A FREE BOUNDARY PROBLEM IN COMBUSTION THEORY

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ABSTRACT. In this paper we consider the following problem arising in combustion theory

$$\begin{cases} \Delta u^\varepsilon - u^\varepsilon_t &= v^\varepsilon f_\varepsilon(u^\varepsilon) \quad \text{in } \mathcal{D}, \\ \Delta v^\varepsilon - v^\varepsilon_t &= v^\varepsilon f_\varepsilon(u^\varepsilon) \quad \text{in } \mathcal{D}, \end{cases}$$

where $\mathcal{D} \subset \mathbb{R}^{N+1}$, $f_\varepsilon(s) = \frac{1}{\varepsilon^2} f(\frac{s}{\varepsilon})$ with f a Lipschitz continuous function with support in $(-\infty, 1]$.

Here v^ε is the mass fraction of some reactant, u^ε the rescaled temperature of the mixture and ε is essentially the inverse of the activation energy. This model is derived in the framework of the theory of equidiffusional premixed flames for Lewis number 1.

We prove that, under suitable assumptions on the functions u^ε and v^ε, we can pass to the limit ($\varepsilon \to 0$) – the so called high activation energy limit – and that the limit function $u = \lim u^\varepsilon = \lim v^\varepsilon$ is a solution of the following free boundary problem

$$\begin{aligned} &\Delta u - u_t = 0 \quad \text{in } \{u > 0\} \\ &|\nabla u| = \sqrt{2M(x,t)} \quad \text{on } \partial\{u > 0\} \end{aligned} \tag{0.1}$$

in a pointwise sense at regular free boundary points and in a viscosity sense. Here $M(x,t) = \int_{-w_0(x,t)}^1 (s + w_0(x,t)) f(s)\,ds$ and $-1 < w_0 = \lim_{\varepsilon\to 0} \frac{v^\varepsilon - u^\varepsilon}{\varepsilon}$.

Since $v^\varepsilon - u^\varepsilon$ is a solution of the heat equation it is fully determined by its initial-boundary datum. In particular, the free boundary condition only (but strongly) depends on the approximation of the initial-boundary datum.

Moreover, if $\mathcal{D} \cap \partial\{u > 0\}$ is a Lipschitz surface, u is a classical solution to (0.1).

1. Introduction

In this paper we consider the following problem arising in combustion theory

$$\begin{cases} \Delta u^\varepsilon - u^\varepsilon_t &= v^\varepsilon f_\varepsilon(u^\varepsilon) \quad \text{in } \mathcal{D}, \\ \Delta v^\varepsilon - v^\varepsilon_t &= v^\varepsilon f_\varepsilon(u^\varepsilon) \quad \text{in } \mathcal{D}, \end{cases} \tag{1.1}$$

where $\mathcal{D} \subset \mathbb{R}^{N+1}$.

This model appears in combustion theory in the analysis of the propagation of curved flames. It is derived in the framework of the theory of equidiffusional premixed flames analyzed in the relevant limit of high activation energy for Lewis number 1. In this application, v^ε represents the fraction of some reactant (and hence it is assumed to be nonnegative), and u^ε its temperature (more precisely, $u^\varepsilon = \lambda(T_f - T^\varepsilon)$ where T_f is the

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flame temperature and λ is a normalization factor). Observe that the term $v^\varepsilon f_\varepsilon(u^\varepsilon)$ acts as an absorption term in the equation (1.1). Since $T^\varepsilon = T_f - (u^\varepsilon/\lambda)$, it is in fact a reaction term for the temperature. In the flame model, such a term represents the effect of the exothermic chemical reaction and f has accordingly a number of properties: it is a nonnegative Lipschitz continuous function which is positive in an interval $(-\infty, \varepsilon)$ and vanishes otherwise (i.e., reaction occurs only when $T > T_f - \frac{\varepsilon}{\lambda}$). The parameter ε is essentially the inverse of the activation energy of the chemical reaction. For the sake of simplicity we will assume that $f_\varepsilon(s) = \frac{1}{\varepsilon^2} f(\frac{s}{\varepsilon})$, where f is a nonnegative, Lipschitz continuous function with support in $(-\infty, 1]$.

