A MIXED SEMILINEAR PARABOLIC PROBLEM
IN A NON-CYLINDRICAL SPACE-TIME DOMAIN

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Abstract

In this paper we prove existence, uniqueness and regularity of the solution to a mixed initial boundary value problem for a semilinear uniformly parabolic equation with principal part in divergence form, in a non-cylindrical space-time domain. We only assume mild regularity on the coefficients and on the non-cylindrical part of the lateral boundary (on which Dirichlet data are given). Also, we only assume mild regularity on the Dirichlet data.

We consider two different situations. One with a bounded domain and one with an unbounded domain.

This problem is of interest in combustion theory. In that situation, the non-cylindrical part of the lateral boundary may be considered as an approximation of a flame front. The second order part of the equation is the Laplace operator. In particular, the results in this paper are used in [LVW] to prove the uniqueness of a "limit" solution to the combustion problem.

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Introduction

In this paper we prove existence, uniqueness and regularity of the solution to a mixed initial boundary value problem for a semilinear uniformly parabolic equation with principal part in divergence form, in a non-cylindrical space-time domain. The problem considered is the following:

To find $u \in C(\overline{\mathcal{R}})$, with $\nabla u \in C(\overline{\mathcal{R}})$, a weak solution to

$$\sum_{i,\,j} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + \sum_{i} b_i \frac{\partial u}{\partial x_i} + c\,u - u_t = \beta(x,t,u) \quad \text{in } \mathcal{R}$$

$$\sum_{i,\,j} a_{ij}\frac{\partial u}{\partial x_j}\eta_i = 0 \quad \text{on } \partial_N\mathcal{R}$$

$$u = \phi \quad \text{on } \partial_D\mathcal{R},$$

where $\mathcal{R} \subset \mathbb{R}^N \times (0,T)$ is a non-cylindrical space-time domain, bounded or unbounded, $\partial_N\mathcal{R}$ is an open subset of the parabolic boundary, $\partial_p\mathcal{R}$, and $\partial_D\mathcal{R} = \partial_p\mathcal{R} \setminus \partial_N\mathcal{R}$.

The non-cylindrical part of $\partial_p\mathcal{R}$ is $\partial_D\mathcal{R}\cap\{t>0\}$. We only assume mild regularity on the coefficients and on $\partial_D\mathcal{R}\cap\{t>0\}$. Also, we assume a minimum smoothness on the Dirichlet datum ϕ.

This problem is of interest in combustion theory. In that situation, the non-cylindrical part of the lateral boundary may be considered as an approximation of a flame front. The second order part of the equation is the Laplace operator. In particular, the results in this paper are used in [LVW] to prove the uniqueness of a "limit" solution to the combustion problem.

In that context, the initial datum $\phi(x,0)$ is only globally Hölder continuous with Hölder continuous spatial gradient near the initial flame front $\overline{\partial_D\mathcal{R}\cap\{t>0\}}\cap\{t=0\}$. The solution u must satisfy that $\nabla u \in C(\overline{\mathcal{R}}\cap\{t>0\})$ and ∇u must be continuous up to time $t=0$ near the flame front $\overline{\partial_D\mathcal{R}\cap\{t>0\}}$. With that regularity of the datum, standard Schauder or Sobolev type results cannot be applied, even if we had a cylindrical space-time domain or $\partial_N\mathcal{R}=\emptyset$. The main difficulty in the paper is not the fact that the domain is non-cylindrical in space-time, since the domain we consider can be reduced to a space-time cylinder. The main point in this work is the proof of the regularity of ∇u up to the boundary with mild regularity assumptions on the data.

We point out that we prove the existence of a weak solution $u \in C^{\gamma,\frac{\gamma}{2}}(\overline{\mathcal{R}})$ with $\nabla u \in C(\mathcal{R}\cup\partial_N\mathcal{R})$ assuming that ϕ is only Hölder continuous. Further continuity of ∇u is obtained in every neighborhood of a point in $\partial_D\mathcal{R}$ where ϕ is smooth enough.

We want to remark that there is a vast literature on mixed boundary value problems for parabolic equations —even in non-cylindrical space-time domains— (see, for instance, [BFO1], [BFO2], [BR], [L], [T]). However, our results cannot be derived from those papers.

The paper is divided into two sections. In Section 1, we prove the result in the case of a bounded domain. In Section 2, we prove the result in an unbounded situation.

Notation and hypotheses.

Throughout the paper N will denote the spatial dimension and, in addition, the following notation will be used:

The symbol ∇ will denote the corresponding operator in the space variables; the symbol ∂_p applied to a domain will denote parabolic boundary.

Let us define the Hölder spaces we are going to use. Let $m \geq 0$ be an integer and $0 < \alpha < 1$. For a space-time cylinder $Q = \Omega \times (0,T) \subset \mathbb{R}^{N+1}$, $C^{m+\alpha,\frac{m+\alpha}{2}}(Q)$ is the parabolic Hölder space denoted by $H^{m+\alpha,\frac{m+\alpha}{2}}(Q)$ in [LSU]. If $\mathcal{D} \subset \mathbb{R}^{N+1}$ is a general domain, then $C^{m+\alpha,\frac{m+\alpha}{2}}(\mathcal{D})$ will denote the space of functions in $C^{m+\alpha,\frac{m+\alpha}{2}}(Q)$ for every space-time cylinder $Q \subset \mathcal{D}$. If $\mathcal{D}$ is bounded, we will say that $u \in C^{m+\alpha,\frac{m+\alpha}{2}}(\overline{\mathcal{D}})$ if there exists a domain $\mathcal{D}'$ with $\overline{\mathcal{D}} \subset \mathcal{D}'$ and a function $u' \in C^{m+\alpha,\frac{m+\alpha}{2}}(\mathcal{D}')$ such that $u = u'$ in $\overline{\mathcal{D}}$. If $\mathcal{D}$ is unbounded, we will say that $u \in C^{m+\alpha,\frac{m+\alpha}{2}}(\overline{\mathcal{D}})$ if $u \in C^{m+\alpha,\frac{m+\alpha}{2}}(\overline{\mathcal{D}'})$ for every bounded domain $\mathcal{D}' \subset \mathcal{D}$.

When $\mathcal{D}$ is bounded, we will denote by $C^{dini}(\overline{\mathcal{D}})$ the set of functions which are continuous in $\overline{\mathcal{D}}$ and such that their modulus of continuity $\omega(r)$ with respect to the parabolic norm $||(x,t)|| = |x| + |t|^{\frac{1}{2}}$ satisfy the Dini condition

$$\int_0^1 \frac{\omega(r)}{r}\, dr < \infty.$$

When $\mathcal{D}$ is unbounded, we will say that $u \in C^{dini}(\overline{\mathcal{D}})$ if $u \in C^{dini}(\overline{\mathcal{D}'})$ for every bounded domain $\mathcal{D}' \subset \mathcal{D}$. On the other hand, $C^{dini}(\mathcal{D})$ will denote the set of functions which belong to $C^{dini}(\overline{\mathcal{D}'})$ for every $\mathcal{D}' \subset\subset \mathcal{D}$.

Throughout the paper we will let $\Omega = \mathbb{R} \times \Sigma$ and $\Sigma \subset \mathbb{R}^{N-1}$ a bounded C^3 domain with interior unit normal η', and we will denote by $\eta = (0, \eta')$ the interior unit normal to $\partial\Omega$. We will denote points in $\overline{\Omega}$ by $x = (x_1, x')$ with $x_1 \in \mathbb{R}$ and $x' \in \overline{\Sigma}$.

On the other hand f will be a continuously differentiable function in $\overline{\Sigma} \times [0,T]$ with $\nabla_{x'} f \in C^{dini}(\overline{\Sigma} \times [0,T])$, such that $\nabla_{x'} f(x',t) \cdot \eta' = 0$ on $\partial\Sigma \times (0,T)$.

Section 1

In this section we consider a bounded non-cylindrical space-time domain and we prove an existence, uniqueness and regularity result, as well as a-priori estimates, for a mixed initial-boundary value problem associated to a uniformly parabolic equation with principal part in divergence form, with a nonlinear forcing term.

In Proposition 1.1 we prove existence and uniqueness for the linear problem, with a minimum smoothness of the data. In Proposition 1.2 we prove a regularity result for the linear problem. Finally, we study the semilinear problem in Theorem 1.1.

