that

$$\left\| B_{\varepsilon_{j_n}/\lambda_n}\big((u^{\varepsilon_{j_n}})_{\lambda_n}\big) - \chi^{\lambda_n} \right\|_{L^1(Q_{r_0/\lambda_n})} = \frac{1}{\lambda_n^{N+2}} \int_{Q_{r_0}} \left| B_{\varepsilon_{j_n}}(u^{\varepsilon_{j_n}}) - \chi \right| dx\,dt < \frac{1}{n}$$

(we denote $Q_r = \{(x,t)/|x| < r, |t| < r^2\}$). This can be done since the subsequence may be chosen in such a way that $B_{\varepsilon_{j_n}}(u^{\varepsilon_{j_n}}) \to \chi$ strongly in $L^1(Q_{r_0})$ for $r_0 > 0$ small. See [18], Proposition 4.1, where this result was proved in the one phase homogeneous case. For the two phase solutions of $P_{\varepsilon_j}(f_{\varepsilon_j})$ the result follows similarly under the present assumptions. We omit the proof here.

Let $\varphi \in L^1 \cap L^\infty$ with compact support. Then, if n is large enough so that supp $\varphi \subset Q_{r_0/\lambda_n}$,

$$\left| \int \left(B_{\varepsilon_{j_n}/\lambda_n} \big((u^{\varepsilon_{j_n}})_{\lambda_n} \big) - \chi_0 \right) \varphi \, dx \, dt \right| \leq$$

$$\leq \int \left| B_{\varepsilon_{j_n}/\lambda_n} \big((u^{\varepsilon_{j_n}})_{\lambda_n} \big) - \chi^{\lambda_n} \right| |\varphi| \, dx \, dt + \left| \int (\chi^{\lambda_n} - \chi_0) \varphi \, dx \, dt \right| \leq \frac{1}{n} \| \varphi \|_\infty + \eta$$

for any $\eta > 0$ if n is large enough.

Now, if $\varphi \in L^1$ with compact support, we also get

$$\int \left(B_{\varepsilon_{j_n}/\lambda_n} \big((u^{\varepsilon_{j_n}})_{\lambda_n} \big) - \chi_0 \right) \varphi \, dx \, dt \to 0$$

since $0 \leq B_{\delta_n}, \chi_0 \leq M$.

Since, $\bar{\alpha} > 0$ there holds that $u_0 > 0$ in $\{x_1 < 0\} \cap \{t \leq 0\}$. Thus, since $u_0 = \lim u^{\delta_n}$ and $\chi_0 = \lim B_{\delta_n}(u^{\delta_n})$, necessarily $\chi_0 = M$ in $\{x_1 < 0\} \cap \{t \leq 0\}$.

On the other hand, since $u > 0$ in $B \cap \{t \leq 0\}$ there holds that $\chi = M$ in $B \cap \{t \leq 0\}$. Thus, $\chi_0 = \lim \chi^{\lambda_n} = M$ in $\{x_1 > 0\} \cap \{t \leq 0\}$.

Thus, $B_{\delta_n}(u^{\delta_n}) \to M$ in $\{t \leq 0\}$. As in the proof of Proposition 5.3 in [9], using that the forcing term f_{δ_n} in the equation $P_{\delta_n}(f_{\delta_n})$ satisfied by $u^{\delta_n} = (u^{\varepsilon_{j_n}})_{\lambda_n}$ converges to 0, we deduce that $\bar{\alpha} = \alpha$.

Now, let $\tilde{u}_0 = \lim u_{\tilde{\lambda}_n}$ with $\tilde{\lambda}_n \to 0$. The argument above shows that $\tilde{u}_0 = \alpha x_1^+ + \tilde{\alpha} x_1^-$ in $\{t \leq 0\}$ with $\tilde{\alpha} \geq 0$. Let us see that necessarily $\tilde{\alpha} > 0$ and thus $\tilde{\alpha} = \alpha$.