For the derivation of the model, we cite [3].

Here we are interested in high activation energy limits (i.e. $\varepsilon \to 0$). These limits, are currently the subject of active investigation, specially in the case $u^\varepsilon = v^\varepsilon$. This is a natural assumption in the case of traveling waves.

The study of the limit as $\varepsilon \to 0$ was proposed in the 30's by Zeldovich and Frank-Kamenetski [13] and has been much discussed in the combustion literature. In the case $u^\varepsilon = v^\varepsilon$ the reaction function $u^\varepsilon f_\varepsilon(u^\varepsilon)$ tends to a Dirac delta, $M_0 \delta(u)$ where $M_0 = \int_0^1 s f(s)\,ds$. In this way the reaction zone where $u^\varepsilon f_\varepsilon(u^\varepsilon)$ acts is reduced to a surface, the flame front, and a free boundary problem arises. The fact that $M_0 > 0$ ensures that a nontrivial combustion process takes place so that a non-empty free boundary actually appears.

Although the convergence of the most relevant propagation modes, i.e. the traveling waves, was already discussed by Zeldovich and Frank-Kamenetski, and an enormous progress in this direction has been made, a rigorous mathematical investigation of the convergence of general solutions is still in progress. Berestycki and his collaborators have rigorously studied the convergence problem for traveling waves and, more generally in the elliptic stationary case, cf. [2] and its references. See also [12]. The study of the limit in the general evolution case for the heat operator has been performed in [7] for the one phase case (this is, with $u^\varepsilon \geq 0$) and in [4, 5, 6] for the two-phase case, where no sign restriction on u^ε is made.

In [7] the authors show that, under certain assumptions on the initial datum and its approximations, for every sequence $\varepsilon_n \to 0$ there exists a subsequence ε_{n_k} and a limit function $u = \lim u^{\varepsilon_{n_k}}$ which solves the following free boundary problem

$$\begin{cases} \Delta u - u_t = 0 & \text{in } \mathcal{D} \cap \{u > 0\} \\ |\nabla u^+| = \sqrt{2M_0} & \text{on } \mathcal{D} \cap \partial\{u > 0\} \end{cases} \tag{1.2}$$

in a weak integral sense. Here $M_0 = \int_0^1 s f(s)\,ds$.

In [5] and [6] the authors show that the free boundary condition for the two phase case (when it is assumed that no reaction takes place if $u^\varepsilon \leq 0$) is

$$|\nabla u^+|^2 - |\nabla u^-|^2 = 2M_0$$

and that the limit function is a solution of the free boundary problem in a pointwise sense at regular free boundary points when $\{u = 0\}$ has zero "parabolic density" and in a viscosity sense in the absence of a zero phase (i.e. when $\{u = 0\}^\circ \cap \mathcal{D} = \emptyset$)

On the other hand, in [10] it was shown for the one phase problem in a cylinder with Neumann boundary conditions, that when a classical solution to the free boundary problem (1.2) exists, it is the limit of the whole family u^ε, not just of a subsequence. Moreover, this classical solution is the limit of u^ε independently of the choice of the approximate initial data $u^\varepsilon(x, 0)$. A similar result has been obtain for the two phase case in [11].

So that a natural question is if a classical solution to the free boundary problem (1.2) is also the limit of u^ε if $(u^\varepsilon, v^\varepsilon)$ is the solution to the system (1.1) and both $u^\varepsilon(x, 0)$ and $v^\varepsilon(x, 0)$ converge to $u(x, 0)$ but $u^\varepsilon(x, 0) \neq v^\varepsilon(x, 0)$.

Or we may ask a more elementary question: Will a sequence of uniformly bounded solutions $(u^\varepsilon, v^\varepsilon)$ with $(v^\varepsilon - u^\varepsilon) \to 0$ as $\varepsilon \to 0$ be such that u^ε converges to a solution of the free boundary problem (1.2)? This is, will the asymptotic limit for activation energy going to infinity, in the case in which $(v^\varepsilon - u^\varepsilon) \to 0$ but $u^\varepsilon \neq v^\varepsilon$, be a solution of the same free boundary problem as in the case in which $u^\varepsilon = v^\varepsilon$?