Throughout this section, we will fix $R > 0$ and let

$$\mathcal{D}_R := \{(x,t) \in \Omega \times (0,T) \,/\, f(x',t) < x_1 < f(x',t) + R\}.$$

We define, as usual, $\partial_p \mathcal{D}_R := \overline{\partial \mathcal{D}_R \setminus \{t = T\}}$ and we let

$$\begin{aligned}
&\partial_N \mathcal{D}_R := \{(x,t) \in \partial_p \mathcal{D}_R \,/\, x' \in \partial\Sigma,\ 0 < t \le T,\ f(x',t) < x_1 < f(x',t) + R\},\\
&\partial_D \mathcal{D}_R := \partial_p \mathcal{D}_R \setminus \partial_N \mathcal{D}_R,\\
&\partial_S \mathcal{D}_R := \{(x,t) \in \partial_p \mathcal{D}_R \,/\, x_1 = f(x',t) \ \text{ or } \ x_1 = f(x',t) + R\}.
\end{aligned}$$

We first state the following existence and uniqueness result for the linear problem:

Proposition 1.1. *Let $\mathcal{D}_R$, $\partial_N \mathcal{D}_R$, $\partial_D \mathcal{D}_R$ and $\partial_S \mathcal{D}_R$ as above. For $i, j = 1, \cdots, N$, let a_{ij}, b_i, c, $g \in L^\infty(\mathcal{D}_R)$. Assume that $a_{ij}(x,t)\xi_i\xi_j \ge \lambda |\xi|^2$ for some $\lambda > 0$ and every $\xi \in \mathbb{R}^N$, $(x,t) \in \mathcal{D}_R$. Let $\phi \in C^{\alpha, \frac{\alpha}{2}}(\overline{\mathcal{D}}_R)$.*

There exists a unique function $u \in C(\overline{\mathcal{D}}_R)$, with $\nabla u \in L^2_{loc}(\overline{\mathcal{D}}_R \setminus \partial_S \mathcal{D}_R)$, such that u is a weak solution to the following mixed initial boundary value problem

$$(1.1) \qquad \begin{aligned}
\sum_{i,j} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + \sum_i b_i \frac{\partial u}{\partial x_i} + c\,u - u_t = g \quad &\text{in } \mathcal{D}_R\\
\sum_{i,j} a_{ij}\frac{\partial u}{\partial x_j}\eta_i = 0 \quad &\text{on } \partial_N \mathcal{D}_R\\
u = \phi \quad &\text{on } \partial_D \mathcal{D}_R.
\end{aligned}$$

Moreover, there exist $0 < \gamma \le \alpha$ and $C > 0$, depending only on α, T, R, λ, $||a_{ij}||_{L^\infty(\mathcal{D}_R)}$, $||b_i||_{L^\infty(\mathcal{D}_R)}$, $||c||_{L^\infty(\mathcal{D}_R)}$, $||\phi||_{C^{\alpha,\frac{\alpha}{2}}(\partial_D \mathcal{D}_R)}$, $||g||_{L^\infty(\mathcal{D}_R)}$, the domain Σ and the function f, such that

$$(1.2) \qquad ||u||_{C^{\gamma,\frac{\gamma}{2}}(\overline{\mathcal{D}}_R)} \le C.$$

Even more, for every $R_0 < R$ there exist $0 < \gamma' \le \alpha$ and $C' > 0$ depending only on α, T, R_0, λ, $||a_{ij}||_{L^\infty(\mathcal{D}_{R_0})}$, $||b_i||_{L^\infty(\mathcal{D}_{R_0})}$, $||c||_{L^\infty(\mathcal{D}_{R_0})}$, $||\phi||_{C^{\alpha,\frac{\alpha}{2}}(\partial_D \mathcal{D}_R \cap \overline{\mathcal{D}}_{R_0})}$, $||g||_{L^\infty(\mathcal{D}_{R_0})}$, the domain Σ, the function f and $||u||_{L^\infty(\mathcal{D}_{R_0})}$, such that

$$(1.3) \qquad ||u||_{C^{\gamma',\frac{\gamma'}{2}}(\overline{\mathcal{D}}_{R_0/2})} \le C',$$

$$(1.4) \qquad ||\nabla u||_{L^2(\mathcal{D}_{R_0/2} \cap \{x_1 > f(x',t) + \varepsilon\})} \le C''.$$

Estimate (1.4) holds for every $\varepsilon > 0$, with $C'' > 0$ depending only on ε and on the same arguments as C', but independent of α and $||\phi||_{C^{\alpha,\frac{\alpha}{2}}(\partial_D \mathcal{D}_R \cap \overline{\mathcal{D}}_{R_0})}$.

Now let $\psi_1(x',t) = \phi(f(x',t), x', t), \psi_2(x',t) = \phi(f(x',t) + R, x', t)$ and assume, in addition, that $\psi_{i_t} \in L^2(\Sigma \times (0,T))$, $\nabla_{x'}\psi_i \in L^2(\Sigma \times (0,T))$ for $i = 1, 2$. Then $\nabla u \in L^2(\mathcal{D}_R)$.

We next state the following regularity result for the linear problem:

Proposition 1.2. *Let $\mathcal{D}_R$, $\partial_N\mathcal{D}_R$, $\partial_D\mathcal{D}_R$, a_{ij}, b_i, c, g, ϕ, ψ_i as in Proposition 1.1. Let $u \in C^{\gamma,\frac{\gamma}{2}}(\overline{\mathcal{D}}_R)$, with $\nabla u \in L^2(\mathcal{D}_R)$, be the unique weak solution to (1.1). Now assume that $a_{ij} \in C^{dini}(\overline{\mathcal{D}}_R)$ and $a_{ij} = \delta_{ij}$ on $\partial_N\mathcal{D}_R$.*

Then, $\nabla u \in C(\mathcal{D}_R \cup \partial_N\mathcal{D}_R)$.

If, in addition, $\psi_1(x',t) \in C^1(\overline{\Sigma}\times(0,T])$, with $\nabla_{x'}\psi_1 \in C^{dini}(\overline{\Sigma}\times(0,T])$ and $\frac{\partial\psi_1}{\partial\eta'} = 0$ on $\partial\Sigma\times(0,T)$, there holds that ∇u is continuous in $\overline{\mathcal{D}}_R \cap \{x_1 < f(x',t) + R\} \cap \{t > 0\}$.

If, moreover, $\psi_1(x',t) \in C^1(\overline{\Sigma}\times[0,T])$, with $\nabla_{x'}\psi_1 \in C^{dini}(\overline{\Sigma}\times[0,T])$, and $\nabla\phi \in C^{dini}(\overline{\mathcal{D}}_{R_0} \cap \{t=0\})$, with $\frac{\partial\phi}{\partial\eta} = 0$ on $\overline{\partial_N\mathcal{D}_{R_0}} \cap \{t=0\}$, there holds that $\nabla u \in C(\overline{\mathcal{D}}_{R_0/2})$ and there exist a constant $C>0$ and an increasing function $\omega(r)$, with $\omega(0^+) = 0$, such that

$$
\begin{aligned}
&||\nabla u||_{L^\infty(\mathcal{D}_{R_0/2})} \le C \quad \textit{and} \\
&|\nabla u(x,t) - \nabla u(y,s)| \le \omega(|x-y| + |t-s|^{1/2}) \textit{ for } (x,t),\,(y,s) \in \overline{\mathcal{D}}_{R_0/2}.
\end{aligned}
\tag{1.5}
$$

With the same regularity of ψ_1 and no regularity assumptions on $\phi(x,0)$, for every $\tau > 0$, (1.5) holds in $\mathcal{D}_{R_0/2} \cap \{t \ge \tau\}$ with C and ω independent of $\phi(x,0)$ but depending on τ.

Analogously, if $\psi_2(x',t) \in C^1(\overline{\Sigma}\times(0,T])$, with $\nabla_{x'}\psi_2 \in C^{dini}(\overline{\Sigma}\times(0,T])$ and $\frac{\partial\psi_2}{\partial\eta'} = 0$ on $\partial\Sigma\times(0,T)$, and with no regularity assumptions on ψ_1 and on $\phi(x,0)$, there holds that ∇u is continuous in $\overline{\mathcal{D}}_R \cap \{x_1 > f(x',t)\} \cap \{t > 0\}$.

Also, if $\psi_i \in C^1(\overline{\Sigma}\times[0,T])$, with $\nabla_{x'}\psi_i \in C^{dini}(\overline{\Sigma}\times[0,T])$, and $\frac{\partial\psi_i}{\partial\eta'} = 0$ on $\partial\Sigma\times(0,T)$ for $i=1,2$ and $\nabla\phi \in C^{dini}(\overline{\mathcal{D}_R} \cap \{t=0\})$ with $\frac{\partial\phi}{\partial\eta} = 0$ on $\overline{\partial_N\mathcal{D}_R} \cap \{t=0\}$, there holds that $\nabla u \in C(\overline{\mathcal{D}}_R)$.

Finally, if $a_{ij} \in C^{1+\mu,\frac{1+\mu}{2}}(\mathcal{D}_R)$, b_i, c, $g \in C^{\mu,\frac{\mu}{2}}(\mathcal{D}_R)$, u is a classical solution in the sense that $u \in C^{2+\mu,1+\frac{\mu}{2}}(\mathcal{D}_R)$.

Let us now prove our first result:

Proof of Proposition 1.1. Let ψ_1, ψ_2 be as in the statement. We will first prove the proposition with the extra assumption that $\psi_{it} \in L^2(\Sigma\times(0,T))$, $\nabla_{x'}\psi_i \in L^2(\Sigma\times(0,T))$ for $i=1,2$.

Let us take the new coordinate system

$$
\begin{cases}
y_1 = x_1 - f(x',t) \\
y' = x'
\end{cases}
$$

and let $\overline{u}(y,t) := u(x,t)$. Then $u \in C(\overline{\mathcal{D}}_R)$ with $\nabla u \in L^2(\mathcal{D}_R)$ is a weak solution of (1.1) if and only if $\overline{u} \in C(\overline{Q})$ with $\nabla\overline{u} \in L^2(Q)$ is a weak solution in

$$Q := (0,R)\times\Sigma\times(0,T)$$

of the following problem

$$
(1.6)\qquad
\begin{aligned}
&\overline{\mathcal{L}}\bar{u} := \sum_{i,\,j} \frac{\partial}{\partial y_i}\left(\bar{a}_{ij}\frac{\partial \overline{u}}{\partial y_j}\right) + \sum_i \bar{b}_i \frac{\partial \overline{u}}{\partial y_i} + \bar{c}\,\overline{u} - \overline{u}_t = \overline{g} \quad \text{in } Q\\
&\sum_{i,\,j} \bar{a}_{ij}\frac{\partial \overline{u}}{\partial y_j}\eta_i = 0 \qquad \text{on } \partial_N Q := (0,R)\times\partial\Sigma\times(0,T)\\
&\overline{u} = \overline{\phi} \qquad\qquad \text{on } \partial_D Q := \partial_p Q \setminus \partial_N Q
\end{aligned}
$$

where $\overline{g}(y,t) = g(x,t)$, $\overline{\phi}(y,t) = \phi(x,t)$, $\bar{c}(y,t) = c(x,t)$,

$$
\begin{aligned}
&\bar{a}_{ij}(y,t) = a_{ij}(x,t) \qquad i,j>1,\\
&\bar{a}_{1j}(y,t) = a_{1j}(x,t) - \sum_{k>1} a_{kj}(x,t) f_{x_k}(x',t) \qquad j>1,\\
&\bar{a}_{i1}(y,t) = a_{i1}(x,t) - \sum_{l>1} a_{il}(x,t) f_{x_l}(x',t) \qquad i>1,\\
&\bar{a}_{11}(y,t) = a_{11}(x,t) + \sum_{k,l>1} a_{kl}(x,t) f_{x_k}(x',t) f_{x_l}(x',t) - \sum_{l>1} a_{1l}(x,t) f_{x_l}(x',t) -\\
&\qquad \sum_{k>1} a_{k1}(x,t) f_{x_k}(x',t)
\end{aligned}
$$

and

$$
\bar{b}_i(y,t) = \begin{cases} b_1(x,t) - \sum\limits_{j>1} b_j(x,t) f_{x_j}(x',t) + f_t(x',t) & \text{for } i=1\\ b_i(x,t) \quad \text{for } i>1. \end{cases}
$$