In fact, by Corollary 2.1, we know that there exists δ independent of the sequences λ_n and $\tilde{\lambda}_n$ such that

$$\delta = \mathcal{W}(r, u_0, 1, \chi_0) = \mathcal{W}(r, \tilde{u}_0, 1, \tilde{\chi}_0) \quad \text{for every} \quad r > 0.$$

Now, Corollary 2.2 states that

$$\delta = \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} 2\chi_0 \, G(x, -t) \, dx \, dt = \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} 2\tilde{\chi}_0 \, G(x, -t) \, dx \, dt.$$

Since, for $\{t \leq 0\}$, $\chi_0 \equiv M$ and $0 \leq \tilde{\chi}_0 \leq M$, necessarily $\tilde{\chi}_0 \equiv M$ for $\{t \leq 0\}$.

As above, this implies that $\tilde{\alpha} = \alpha$.

Thus, if $\bar{\alpha} > 0$,

(3.4) $$u(x,t) = \alpha x_1^+ + \alpha x_1^- + o(|x| + |t|^{1/2}) \quad \text{for} \quad \{t \leq 0\}.$$

Observe that in this case since we showed that $\alpha = \bar{\alpha}$ there holds that $\alpha > 0$.

B.2 Assume $\bar{\alpha} = 0$. This is, $u_0(x,t) = \alpha x_1^+$ for $\{t \leq 0\}$. Let us see that

(3.5) $$u(x,t) = \alpha x_1^+ + o(|x| + |t|^{1/2}) \quad \text{for} \quad \{t \leq 0\}.$$

In fact, let $\tilde{u}_0 = \lim u_{\tilde{\lambda}_n}$ with $\tilde{\lambda}_n \to 0$. Then, $\tilde{u}_0 = \alpha x_1^+ + \tilde{\alpha} x_1^-$ in $\{t \leq 0\}$ with $\tilde{\alpha} \geq 0$.

If $\tilde{\alpha} > 0$, the argument in **B.1** with $\bar{\alpha}$ replaced by $\tilde{\alpha}$ shows that (3.4) holds with $\alpha > 0$. In particular, $u_0(x,t) = \alpha x_1^+ + \alpha x_1^-$ for $\{t \leq 0\}$ with $\alpha > 0$ which is a contradiction.

Thus, every blow up limit is αx_1^+, for $\{t \leq 0\}$ and (3.5) holds.

This ends Step B.

Step C Conclusion of the Case I. We have proved that there exist $\alpha \geq 0$ and $\sigma \in \mathbb{R}$ such that

$$u(x,t) = \alpha x_1^+ + \sigma x_1^- + o(|x| + |t|^{1/2}) \quad \text{in} \quad \{t \leq 0\}.$$

Let us characterize the values of α and σ.

In Case B.1 every blow up limit is $\alpha x_1^+ + \alpha x_1^-$, for $\{t \leq 0\}$. As in [13], Proposition 5.2 we deduce that $0 < \alpha \leq \sqrt{2M}$.

In Case B.2 every blow up limit is αx_1^+, for $\{t \leq 0\}$. As in [13], Proposition 5.1 we deduce that $\alpha = 0$ or $\alpha = \sqrt{2M}$.

Thus, if $\alpha > \sqrt{2M}$, necessarily $\sigma < 0$. And (1) in the statement holds.

If $0 \leq \alpha \leq \sqrt{2M}$ there holds that $\sigma \geq 0$ since if not, $\sigma < 0$ with $\alpha^2 - \sigma^2 = 2M$, so that $\alpha > \sqrt{2M}$.

If $\alpha = \sqrt{2M}$ both cases B.1 and B.2 are possible. Thus, $\sigma = 0$ or $\sigma = \alpha = \sqrt{2M}$. This proves (2) in the statement.

If $0 < \alpha < \sqrt{2M}$ the only possible case is B.1. Thus, $\sigma = \alpha$.