Note that the differential equation for $\overline{u}$ has bounded coefficients and right hand side $\overline{g}$. On the other hand, it is uniformly parabolic (with parabolicity constant depending only on λ and $\|\nabla_{x'} f\|_{L^\infty}$). Let $\overline{\psi}_i(y',t) = \psi_i(x',t)$ for $i=1,2$, and

$$
\overline{\psi}(y,t) = \overline{\psi}_1(y',t)\left(1-\frac{y_1}{R}\right) + \overline{\psi}_2(y',t)\frac{y_1}{R}
$$

and let $\bar{v}(y,t) \in C\left([0,T];L^2((0,R)\times\Sigma)\right)$, with $\nabla\bar{v} \in L^2(Q)$, be the unique solution to

$$
\begin{aligned}
&(\overline{\mathcal{L}}\bar{v},\varphi) = (\mathcal{F},\varphi) \quad \forall\varphi\in C^\infty(\overline{Q}),\ \varphi=0 \text{ in } \partial Q\setminus\partial_N Q,\\
&\bar{v} = 0 \quad \text{on } \{y_1=0\} \text{ and } \{y_1=R\},\\
&\bar{v} = \bar{\phi} - \overline{\psi} \quad \text{on } \{t=0\},
\end{aligned}
$$

where

$$
\begin{aligned}
&(\overline{\mathcal{L}}\bar{v},\varphi) := -\sum_{i,\,j}\int_Q \bar{a}_{ij}\frac{\partial\bar{v}}{\partial y_j}\frac{\partial\varphi}{\partial y_i} + \sum_i\int_Q \bar{b}_i\frac{\partial\bar{v}}{\partial y_i}\varphi + \int_Q \bar{c}\,\bar{v}\varphi + \int_Q \bar{v}\varphi_t,\\
&(\mathcal{F},\varphi) := \int_Q g_1\varphi + \sum_i\int_Q (g_2)_i\frac{\partial\varphi}{\partial y_i},\\
&g_1 := \bar{g} - \sum_i \bar{b}_i\frac{\partial\overline{\psi}}{\partial y_i} - \bar{c}\,\overline{\psi} + \overline{\psi}_t,\\
&(g_2)_i := \sum_j \bar{a}_{ij}\frac{\partial\overline{\psi}}{\partial y_j}.
\end{aligned}
$$

(For the existence and uniqueness of the solution $\bar{v}$ see, for instance, Theorem X.9 in [B]).

Let $\bar{u} = \bar{v} + \overline{\psi}$. Then $\overline{u} \in C\left([0,T]; L^2((0,R)\times\Sigma)\right)$ with $\nabla\overline{u} \in L^2(Q)$ is a weak solution to (1.6).

Let $R_0 < R$. Let us prove that there exist $0 < \gamma' \leq \alpha$ and $C > 0$ depending only on the L^∞ norm of the coefficients of the equation in (1.6) in the domain $Q \cap \{y_1 < R_0\}$, the constants α, λ, T and R_0, the domain Σ, $||\overline{\phi}||_{C^{\alpha,\frac{\alpha}{2}}(\partial_D Q\cap\{y_1\leq R_0\})}$, $||\nabla_{x'} f||_{L^\infty}$, $||\overline{g}||_{L^\infty(Q\cap\{y_1<R_0\})}$ and $||\overline{u}||_{L^\infty(Q\cap\{y_1<R_0\})}$, such that

$$||\overline{u}||_{C^{\gamma',\frac{\gamma'}{2}}(\overline{Q}\cap\{y_1\leq R_0/2\})} \leq C.$$

This will prove that (1.3) holds.

We will make use of Theorem 10.1, Chap. III in [LSU].

So let us first take for $S > 0$ the set $Q_{\delta,S} := (0,S)\times\Sigma_\delta\times(0,T)$ where $\Sigma_\delta := \{y' \in \Sigma \,/\, \mathrm{dist}(y',\partial\Sigma) > \delta\}$ and $\delta > 0$ is small to be fixed later.

By Thm. 10.1, Chap. III in [LSU] there exist $0 < \gamma' \leq \alpha$ and $C > 0$ depending only on α, λ, T, R_0, Σ, $||\nabla_{x'} f||_{L^\infty}$, the L^∞ norm of the coefficients of the equation in (1.6) and free term $\bar{g}$ in $Q_{\delta/2,R_0}$, $||\overline{\phi}||_{C^{\alpha,\frac{\alpha}{2}}(\partial_D Q\cap\overline{Q}_{\delta/2,R_0})}$ and $||\overline{u}||_{L^\infty(Q_{\delta/2,R_0})}$ such that

$$||\overline{u}||_{C^{\gamma',\frac{\gamma'}{2}}(\overline{Q}_{\delta,R_0/2})} \leq C.$$

Next, let $(y_0,t_0) \in [0,R]\times\partial\Sigma\times[0,T]$. We will straighten up $\partial_N Q$. For that purpose we denote $y_0 = (y_{0_1}, y_0')$, with $y_{0_1} \in [0,R]$ and $y_0' \in \partial\Sigma$ and we take $\mathcal{O} \subset \mathbb{R}^{N-1}$ a neighborhood of y_0' such that $[0,R]\times(\partial\Sigma\cap\mathcal{O})$ is parameterized in the variables $(z_1,\cdots,z_{N-1})$ by

$$\begin{aligned} &y_1 = z_1, && 0 \leq z_1 \leq R,\\ &y' = \sigma'(z_2,\cdots,z_{N-1}), && (z_2,\cdots,z_{N-1}) \in \mathcal{N} \subset \mathbb{R}^{N-2}. \end{aligned}$$

Here $\mathcal{N}$ is the ball in $\mathbb{R}^{N-2}$ with center in the origin and radius $r = 1$. Since Σ is a C^3 domain, we may take as σ' a C^3 regular parameterization. Also η', the interior unit normal to Σ, is a C^2 function of the point $y' \in \partial\Sigma$.

Then, we may choose the neighborhood $\mathcal{O}$ so that every point $y \in [0,R]\times\mathcal{O}$ can be written in a unique way as $y = h^{-1}(z)$, where

$$\begin{cases} y_1 = z_1\\ y' = \sigma'(z_2,\cdots,z_{N-1}) + \eta'(\sigma'(z_2,\cdots,z_{N-1}))\,z_N, \end{cases}$$

with $z \in [0,R]\times\mathcal{N}\times\{|z_N| < 2\delta\}$, for some $\delta > 0$, and h a C^2 invertible function with nonvanishing Jacobian in $[0,R]\times\overline{\mathcal{O}}$ and $h([0,R]\times\mathcal{O}) = [0,R]\times\mathcal{N}\times\{|z_N| < 2\delta\}$ and $h([0,R]\times(\mathcal{O}\cap\Sigma)) = [0,R]\times\mathcal{N}\times\{0 < z_N < 2\delta\}$.

Here δ can be chosen independent of (y_0,t_0) and R, and will remain fixed from now on.

Let $\overline{\overline{u}}(z,t) := \overline{u}(y,t)$ for $z_N \geq 0$ and $\tilde{Q}_+ := \{(z,t) \in \tilde{Q} \,/\, z_N > 0\}$ where

$$\tilde{Q} := (0,R) \times \mathcal{N} \times \{|z_N| < 2\delta\} \times (0,T).$$

Then $\overline{\overline{u}} \in C\left([0,T]; L^2((0,R) \times \mathcal{N} \times \{0 < z_N < 2\delta\})\right)$, with $\nabla\overline{\overline{u}} \in L^2(\tilde{Q}_+)$, is a weak solution in $\tilde{Q}_+$ to

$$\overline{\overline{\mathcal{L}}}\,\overline{\overline{u}} := \sum_{i,\,j} \frac{\partial}{\partial z_i}\left(\bar{\bar{a}}_{ij}\frac{\partial \overline{\overline{u}}}{\partial z_j}\right) + \sum_j \bar{\bar{b}}_j \frac{\partial \overline{\overline{u}}}{\partial z_j} + \bar{\bar{c}}\overline{\overline{u}} - \overline{\overline{u}}_t = \bar{\bar{g}} \tag{1.7}$$

which is a uniformly parabolic equation with principal part in divergence form with bounded coefficients and free term. Here

$$\bar{\bar{a}}_{ij}(z,t) = \sum_{k,\,l} \bar{a}_{kl}(y,t)\frac{\partial h_i}{\partial y_k}(y)\frac{\partial h_j}{\partial y_l}(y).$$

Moreover, if we let $\partial_N\tilde{Q}_+ := (0,R) \times \mathcal{N} \times \{z_N = 0\} \times (0,T)$, then $\overline{\overline{u}}$ satisfies weakly that

$$\sum_j \bar{\bar{a}}_{Nj}\frac{\partial \overline{\overline{u}}}{\partial z_j} = 0 \qquad \text{on } \partial_N\tilde{Q}_+,$$

(note that e_N is the interior normal direction to $\tilde{Q}_+$ on $\{z_N = 0\}$).

We extend $\overline{\overline{u}}$ to $\{z_N < 0\}$ by reflection. This is, we define for $z_N < 0$

$$\overline{\overline{u}}(z,t) = \overline{\overline{u}}(z_1, z_2, \cdots, -z_N, t).$$

In this way, $\overline{\overline{u}} \in C\left([0,T]; L^2((0,R) \times \mathcal{N} \times \{|z_N| < 2\delta\})\right)$, with $\nabla\overline{\overline{u}} \in L^2(\tilde{Q})$, becomes a weak solution in the domain $\tilde{Q}$ of equation (1.7), where we have, for $z_N < 0$,

$$\bar{\bar{a}}_{ij}(z,t) = \begin{cases} \bar{\bar{a}}_{ij}(z_1, \cdots, z_{N-1}, -z_N, t) & \text{if } i < N, j < N, \quad \text{or} \quad i = j = N, \\ -\bar{\bar{a}}_{ij}(z_1, \cdots, z_{N-1}, -z_N, t) & \text{if } i = N, j < N, \quad \text{or} \quad i < N, j = N, \end{cases}$$

$$\bar{\bar{b}}_j(z,t) = \begin{cases} \bar{\bar{b}}_j(z_1, \cdots, z_{N-1}, -z_N, t) & \text{if } j < N, \\ -\bar{\bar{b}}_j(z_1, \cdots, z_{N-1}, -z_N, t) & \text{if } j = N, \end{cases}$$

and

$$\bar{\bar{c}}(z,t) = \bar{\bar{c}}(z_1, \cdots, z_{N-1}, -z_N, t), \qquad \bar{\bar{g}}(z,t) = \bar{\bar{g}}(z_1, \cdots, z_{N-1}, -z_N, t).$$

Thus, $\overline{\overline{u}}$ is a weak solution in $\tilde{Q}$ of a uniformly parabolic equation with principal part in divergence form with bounded coefficients and free term. We remark that the L^∞ norm of the coefficients and free term as well as the parabolicity constant in equation (1.7) in the domain $\tilde{Q} \cap \{z_1 < R_0\}$ depend only on the corresponding bounds for the equation in (1.6) in the domain $Q \cap \{y_1 < R_0\}$ and on the domain Σ.

We apply again Thm. 10.1, Chap. III in [LSU] to conclude that there exist $0 < \gamma' \le \alpha$ and $C > 0$ such that

$$||\overline{\overline{u}}||_{C^{\gamma', \frac{\gamma'}{2}}(\overline{\tilde{Q}_{\frac{1}{2}}})} \le C.$$

Here $\tilde{Q}_{\frac{1}{2}} = (0, R_0/2) \times \mathcal{N}_{\frac{1}{2}} \times \{|z_N| < \delta\} \times (0,T)$, where $\mathcal{N}_{\frac{1}{2}}$ is the ball in $\mathbb{R}^{N-2}$ with center in the origin and radius $r = \frac{1}{2}$.

The constants γ' and C depend only on α, λ, T, R_0, Σ, $||\nabla_{x'} f||_{L^\infty}$, the L^∞ norm of the coefficients of the equation in (1.7) and free term in $\tilde{Q} \cap \{z_1 < R_0\}$, $||\bar{\bar{u}}||_{L^\infty(\tilde{Q}\cap\{z_1<R_0\})}$, $||\overline{\overline{\phi}}||_{C^{\alpha,\frac{\alpha}{2}}(\overline{\tilde{Q}}_+\cap\{z_1\le R_0\}\cap(\{t=0\}\cup\{z_1=0\}))}$. Here $\overline{\overline{\phi}}(z,t) := \overline{\phi}(y,t)$ for $(y,t) \in \partial_D Q$.

Therefore (1.3) holds.

On the other hand, reasoning as above and using that $||u||_{L^\infty(\mathcal{D}_R)}$ is bounded by a constant depending only on $||\phi||_{L^\infty(\partial_D\mathcal{D}_R)}$, $||c||_{L^\infty(\mathcal{D}_R)}$, $||g||_{L^\infty(\mathcal{D}_R)}$ and T, we see that there exists $0 < \gamma \le \alpha$ such that $u \in C^{\gamma,\frac{\gamma}{2}}(\overline{\mathcal{D}}_R)$. Moreover, there exists $C > 0$ depending only on α, T, R, λ, the L^∞ norm of the coefficients and free term in (1.1), $||\phi||_{C^{\alpha,\frac{\alpha}{2}}(\partial_D\mathcal{D}_R)}$, the domain Σ and the function f such that

$$||u||_{C^{\gamma,\frac{\gamma}{2}}(\overline{\mathcal{D}}_R)} \le C.$$

Let us now prove that (1.4) holds. In fact, from the energy estimate (2.18) in [LSU], Chap. III, applied to the function $\overline{u}$ and test functions depending only on (y_1, t), it follows that for every $\varepsilon > 0$ and $R_0 < R$ there exists $C > 0$ such that

$$||\nabla\overline{u}||_{L^2(Q\cap\{\varepsilon<y_1<R_0/2\})} \le C.$$

The constant C depends only on ε, the L^∞ norm of the coefficients of the equation in (1.6) in the domain $Q \cap \{y_1 < R_0\}$, the constants λ, T and R_0, the domain Σ, $||\nabla_{x'} f||_{L^\infty}$, $||\overline{g}||_{L^\infty(Q\cap\{y_1<R_0\})}$ and $||\overline{u}||_{L^\infty(Q\cap\{y_1<R_0\})}$. This implies that (1.4) holds.

In order to complete the proof, we have to see that the results in the statement hold when we do not have the extra assumption that $\psi_{it} \in L^2(\Sigma\times(0,T))$, $\nabla_{x'}\psi_i \in L^2(\Sigma\times(0,T))$ for $i = 1, 2$. In this case, we approximate the Dirichlet datum ϕ by a sequence of smooth functions, we solve problem (1.1) with these smooth data and then, we pass to the limit, using the estimates (1.2) and (1.4). This shows that such a solution exists.

Finally, to prove uniqueness, we let $u_i \in C(\overline{\mathcal{D}}_R)$, with $\nabla u_i \in L^2_{loc}(\overline{\mathcal{D}}_R \setminus \partial_S\mathcal{D}_R)$ be weak solutions to (1.1), for $i = 1, 2$. We can take $L > 0$ large enough so that, for every $\varepsilon > 0$, the function $w = (u_1 - u_2)e^{-Lt} - \varepsilon$ is a weak solution to problem (1.1) with $c < 0$, $g > 0$ and Dirichlet datum $\phi = -\varepsilon$. Then, changing variables as in (1.6) and looking at the corresponding function $\overline{w}$, we deduce that $w \le 0$ in $\mathcal{D}_R \cap \{f(x',t) + \delta < x_1 < f(x',t) + R - \delta\}$, for any $\delta > 0$ small, and therefore $w \le 0$ in $\mathcal{D}_R$. Since ε is arbitrary, then $u_1 \le u_2$ in $\mathcal{D}_R$. Analogously, we obtain $u_2 \le u_1$ in $\mathcal{D}_R$, which completes the proof. □

Remark 1.1. Proposition 1.1 is still true if instead of assuming that Σ is a C^3 domain we only require that Σ be Lipschitz.

In fact, in the proof of the Hölder continuity of the solution u we may straighten up $\partial_N Q = (0,R)\times\partial\Sigma\times(0,T)$ in a neighborhood of (y_0,t_0) by taking $y = h^{-1}(z)$ in this case as

$$\begin{cases} y_1 = z_1, \\ y_i = z_i \qquad 2\le i\le N-1, \\ y_N = z_N + G(z_2,\cdots,z_{N-1}), \end{cases}$$

with G a Lipschitz continuous function in a neighborhood of $({y_0}_2,\cdots,{y_0}_{N-1})$. Here we are assuming that, in a neighborhood of ${y_0}'$, $\partial\Sigma$ is the graph of the function G in the direction y_N.

The change of variables chosen in the proof of Proposition 1.1 will allow us to get further regularity of the solution u in Proposition 1.2.

Clearly, in order to obtain the Hölder continuity of the solution u, we can also relax the regularity assumption on the function $f(x',t)$ in the definition of the domain $\mathcal{D}_R$ by assuming that f is only Lipschitz continuous.

Finally, the assumption that $\nabla_{x'} f(x',t)\cdot\eta' = 0$ on $\partial\Sigma\times(0,T)$ is not needed in Proposition 1.1.

We now prove the regularity result for the linear problem:

Proof of Proposition 1.2. Throughout the proof we will use the same notation as in the proof of Proposition 1.1.

Let us now assume that $a_{ij}\in C^{dini}(\overline{\mathcal{D}}_R)$ and $a_{ij} = \delta_{ij}$ on $\partial_N\mathcal{D}_R$. In order to prove that $\nabla u\in C(\mathcal{D}_R\cup\partial_N\mathcal{D}_R)$ we consider a point $(y_0,t_0)\in(0,R)\times\partial\Sigma\times[0,T]$ and the corresponding function $\overline{\overline{u}}(z,t)$ which is defined and continuous in $\overline{\tilde{Q}}$ with $\nabla\overline{\overline{u}}\in L^2(\tilde{Q})$. Also, $\overline{\overline{u}}$ is a weak solution to (1.7) in $\tilde{Q}$.