If $\alpha = 0$, the only possible case is B.2. This is, $\sigma = \alpha = 0$. This ends the proof of (3) in the statement.

CASE II: Assume now $B \cap \{t \leq 0\} \subset \{u \leq 0\}$ and e_1 is the exterior unit spatial normal to B at $(0,0)$.

Since $u^+ = 0$ in $B \cap \{t \leq 0\}$ we get by applying Lemma A.2 in [13] that there exists $\alpha \geq 0$ such that

$$u^+(x,t) = \alpha x_1^+ + o(|x| + |t|^{1/2}) \quad \text{in} \quad \{t \leq 0\}.$$

If $\alpha > 0$ we get by applying the same ideas as in [14], Theorem 7.2, that the asymptotic development (3.1) holds with $\sigma \leq 0$ and $\alpha^2 - \sigma^2 = 2M$.

If, on the other hand, $\alpha = 0$ we let $\lambda_n \to 0$ and $u_0 = \lim u_{\lambda_n}$. Then, $u_0 \leq 0$ in $\mathbb{R}^N \times (-\infty, 0]$, $\Delta u_0 - \partial_t u_0 \geq 0$ and $u_0(0,0) = 0$. Therefore, $u_0 \equiv 0$ in $\{t \leq 0\}$.

Then, the blow up limit u_0 is independent of the sequence λ_n, for $\{t \leq 0\}$. Therefore, the asymptotic development (3.1) holds with $\alpha = \sigma = 0$. $\qquad \square$

4. Application II: Regularity of the free boundary

In this section we present further applications of the results in Section 2. They deal with the regularity of the boundary of $\{u > 0\}$ in the stationary case including, in particular, regularity results for energy minimizers, and also with regularity results for travelling waves of a combustion model.

First, we consider a family u^{ε_j} of stationary solutions to $P_{\varepsilon_j}(f_{\varepsilon_j})$ such that u^{ε_j} and f_{ε_j} are uniformly bounded in L^∞ norm. In [14] we prove that u^{ε_j} are uniformly bounded in *Lip* norm. So that, the results of the present paper apply to this family. Let $u = \lim u^{\varepsilon_j}$ uniformly on compact subsets as $\varepsilon_j \to 0$. In [14] we prove that u is a solution to

$$\Delta u = f\chi_{\{u \not\equiv 0\}} \quad \text{in } \{u > 0\} \cup \{u \leq 0\}^\circ$$

where $f = \lim f_{\varepsilon_j}$ $*$-weakly in L^∞.

Moreover, in [14] we prove that, under suitable assumptions, $\partial\{u > 0\}$ is smooth and u is a classical solution to the following free boundary problem

$$(E(f)) \qquad \begin{aligned} \Delta u &= f\chi_{\{u \not\equiv 0\}} & &\text{in } \{u > 0\} \cup \{u \leq 0\}^\circ, \\ |\nabla u^+|^2 - |\nabla u^-|^2 &= 2M & &\text{on } \partial\{u > 0\}. \end{aligned}$$

The purpose of this section is to state some theorems on the regularity of the free boundary $\partial\{u > 0\}$ that are proved in [14] for which the results in this paper are an essential tool.

In fact, assume u is defined in $B_R(x_0)$ with $x_0 \in \partial\{u > 0\}$. Let $\chi = \lim B_{\varepsilon_j}(u^{\varepsilon_j})$ $*$-weakly in $L^\infty(B_R(x_0))$ and

$$(4.1) \qquad \delta(x_0) = \lim_{r \to 0^+} \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} \left(|\nabla(u\psi)|^2 + 2\psi^2\chi + \frac{1}{2}\frac{(u\psi)^2}{t} \right) G(x - x_0, -t)\, dx\, dt,$$

where $G(x, t) = \frac{1}{(4\pi t)^{N/2}} \exp(-\frac{|x|^2}{4t})$ and ψ is any function satisfying that $\psi \in C_0^\infty(B_R(x_0))$, $0 \le \psi \le 1$, $\psi \equiv 1$ in $B_{R/2}(x_0)$.