Let us see that the principal coefficients in (1.7), $\bar{\bar{a}}_{ij}$, belong to $C^{dini}(\overline{\tilde{Q}})$. In fact, since $\bar{\bar{a}}_{ij}$ are Dini continuous in $\{z_N\ge 0\}\cup\{z_N\le 0\}$ we only need to verify that $\bar{\bar{a}}_{iN}(z_1,\cdots,z_{N-1},0,t) = 0$ for $i<N$. There holds that

$$\begin{aligned}\bar{\bar{a}}_{iN}(z_1,\cdots,&z_{N-1},0,t) = \nabla h_i\cdot\nabla h_N + \\ &\frac{\partial h_N}{\partial y_1}\left(|\nabla_{y'}f|^2\frac{\partial h_i}{\partial y_1} - \sum_{k>1} f_{y_k}\frac{\partial h_i}{\partial y_k}\right) - \frac{\partial h_i}{\partial y_1}\left(\sum_{k>1} f_{y_k}\frac{\partial h_N}{\partial y_k}\right).\end{aligned}$$

From the fact that $h\big(z_1,\sigma'(z_2,\cdots,z_{N-1}) + \eta'(\sigma'(z_2,\cdots,z_{N-1}))z_N\big) = z$ we deduce that, on $(0,R)\times\partial\Sigma$, ∇h_i is tangent to $(0,R)\times\partial\Sigma$ for $i<N$ and $\nabla h_N = \eta$. Therefore, $\nabla h_i\cdot\nabla h_N = 0$ on $(0,R)\times\partial\Sigma$ for $i<N$.

Since $\eta_1 = 0$, there holds that $\frac{\partial h_N}{\partial y_1} = 0$. Finally, we use the fact that $\nabla_{y'}f\cdot\eta' = 0$ on $\partial\Sigma\times(0,T)$ and conclude that we have $\bar{\bar{a}}_{iN} = 0$ on $\{z_N = 0\}$ for $i<N$.

We can now apply Theorem 1.3.1 in [CK] in $\tilde{Q}$ to deduce that $\nabla\overline{\overline{u}}$ is continuous in $(0,R)\times\mathcal{N}\times\{|z_N|<2\delta\}\times(0,T]$.

On the other hand, a direct application of Theorem 1.3.1 in [CK] to u gives the continuity of ∇u in $\mathcal{D}_R$.

Now assume that $\psi_1(x',t) = \phi(f(x',t),x',t) \in C^1(\overline{\Sigma}\times(0,T])$ with $\nabla_{x'}\psi_1 \in C^{dini}(\overline{\Sigma}\times(0,T])$ and $\frac{\partial\psi_1}{\partial\eta'} = 0$ on $\partial\Sigma\times(0,T)$.

Let

$$\overline{w}(y,t) = \overline{u}(y,t) - \overline{\psi}_1(y',t).$$

Then, $\overline{w} \in C^{\gamma,\frac{\gamma}{2}}(\overline{Q})$, with $\nabla\overline{w} \in L^2(Q)$, is a weak solution to

$$\begin{aligned}&\overline{\mathcal{L}}\,\overline{w} = \bar{g} - \overline{\mathcal{L}}\,\overline{\psi}_1 \quad \text{in } Q\\ &\overline{w} = 0 \quad \text{on } \{y_1 = 0\}.\end{aligned}$$

We may apply Theorems 1.3.1 and 1.4.10 in [CK] to deduce that $\nabla\overline{w} \in C(\overline{Q}_{\delta,R}\cap\{y_1 < R\}\cap\{0 < t \le T\})$.

On the other hand, let $(y_0,t_0) \in [0,R]\times\partial\Sigma\times[0,T]$ and consider the function

$$\overline{\overline{w}}(z,t) = \begin{cases} \overline{w}(h^{-1}(z),t) & \text{for } z_N \ge 0\\ \overline{\overline{w}}(z_1,\cdots,z_{N-1},-z_N,t) & \text{for } z_N < 0.\end{cases}$$

Then, $\overline{\overline{w}} \in C^{\gamma,\frac{\gamma}{2}}(\tilde{Q})$, with $\nabla\overline{\overline{w}} \in L^2(\tilde{Q})$, is a weak solution to

$$\begin{aligned}&\overline{\overline{\mathcal{L}}}\,\overline{\overline{w}} = \bar{\bar{g}} - \overline{\overline{\mathcal{L}}}\,\overline{\overline{\psi}}_1 \quad \text{in } \tilde{Q},\\ &\overline{\overline{w}} = 0 \quad \text{on } \{z_1 = 0\}.\end{aligned}$$

Applying again Theorems 1.3.1 and 1.4.10 in [CK] we deduce that $\nabla\overline{\overline{w}}$ is continuous in $[0,R)\times\mathcal{N}\times\{|z_N| < 2\delta\}\times(0,T]$.

We proceed analogously if ψ_2 is regular enough.

Finally, if we have enough regularity of the data up to $\{t=0\}$, we conclude that ∇u is continuous up to $t = 0$. In fact, in this case we consider the function

$$\overline{p}(y,t) = \overline{u}(y,t) - \overline{\psi}_1(y',t) - \overline{\phi}(y,0) + \overline{\psi}_1(y',0).$$

Then $\overline{p} \in C^{\gamma,\frac{\gamma}{2}}(\overline{Q}\cap\{y_1 \le R_0\})$, with $\nabla\overline{p} \in L^2(Q\cap\{y_1 < R_0\})$ is a weak solution to

$$\begin{aligned}&\overline{\mathcal{L}}\,\overline{p} = \bar{g} - \overline{\mathcal{L}}\,(\overline{\psi}_1 + \overline{\phi}(y,0) - \overline{\psi}_1(y',0)) \quad \text{in } Q\cap\{y_1 < R_0\}\\ &\overline{p} = 0 \quad \text{on } \{y_1 = 0\}\\ &\overline{p} = 0 \quad \text{on } \{t = 0\}.\end{aligned}$$

Applying Theorems 1.3.1, 1.4.3 and 1.4.10, Corollary 1.2.22 and Remark 1.4.11 in [CK], we find that there exist $C > 0$ and an increasing function $\omega(r)$ with $\omega(0^+) = 0$ which are independent of R and only depend on $\overline{p}$ through $||\overline{p}||_{L^\infty(Q_{\delta/2,R_0})}$, and such that

$$||\nabla\overline{p}||_{L^\infty(Q_{\delta,\frac{R_0}{2}})} \le C \quad \text{and}$$

$$|\nabla\overline{p}(y_1,t_1) - \nabla\overline{p}(y_2,t_2)| \le \omega(|y_1-y_2| + |t_1-t_2|^{\frac12}) \text{ for } (y_1,t_1),\ (y_2,t_2) \in \overline{Q}_{\delta,\frac{R_0}{2}}.$$

We define the function $\overline{\overline{p}}(z,t)$ in a similar way as $\overline{\overline{w}}(z,t)$. Then $\overline{\overline{p}}$ is a weak solution to

$$
\begin{aligned}
&\overline{\overline{\mathcal{L}}}\overline{\overline{p}} = \bar{\bar{g}} - \overline{\overline{\mathcal{L}}}\,(\overline{\overline{\psi}}_1 + \overline{\overline{\phi}}(z,0) - \overline{\overline{\psi}}_1(z',0)) \quad \text{in } \tilde{Q} \cap \{z_1 < R_0\},\\
&\overline{\overline{p}} = 0 \quad \text{on } \{z_1 = 0\}\\
&\overline{\overline{p}} = 0 \quad \text{on } \{t = 0\}.
\end{aligned}
$$

Applying Theorems 1.3.1, 1.4.3 and 1.4.10, Corollary 1.2.22 and Remark 1.4.11 in [CK] once more, we find that

$$
||\nabla\overline{\overline{p}}||_{L^\infty(\tilde{Q}_{1/2})} \le C \quad \text{and}
$$

$$
|\nabla\overline{\overline{p}}(y_1,t_1) - \nabla\overline{\overline{p}}(y_2,t_2)| \le \omega(|y_1-y_2| + |t_1-t_2|^{\frac{1}{2}}) \text{ for } (y_1,t_1),\ (y_2,t_2) \in \overline{\tilde{Q}}_{1/2}.
$$

So that (1.5) holds.

Finally, if also ψ_2 is smooth up to $t=0$ and $\phi(x,0)$ is smooth up to $x_1 = f(x',0)+R$, we find that $\nabla u \in C(\overline{\mathcal{D}}_R)$.

From classical Schauder estimates we deduce that, when $a_{ij} \in C^{1+\mu,\frac{1+\mu}{2}}(\mathcal{D}_R)$, and $b_i,\ c,\ g \in C^{\mu,\frac{\mu}{2}}(\mathcal{D}_R)$, there holds that $u \in C^{2+\mu,1+\frac{\mu}{2}}(\mathcal{D}_R)$.

This completes the proof. $\square$

Remark 1.2. We want to point out that, clearly, in Proposition 1.2 we do not need the assumption that $\psi_{i_t} \in L^2(\Sigma\times(0,T))$, $\nabla_{x'}\psi_i \in L^2(\Sigma\times(0,T))$ for $i=1,2$ to prove that $\nabla u \in C(\mathcal{D}_R \cup \partial_N\mathcal{D}_R)$.

On the other hand, in the same proposition, the assumption that $\psi_{2t} \in L^2(\Sigma\times(0,T))$, $\nabla_{x'}\psi_2 \in L^2(\Sigma\times(0,T))$ is not necessary to prove that ∇u is continuous in $\overline{\mathcal{D}}_R \cap \{x_1 < f(x',t)+R\}$. This follows from the fact that it is enough to have $\psi_{1t} \in L^2(\Sigma\times(0,T))$, $\nabla_{x'}\psi_1 \in L^2(\Sigma\times(0,T))$ to show that the solution to (1.1) satisfies $\nabla u \in L^2(\mathcal{D}_{R_0})$ $\forall R_0 < R$. And this can be seen by approximating the Dirichlet datum by smooth functions as in Proposition 1.1 and then using the estimates that are proven in Theorem 2.1.

Remark 1.3. We point out that the constant C and the function $\omega(r)$ appearing in (1.5) depend only on T, R_0, λ, $||a_{ij}||_{L^\infty(\mathcal{D}_{R_0})}$, $||b_i||_{L^\infty(\mathcal{D}_{R_0})}$, $||c||_{L^\infty(\mathcal{D}_{R_0})}$, $||\nabla\phi||_{L^\infty(\overline{\mathcal{D}}_{R_0}\cap\{t=0\})}$, $||\psi_1||_{C^1(\overline{\Sigma}\times[0,T])}$, the Dini modulus of continuity of $\nabla_{x'}\psi_1$ in $\overline{\Sigma}\times[0,T]$, of $\nabla\phi$ on $\overline{\mathcal{D}}_{R_0}\cap\{t=0\}$ and of a_{ij} in $\overline{\mathcal{D}}_{R_0}$, $||g||_{L^\infty(\mathcal{D}_{R_0})}$, the domain Σ, the function f and $||u||_{L^\infty(\mathcal{D}_{R_0})}$.

It is now easy to prove our first existence, uniqueness and regularity result for the semilinear problem.

Theorem 1.1. *Let $\mathcal{D}_R$, $\partial_N\mathcal{D}_R$, $\partial_D\mathcal{D}_R$, $\partial_S\mathcal{D}_R$, a_{ij}, b_i, c and ϕ as in Proposition 1.1. Let $\beta(x,t,u) \in L^\infty(\mathcal{D}_R\times\mathbb{R})$ be such that $\beta(x,t,\cdot)$ is locally Lipschitz continuous in $\mathbb{R}$ uniformly for $(x,t)\in\mathcal{D}_R$.*

There exists a unique function $u \in C^{\gamma,\frac{\gamma}{2}}(\overline{\mathcal{D}}_R)$ for some $0 < \gamma \le \alpha$, with $\nabla u \in L^2_{loc}(\overline{\mathcal{D}}_R \setminus \partial_S \mathcal{D}_R)$, such that u is a weak solution to the following problem

$$
\begin{aligned}
\sum_{i,j} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + \sum_i b_i \frac{\partial u}{\partial x_i} + c\,u - u_t = \beta(x,t,u) \quad &\text{in } \mathcal{D}_R \\
\sum_{i,j} a_{ij}\frac{\partial u}{\partial x_j}\eta_i = 0 \quad &\text{on } \partial_N \mathcal{D}_R \\
u = \phi \quad &\text{on } \partial_D \mathcal{D}_R.
\end{aligned} \tag{1.8}
$$

Now let $\psi_1(x',t) = \phi(f(x',t),x',t), \psi_2(x',t) = \phi(f(x',t)+R,x',t)$ and assume, in addition, that $\psi_{it} \in L^2(\Sigma \times (0,T))$, $\nabla_{x'}\psi_i \in L^2(\Sigma \times (0,T))$ for $i = 1,2$. Then $\nabla u \in L^2(\mathcal{D}_R)$.

Moreover, Propositions 1.1 and 1.2 apply to u. In particular, further assumptions on the coefficients a_{ij} and on the Dirichlet data ψ_i and/or ϕ give regularity results for ∇u up to the corresponding subset of the Dirichlet boundary, $\partial_D \mathcal{D}_R$.

If, in addition, $\beta(\cdot,\cdot,u) \in C^{\mu,\frac{\mu}{2}}(\mathcal{D}_R)$ uniformly for u in compact subsets of $\mathbb{R}$, $a_{ij} \in C^{1+\mu,\frac{1+\mu}{2}}(\mathcal{D}_R)$, $b_i,\ c \in C^{\mu,\frac{\mu}{2}}(\mathcal{D}_R)$, there holds that $u \in C^{2+\mu,1+\frac{\mu}{2}}(\mathcal{D}_R)$.

Proof. We use Proposition 1.1 and Schauder's fixed point Theorem.

In fact, let $B = ||\beta||_{L^\infty}$, and let γ and C be the constants given by Proposition 1.1 when ϕ is fixed and $||g||_{L^\infty(\mathcal{D}_R)} \le B$.

Let $0 < \nu < \gamma$ and let $K_R > 0$ be such that $||\cdot||_{C^{\nu,\frac{\nu}{2}}(\overline{\mathcal{D}}_R)} \le K_R ||\cdot||_{C^{\gamma,\frac{\gamma}{2}}(\overline{\mathcal{D}}_R)}$. Let us consider the set

$$\mathcal{B} = \{v \in C^{\nu,\frac{\nu}{2}}(\overline{\mathcal{D}}_R) \,/\, ||v||_{C^{\nu,\frac{\nu}{2}}(\overline{\mathcal{D}}_R)} \le K_R\, C\}.$$

Let $\mathcal{T}$ be defined on $\mathcal{B}$ by $\mathcal{T}v := u$ where u is the unique solution given by Proposition 1.1 when $g(x,t) = \beta(x,t,v(x,t))$. Then

$$||\mathcal{T}v||_{C^{\nu,\frac{\nu}{2}}(\overline{\mathcal{D}}_R)} \le K_R ||\mathcal{T}v||_{C^{\gamma,\frac{\gamma}{2}}(\overline{\mathcal{D}}_R)} \le K_R\, C.$$

Therefore, $\mathcal{T}$ maps $\mathcal{B}$ continuously into a compact subset of $\mathcal{B}$. So that $\mathcal{T}$ has a fixed point u which clearly is a solution to (1.8).

In order to prove uniqueness, we let u_1 and u_2 be solutions to (1.8). Then $w = u_1 - u_2$ is a solution to (1.1) with a different coefficient c (which depends on u_1 and u_2), and with $g = \phi = 0$. By Proposition 1.1, $w = 0$.

If, in addition, $\beta(\cdot,\cdot,u) \in C^{\mu,\frac{\mu}{2}}(\mathcal{D}_R)$ uniformly for u in compact subsets of $\mathbb{R}$, there holds that u is a solution of (1.1) with $g \in C^{\gamma',\frac{\gamma'}{2}}(\mathcal{D}_R)$, with $\gamma' = \min\{\mu,\gamma\}$. Then, if $a_{ij} \in C^{1+\mu,\frac{1+\mu}{2}}(\mathcal{D}_R)$, $b_i,\ c \in C^{\mu,\frac{\mu}{2}}(\mathcal{D}_R)$, there holds that $u \in C^{2+\gamma',1+\frac{\gamma'}{2}}(\mathcal{D}_R)$, so that $g \in C^{\mu,\frac{\mu}{2}}(\mathcal{D}_R)$ and we deduce that $u \in C^{2+\mu,1+\frac{\mu}{2}}(\mathcal{D}_R)$. $\square$

Section 2

In this section we prove an existence, uniqueness and regularity result for the semilinear problem considered in Section 1 in an unbounded domain.

We will denote throughout this section

$$\mathcal{D} := \{(x,t) \in \Omega \times (0,T) \,/\, x_1 > f(x',t)\},$$

and correspondingly, we will let

$$\begin{aligned}
&\partial_N \mathcal{D} := \{(x,t) \in \partial_p \mathcal{D} \,/\, x' \in \partial\Sigma, 0 < t \le T, x_1 > f(x',t)\},\\
&\partial_D \mathcal{D} := \partial_p \mathcal{D} \setminus \partial_N \mathcal{D},\\
&\partial_S \mathcal{D} := \{(x,t) \in \partial_p \mathcal{D} \,/\, x_1 = f(x',t)\}.
\end{aligned}$$

The following result holds in $\mathcal{D}$:

Theorem 2.1. *Let $\mathcal{D}$, $\partial_N\mathcal{D}$, $\partial_D\mathcal{D}$ and $\partial_S\mathcal{D}$ as above. For $i,j = 1,\cdots,N$, let a_{ij}, b_i, $c \in L^\infty(\mathcal{D})$. Assume that $a_{ij}(x,t)\xi_i\xi_j \ge \lambda|\xi|^2$ for some $\lambda > 0$ and every $\xi \in \mathbb{R}^N$, $(x,t) \in \mathcal{D}$. Let $\phi \in C^{\alpha,\frac{\alpha}{2}}(\overline{\mathcal{D}}) \cap L^\infty(\partial_D\mathcal{D})$ and let $\beta(x,t,u) \in L^\infty(\mathcal{D}\times\mathbb{R})$ be such that $\beta(x,t,\cdot)$ is Lipschitz continuous in $\mathbb{R}$ uniformly for $(x,t) \in \mathcal{D}$.*

There exists a unique function $u \in L^\infty(\mathcal{D}) \cap C(\overline{\mathcal{D}})$, with $\nabla u \in L^2_{\text{loc}}(\overline{\mathcal{D}} \setminus \partial_S\mathcal{D})$, such that u is a weak solution to the following problem

$$\begin{aligned}
&\sum_{i,\,j} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + \sum_i b_i\frac{\partial u}{\partial x_i} + c\,u - u_t = \beta(x,t,u) \quad \text{in } \mathcal{D}\\
&\sum_{i,j} a_{ij}\frac{\partial u}{\partial x_j}\eta_i = 0 \qquad \text{on } \partial_N\mathcal{D}\\
&u = \phi \qquad \text{on } \partial_D\mathcal{D}.
\end{aligned} \tag{2.1}$$

Let $\psi_1(x',t) = \phi(f(x',t),x',t)$. Assume that $\nabla_{x'}\psi_1$, $\psi_{1t} \in L^2(\Sigma\times(0,T))$. Then, $\nabla u \in L^2_{\text{loc}}(\overline{\mathcal{D}})$.