In the next three theorems, we assume that, in $B_R(x_0)$, u^+ is uniformly nondegenerate. This property holds in many applications (see, for instance, Theorems 4.4 and 4.5). By uniform nondegeneracy we mean that there exists $c > 0$ such that

$$(4.2) \qquad\qquad\qquad u^+(x) \ge c\operatorname{dist}(x, \{u \le 0\}).$$

As mentioned before, we have the following result,

Theorem 4.1 (Theorem 9.7 in [14])**.** *There holds that $\delta(x_0) = 3M$ if and only if the free boundary is $C^{1,\alpha}$ in a neighborhood of x_0. This implies that u is a classical solution to the free boundary problem $E(f)$ in a neighborhood of x_0.*

In the proof of the next two theorems we use the result on asymptotic developments (Theorem 3.1). In case $N = 2$ we use that blow up limits are homogeneous as proved in Corollary 2.1

Theorem 4.2 (Theorem 9.5 in [14])**.** *If x_0 is a regular point from the right and $\delta(x_0) < 6M$ then, the free boundary is a $C^{1,\alpha}$ surface in a neighborhood of x_0. Moreover, u is a classical solution to the free boundary problem $E(f)$ in a neighborhood of x_0.*
If $N = 2$ the same result holds without assuming that x_0 is a regular point from the right.

Theorem 4.3 (Theorem 9.6 in [14])**.** *If x_0 is a regular point from the right and, in addition, $\limsup_{r \to 0} \frac{|B_r(x_0) \cap \{u \le 0\}|}{|B_r(x_0)|} > 0$ then, the free boundary is a $C^{1,\alpha}$ surface in a neighborhood of x_0. Moreover, u is a classical solution to the free boundary problem $E(f)$ in a neighborhood of x_0.*
If $N = 2$ the same result holds without assuming that x_0 is a regular point from the right.

We say that a free boundary point x_0 is regular from the right if there is a ball contained in $\{u > 0\}$ that is tangent to the free boundary at x_0. Note that regular points from the right are dense in the free boundary.

We recall that the examples in [13] show that the free boundary condition may not hold at any free boundary point. In fact, u^+ may degenerate or the density of $\{u \le 0\}$ may be zero at a boundary point. These situations may appear even at points that are regular from the right. Thus, some extra assumption is needed if one wants to show that u is a solution to $E(f)$.

The results in the present paper are also used in [14] to obtain the following regularity results for energy minimizers and for travelling waves of a combustion model.

The first of these results is

Theorem 4.4 (Theorem 10.2 in [14])**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let $\phi_\varepsilon \in H^1(\Omega)$ be such that $\|\phi_\varepsilon\|_{H^1(\Omega)} \le \mathcal{A}_1$. Let $f^\varepsilon \in L^\infty(\Omega)$ such that $\|f^\varepsilon\|_{L^\infty(\Omega)} \le \mathcal{A}_2$. Let $u^\varepsilon \in H^1(\Omega)$ be a minimizer of the energy*

$$J_\varepsilon(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 + B_\varepsilon(v) + f^\varepsilon v$$

among functions $v \in H^1(\Omega)$ such that $v = \phi_\varepsilon$ on $\partial\Omega$. Here $B_\varepsilon(s) = \int_0^s \beta_\varepsilon(\tau)\, d\tau$.

Then, the functions u^ε are stationary solutions to $P_\varepsilon(f_\varepsilon)$ in Ω and they are uniformly bounded in $Lip(\Omega') \subset\subset \Omega$ for every $\Omega' \subset\subset \Omega$.