Moreover, if $a_{ij} \in C^{dini}(\overline{\mathcal{D}})$, $a_{ij} = \delta_{ij}$ on $\partial_N\mathcal{D}$, there holds that $\nabla u \in C(\mathcal{D} \cup \partial_N\mathcal{D})$. If, in addition, $\psi_1 \in C^1(\overline{\Sigma}\times[0,T])$, $\nabla_{x'}\psi_1 \in C^{dini}(\overline{\Sigma}\times[0,T])$ and $\frac{\partial\psi_1}{\partial\eta'} = 0$ on $\partial\Sigma\times(0,T)$, there holds that $\nabla u \in C(\overline{\mathcal{D}} \cap \{t > 0\})$. And if, moreover, $\nabla\phi \in C^{dini}(\overline{\mathcal{D}} \cap \{t = 0\})$, with $\frac{\partial\phi}{\partial\eta} = 0$ on $\overline{\partial_N\mathcal{D}} \cap \{t = 0\}$, there holds that $\nabla u \in C(\overline{\mathcal{D}})$.

On the other hand, if $a_{ij} \in C^{1+\mu,\frac{1+\mu}{2}}(\mathcal{D})$, b_i, $c \in C^{\mu,\frac{\mu}{2}}(\mathcal{D})$ and $\beta(\cdot,\cdot,u) \in C^{\mu,\frac{\mu}{2}}(\mathcal{D})$ uniformly for u in compact subsets of $\mathbb{R}$, there holds that $u \in C^{2+\mu,1+\frac{\mu}{2}}(\mathcal{D})$.

Proof. Let $R > 0$ and let $u_R \in C(\overline{\mathcal{D}}_R)$, with $\nabla u_R \in L^2_{\rm loc}(\overline{\mathcal{D}}_R \backslash \partial_S \mathcal{D}_R)$, be the unique weak solution to

$$\sum_{i,j} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + \sum_i b_i \frac{\partial u}{\partial x_i} + c\,u - u_t = \beta(x,t,u) \quad \text{in } \mathcal{D}_R$$

$$\sum_{i,j} a_{ij}\frac{\partial u}{\partial x_j}\eta_i = 0 \qquad \text{on } \partial_N \mathcal{D}_R$$

$$u = \phi_R \qquad \text{on } \partial_D \mathcal{D}_R$$

with $\phi_R(x,t) = \phi(x,t)\xi(\frac{4}{R}(x_1 - f(x',t)))$. Here $\xi \in C^\infty(\mathbb{R})$ is such that $0 \le \xi \le 1$, $\xi \equiv 1$ for $x_1 < 1$ and $\xi \equiv 0$ for $x_1 > 2$.

So that $\phi_R \in C^{\alpha,\frac{\alpha}{2}}(\overline{\mathcal{D}}_R)$ with $||\phi_R||_{L^\infty(\partial_D \mathcal{D}_R)} \le ||\phi||_{L^\infty(\partial_D \mathcal{D})}$ and $\phi_R(x,t) = 0$ for $x_1 = f(x',t) + R$.

By using the maximum principle for weak solutions to (1.6) we deduce that

$$||u_R||_{L^\infty(\mathcal{D}_R)} \le C_1(T, ||\phi||_{L^\infty(\partial_D \mathcal{D})}, ||\beta||_{L^\infty(\mathcal{D}\times\mathbb{R})}, ||c||_{L^\infty(\mathcal{D})}).$$

Thus, by Proposition 1.1, there holds that for every $0 < R_0 < R/8$ there exist $0 < \gamma \le \alpha$, $C_2 > 0$ depending only on R_0, λ, $||a_{ij}||_{L^\infty(\mathcal{D})}$, $||b_i||_{L^\infty(\mathcal{D})}$, $||c||_{L^\infty(\mathcal{D})}$, $||\beta||_{L^\infty(\mathcal{D}\times\mathbb{R})}$, $||\phi||_{C^{\alpha,\frac{\alpha}{2}}(\partial_D \mathcal{D})}$ T, Σ, f and α such that

$$||u_R||_{C^{\gamma,\frac{\gamma}{2}}(\overline{\mathcal{D}}_{R_0})} \le C_2.$$

Therefore, for every R_0 there exists a subsequence u_{R_j}, with $R_j \to \infty$, converging uniformly in $\overline{\mathcal{D}}_{R_0}$. By considering successively domains $\mathcal{D}_{R_0}$ with $R_0 \to \infty$ and a standard diagonal process we see that there exist a sequence u_{R_n} with $R_n \to \infty$ and a continuous function u such that

$$u_{R_n} \to u \qquad \text{uniformly on compact subsets of } \overline{\mathcal{D}}.$$

On the other hand, by (1.4), for every $\delta > 0$, $0 < R_0 < R/2$, there exists $C_3 > 0$ independent of R such that

$$||\nabla u_R||_{L^2(\mathcal{D}_{R_0} \cap \{x_1 > f(x',t)+\delta\})} \le C_3.$$

So that, for every $R_0 > 0$, $\delta > 0$,

$$\nabla u_{R_n} \to \nabla u \qquad \text{weakly in } L^2(\mathcal{D}_{R_0} \cap \{x_1 > f(x',t) + \delta\})$$

and we conclude that $u \in C(\overline{\mathcal{D}})$, with $\nabla u \in L^2_{\rm loc}(\overline{\mathcal{D}} \setminus \partial_S \mathcal{D})$, is a weak solution of (2.1).

Uniqueness follows as in Proposition 1.1. In fact, if u_1 and u_2 are bounded weak solutions to (2.1), for every $\varepsilon > 0$ the function $w = (u_1 - u_2)e^{-Lt} - \varepsilon$ satisfies that

$w \in C(\overline{\mathcal{D}} \cap \{x_1 \geq f(x',t)+\delta\})$, with $\nabla w \in L^2_{\rm loc}(\overline{\mathcal{D}} \cap \{x_1 \geq f(x',t)+\delta\})$ is a bounded weak solution to

$$\sum_{i,j} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + \sum_i b_i \frac{\partial u}{\partial x_i} + \tilde c\, u - u_t = \tilde g \qquad \text{in } \mathcal{D} \cap \{x_1 > f(x',t)+\delta\}$$

$$\sum_{i,j} a_{ij}\frac{\partial u}{\partial x_j}\eta_i = 0 \qquad \text{on } \partial_N\mathcal{D} \cap \{x_1 > f(x',t)+\delta\}$$

such that, when $\delta > 0$ is small enough, $w \leq 0$ on $\overline{\mathcal{D}} \cap \{x_1 \geq f(x',t)+\delta\} \cap \{t=0\}$ and on $\overline{\mathcal{D}} \cap \{x_1 = f(x',t)+\delta\}$.

We choose $L > 0$ large, so that $\frac12 L$ is a bound of $||c||_{L^\infty(\mathcal{D})}$ and of the Lipschitz constant of β. Thus, $\tilde c = c - \frac{\beta(x,t,u_1)-\beta(x,t,u_2)}{u_1-u_2} - L < 0$ and $\tilde g = -\tilde c\,\varepsilon > 0$.

Therefore, $w \leq 0$ in $\mathcal{D} \cap \{x_1 > f(x',t)+\delta\}$.

In fact, we apply the maximum principle for bounded weak solutions to (1.6) with $Q = \{(y,t)\,/\,y_1 > \delta\,,\, y' \in \Sigma\,,\, 0 < t < T\}$, $\bar c < 0$ and $\bar g > 0$.

Letting $\delta \to 0$ and then $\varepsilon \to 0$ we deduce that $u_1 \leq u_2$ in $\mathcal{D}$.

Now, let ψ_1 as in the statement and assume that $\nabla_{x'}\psi_1, {\psi_1}_t \in L^2(\Sigma \times (0,T))$. Let us prove that there exists a constant $C_4 > 0$, independent of R such that, for every $R_0 < R/2$,

$$||\nabla u_R||_{L^2(\mathcal{D}_{R_0})} \leq C_4. \tag{2.2}$$

In fact, we will prove a similar inequality for the solution $\bar u_R$ of (1.6) in $Q_{R_0} = (0,R_0) \times \Sigma \times (0,T)$.

We will follow the notation of Proposition 1.1.

Let $\bar v_R(y,t) = \bar u_R(y,t) - \bar\psi_1(y',t)$. Then, for every $\Psi \in C(\overline{Q}_R)$ such that $\Psi = 0$ on $y_1 = 0$, $y_1 = R$, $t = \tau$, $t = T$, with $\nabla\Psi$, $\Psi_t \in L^2(Q_R)$ there holds that in $Q_{R,\tau} = (0,R) \times \Sigma \times (\tau,T)$,

$$\begin{aligned}\int_{Q_{R,\tau}} \bar a_{ij}\frac{\partial \bar v_R}{\partial y_j}\Psi_{y_i} - \int_{Q_{R,\tau}} \bar b_i \frac{\partial \bar v_R}{\partial y_i}\Psi - \int_{Q_{R,\tau}} \bar c \bar v_R \Psi - \int_{Q_{R,\tau}} \bar v_R \Psi_t \\ = -\int_{Q_{R,\tau}} \bar\beta(y,t,\bar u_R)\Psi - \int_{Q_{R,\tau}} \bar a_{ij}\frac{\partial\bar\psi_1}{\partial y_j}\Psi_{y_i} + \int_{Q_{R,\tau}} \bar b_i \frac{\partial \bar\psi_1}{\partial y_i}\Psi \\ + \int_{Q_{R,\tau}} \bar c\bar\psi_1\Psi + \int_{Q_{R,\tau}} \bar\psi_1\Psi_t.\end{aligned}$$

Replacing $\Psi(y,t)$ by $\Psi^h(y,t) = \frac1h\int_{t-h}^t \Psi(y,s)ds$ with $2h < \tau$ and assuming that $\Psi \equiv 0$ for $t \geq T-\tau$ and for $t \leq \tau$, we have (here we let for each function $g(y,t)$, $g_h(y,t) = \frac1h\int_t^{t+h} g(y,s)ds$ and $Q_{R,\tau,h} = (0,R) \times \Sigma \times (\tau, T-h)$),

$$\begin{aligned}\int_{Q_{R,\tau,h}} (\bar a_{ij}\frac{\partial \bar v_R}{\partial y_j})_h\Psi_{y_i} - \int_{Q_{R,\tau,h}} (\bar b_i \frac{\partial \bar v_R}{\partial y_i})_h\Psi - \int_{Q_{R,\tau,h}} (\bar c \bar v_R)_h \Psi + \int_{Q_{R,\tau}} \frac{\partial(\bar v_R)_h}{\partial t}\Psi \\ = -\int_{Q_{R,\tau,h}} \bar\beta(y,t,\bar u_R)_h\Psi - \int_{Q_{R,\tau,h}} (\bar a_{ij}\frac{\partial\bar\psi_1}{\partial y_j})_h\Psi_{y_i} + \int_{Q_{R,\tau,h}} (\bar b_i \frac{\partial \bar\psi_1}{\partial y_i})_h\Psi \\ + \int_{Q_{R,\tau,h}} (\bar c\bar\psi_1)_h\Psi - \int_{Q_{R,\tau,h}} (\frac{\partial\bar\psi_1}{\partial t})_h\Psi.\end{aligned}$$

Let us take $\Psi = (\bar v_R)_h \varphi^2(y_1)\,\eta_\varepsilon(t)$ where $\varphi \in C^\infty(\mathbb{R})$, $\varphi \equiv 1$ if $y_1 \le R_0$ and $\varphi \equiv 0$ if $y_1 \ge 2R_0$, and $\eta_\varepsilon \in C^\infty(\mathbb{R})$, $\eta_\varepsilon \equiv 1$ if $t \in (\tau+\varepsilon, T-\tau-\varepsilon)$, $\eta_\varepsilon \equiv 0$ if $t \le \tau$ or $t \ge T-\tau$ and $|\eta_\varepsilon'| = \frac{1}{\varepsilon}$ in $(\tau, \tau+\varepsilon)$ and in $(T-\tau-\varepsilon, T-\tau)$. Then, since

$$\begin{aligned}
\int_{Q_{R,\tau,h}} \frac{\partial(\bar v_R)_h}{\partial t}(\bar v_R)_h\varphi^2\eta_\varepsilon &= \frac12\int_{Q_{R,\tau,h}} \frac{\partial(\bar v_R)_h^2}{\partial t}\varphi^2\eta_\varepsilon \\
&= -\frac12\int_{Q_{R,\tau,h}} (\bar v_R)_h^2\varphi^2\eta_\varepsilon' + \frac12\int_{(0,R)\times\Sigma}\left((\bar v_R)_h^2\varphi^2\eta_\varepsilon\right)_\tau^{T-h} \\
&= -\frac12\int_{Q_{R,\tau,h}} (\bar v_R)_h^2\varphi^2\eta_\varepsilon'
\end{aligned}$$

we get, after we let $h \to 0$,

$$\begin{aligned}
&\int_{Q_{R,\tau}} \bar a_{ij}\frac{\partial \bar v_R}{\partial y_j}\frac{\partial \bar v_R}{\partial y_i}\varphi^2\eta_\varepsilon + 2\int_{Q_{R,\tau}} \bar a_{1,j}\frac{\partial \bar v_R}{\partial y_j}\varphi\varphi'\bar v_R\,\eta_\varepsilon - \int_{Q_{R,\tau}} \bar b_i\frac{\partial \bar v_R}{\partial y_i}\bar v_R\varphi^2\eta_\varepsilon \\
&- \int_{Q_{R,\tau}} \bar c\bar v_R^2\varphi^2\eta_\varepsilon - \frac12\int_{Q_{R,\tau}} \bar v_R^2\varphi^2\eta_\varepsilon' = -\int_{Q_{R,\tau}} \bar\beta(y,t,\bar u_R)\bar v_R\varphi^2\eta_\varepsilon \\
&- \int_{Q_{R,\tau}} \bar a_{ij}\frac{\partial \bar\psi_1}{\partial y_j}\frac{\partial \bar v_R}{\partial y_i}\varphi^2\eta_\varepsilon - 2\int_{Q_{R,\tau}} \bar a_{1,j}\frac{\partial\bar\psi_1}{\partial y_j}\varphi\varphi'\bar v_R\,\eta_\varepsilon + \int_{Q_{R,\tau}} \bar b_i\frac{\partial\bar\psi_1}{\partial y_i}\bar v_R\varphi^2\eta_\varepsilon \\
&+ \int_{Q_{R,\tau}} \bar c\bar\psi_1\bar v_R\varphi^2\eta_\varepsilon - \int_{Q_{R,\tau}} \frac{\partial\bar\psi_1}{\partial t}\bar v_R\varphi^2\eta_\varepsilon.
\end{aligned} \tag{2.3}$$

We notice at this point that, as $\varepsilon \to 0$, we have

$$-\frac12\int_{Q_{R,\tau}} \bar v_R^2\varphi^2\eta_\varepsilon' \to \frac12\int_{(0,R)\times\Sigma}\left(\bar v_R^2\varphi^2\right)_\tau^{T-\tau}.$$

Using the ellipticity of (a_{ij}) and Young's inequality we get from (2.3) after we let $\varepsilon \to 0$,

$$\begin{aligned}
\int_{Q_{R,\tau}} |\nabla\bar v_R|^2\varphi^2 \le{}& L\int_{Q_{R,\tau}} \bar v_R^2{\varphi'}^2 + L\int_{Q_{R,\tau}}\bar v_R^2\varphi^2 \\
&+ \frac12\int_{(0,R)\times\Sigma}\bar v_R^2\varphi^2(y,T-\tau) + \frac12\int_{(0,R)\times\Sigma}\bar v_R^2\varphi^2(y,\tau) + L\int_{Q_{R,\tau}}|\nabla\bar\psi_1|^2\varphi^2 \\
&+ L\int_{Q_{R,\tau}}|\nabla\bar\psi_1|^2{\varphi'}^2 + L\int_{Q_{R,\tau}}|\bar\psi_1||\bar v_R|\varphi^2 + L\int_{Q_{R,\tau}}\left|\frac{\partial\bar\psi_1}{\partial t}\right||\bar v_R|\varphi^2.
\end{aligned}$$

Here $L = L(||\bar a_{ij}||_{L^\infty}, ||\bar b_i||_{L^\infty}, ||\bar c||_{L^\infty}, ||\bar\beta||_{L^\infty}, \lambda)$.

Therefore,

$$||\nabla\bar v_R||_{L^2(Q_{R_0})} \le \bar C_4$$

where $\bar C_4$ depends on the same constants as L and on $||\phi||_{L^\infty(\partial_D\mathcal{D})}$, T, $||\nabla\bar\psi_1||_{L^2(\Sigma\times(0,T))}$, $||\frac{\partial\bar\psi_1}{\partial t}||_{L^2(\Sigma\times(0,T))}$ and R_0.

We deduce that (2.2) holds with C_4 independent of R.

Therefore, $\nabla u \in L^2_{\mathrm{loc}}(\overline{\mathcal{D}})$.

The continuity of ∇u in $\mathcal{D} \cup \partial_N \mathcal{D}$ when a_{ij} are smooth, $a_{ij} = \delta_{ij}$ on $\partial_N \mathcal{D}$ and in $\overline{\mathcal{D}} \cap \{t > 0\}$ when, in addition, ψ_1 is smooth and $\frac{\partial \psi_1}{\partial \eta'} = 0$ on $\partial\Sigma \times (0, T)$, as well as the continuity of ∇u in $\overline{\mathcal{D}}$ when, moreover, ϕ is smooth and $\frac{\partial \phi}{\partial \eta} = 0$ on $\overline{\partial_N \mathcal{D}} \cap \{t = 0\}$ are consequences of the results of Proposition 1.2.

On the other hand, as in Theorem 1.1, standard Schauder regularity results imply that $u \in C^{2+\mu, 1+\frac{\mu}{2}}(\mathcal{D})$ when $a_{ij} \in C^{1+\mu, \frac{1+\mu}{2}}(\mathcal{D})$, $b_i \in C^{\mu, \frac{\mu}{2}}(\mathcal{D})$, $c \in C^{\mu, \frac{\mu}{2}}(\mathcal{D})$ and $\beta(\cdot, \cdot, u) \in C^{\mu, \frac{\mu}{2}}(\mathcal{D})$ uniformly for u in compact subsets of $\mathbb{R}$. $\square$

Remark 2.1. If u is as in Theorem 2.1, it can be seen that there exists $0 < \gamma \leq \alpha$ such that $u \in C^{\gamma, \frac{\gamma}{2}}(\overline{\mathcal{D}})$. In fact, far from $\partial_S \mathcal{D}$ and $\partial_N \mathcal{D}$, the result is standard. In a neighborhood of $\partial_S \mathcal{D}$, the result follows by construction of u (it is a consequence of (1.3)). And finally, in a neighborhood of $\partial_N \mathcal{D} \cap \{x_1 > f(x', t) + \delta\}$, we proceed by straightening $\partial_N \mathcal{D}$ with similar arguments as in Proposition 1.1.

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