Let $u = \lim u^{\varepsilon_j}$ and $f = \lim f_{\varepsilon_j}$ with $\varepsilon_j \to 0$. Then, there is a subset $\mathcal{R}$ of the free boundary $\Omega \cap \partial\{u > 0\}$ which is locally a $C^{1,\alpha}$ surface and u is a classical solution to the free boundary problem $E(f)$ in a neighborhood of $\mathcal{R}$. Moreover, $\mathcal{R}$ is open and dense in $\Omega \cap \partial\{u > 0\}$ and the remainder of the free boundary has $(N - 1)-$dimensional Hausdorff measure zero.

In dimensions 2 and 3 we have $\mathcal{R} = \Omega \cap \partial\{u > 0\}$.

In addition, in any dimension, if $u \geq 0$ and $f \in C^{k,\alpha}_{\mathrm{loc}}$ (resp. analytic) then, $\mathcal{R} \in C^{k+2,\alpha}_{\mathrm{loc}}$ (resp. analytic).

The last application we present is the following

Theorem 4.5 (Theorem 10.1 in [14]). *Let $x = (x_1, y) \in \Omega = \mathbb{R} \times \Sigma$, with $\Sigma \subset \mathbb{R}^{N-1}$ a smooth bounded domain, let a be a continuous positive function on $\overline{\Sigma}$ and let $0 < \sigma < 1$ be given.*

Consider travelling wave solutions to the following combustion model

$$(4.3) \qquad \Delta v^\varepsilon - a(y)v^\varepsilon_t = \beta_\varepsilon(v^\varepsilon),$$

where β_ε is as before with $\beta'(0) > 0$. This is, $v^\varepsilon(x, t) = u^\varepsilon(x_1 + c^\varepsilon t, y)$, with u^ε solutions to

$$\Delta u^\varepsilon - c^\varepsilon a(y)u^\varepsilon_{x_1} = \beta_\varepsilon(u^\varepsilon) \qquad in \ \Omega,$$

$$(4.4) \qquad u^\varepsilon(-\infty, y) = (1 - \sigma)^{-1}, \qquad u^\varepsilon(+\infty, y) = 0 \qquad in \ \Sigma,$$

$$\frac{\partial u^\varepsilon}{\partial \eta} = 0 \qquad on \ \mathbb{R} \times \partial\Sigma,$$

for some suitable c^ε.

Let $u = \lim u^{\varepsilon_j}$ ($\varepsilon_j \to 0$). Then, there is a subset $\mathcal{R}$ of the free boundary $\Omega \cap \partial\{u > 0\}$ which is locally a $C^{1,\alpha}$ surface and u is a classical solution to the free boundary problem $E(f)$ in a neighborhood of $\mathcal{R}$ ($f = ca(y)u_{x_1}$ with $c = \lim c^{\varepsilon_j}$). Moreover, $\mathcal{R}$ is open and dense in $\Omega \cap \partial\{u > 0\}$ and the remainder of the free boundary has $(N - 1)-$dimensional Hausdorff measure zero.

In dimension 2 we have $\mathcal{R} = \Omega \cap \partial\{u > 0\}$.

In addition, in any dimension, if $a \in C^{k,\alpha}_{\mathrm{loc}}$ (resp. analytic) then, $\mathcal{R} \in C^{k+2,\alpha}_{\mathrm{loc}}$ (resp. analytic).

We remark that this problem was first studied in [2], where the authors obtained existence of $(u^\varepsilon, c_\varepsilon)$, strict monotonicity in the x_1 direction, uniform Lipschitz estimates and uniform nondegeneracy of the family u^ε, as well as uniform estimates of the velocities c_ε.

The proof of Theorem 4.5 relies on the fact that the density of the zero set is positive at every free boundary point. We obtain this density property by a contradiction argument by means of a delicate discussion on the consequences of the equality (2.19) in Corollary 2.2. In dimension 2 we also use that blow up limits are homogeneous as proved in Corollary 2.1.

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