In order to understand the relation between both assertions it is important to point out that in the case under consideration this is, when Lewis number is 1, the function $w^\varepsilon = v^\varepsilon - u^\varepsilon$ is a solution of the heat equation. So that it is fully determined by its initial-boundary datum. Moreover, the system (1.1) may be rewritten as a single equation for u^ε, namely

$$\Delta u^\varepsilon - u_t^\varepsilon = (u^\varepsilon + w^\varepsilon) f_\varepsilon(u^\varepsilon). \tag{1.3}$$

In this paper we consider the case in which $w^\varepsilon/\varepsilon$ converges to a function w_0 (so that in particular, $v^\varepsilon - u^\varepsilon \to 0$). In this way, at least formally, the reaction term still converges to a delta function and a free boundary problem appears. *But we prove that the free boundary condition strongly depends on the limit function w_0, so that it is different for different approximations of the initial-boundary datum of u.*

In fact, we prove that for every sequence $\varepsilon_n \to 0$ there exists a subsequence ε_{n_k} and a limit function $u = \lim u^{\varepsilon_{n_k}}$ which is a solution of the following free boundary problem

$$\begin{cases} \Delta u - u_t = 0 & \text{in } \mathcal{D} \cap \{u > 0\} \\ |\nabla u^+| = \sqrt{2M(x,t)} & \text{on } \mathcal{D} \cap \partial\{u > 0\} \end{cases} \tag{1.4}$$

where $M(x,t) = \int_{-w_0(x,t)}^1 \Big(s + w_0(x,t)\Big) f(s)\,ds$.

The presence of the function w_0 in the limit of integration gives the necessary positive sign of the function $M(x,t)$.

In conclusion, *the combustion problem is very unstable in the sense that the asymptotic limit for activation energy going to infinity depends on order ε perturbations of the initial-boundary data.*

In this paper we prove that the limit function u is a "viscosity" solution to (1.4), so that, as a consequence of our results and of the regularity results for viscosity solutions to (1.4) of [8], we deduce that, when the free boundary of a limit function u is given by $x_1 = g(x', t)$, $x = (x_1, x')$ with g Lipschitz continuous, u is a classical solution.

We want to stress, that because of our assumption that $v^\varepsilon - u^\varepsilon \to 0$ and since $v^\varepsilon \geq 0$, the limit function u must be nonnegative, so our result is new even in the case $u^\varepsilon = v^\varepsilon$.

In particular, as a consequence of our results we see that limit functions u with $u^\varepsilon(x,0)$ constructed as in [7], and $v^\varepsilon(x,0)$ small perturbations of $u^\varepsilon(x,0)$ are viscosity solutions to (1.4). In this construction, w_0 is any constant such that $w_0 \geq -\eta$ where $\eta > 0$ is small enough.

Notation, hypotheses and outline of the paper

Throughout this paper N will denote the spatial dimension and, in addition, the following notation will be used:

For any $x_0 \in \mathbb{R}^N$, $t_0 \in \mathbb{R}$ and $\tau > 0$

$$\begin{aligned}
B_\tau(x_0) &\equiv \{x \in \mathbb{R}^N / \ |x - x_0| < \tau\}, \\
B_\tau(x_0, t_0) &\equiv \{(x,t) \in \mathbb{R}^{N+1} / \ |x - x_0|^2 + |t - t_0|^2 < \tau^2\}, \\
Q_\tau(x_0, t_0) &\equiv B_\tau(x_0) \times (t_0 - \tau^2, t_0 + \tau^2), \\
Q_\tau^-(x_0, t_0) &\equiv B_\tau(x_0) \times (t_0 - \tau^2, t_0],
\end{aligned}$$

and for any set $K \subset \mathbb{R}^{N+1}$

$$\begin{aligned}
\mathcal{N}_\tau(K) &\equiv \bigcup_{(x_0,t_0)\in K} Q_\tau(x_0, t_0), \\
\mathcal{N}_\tau^-(K) &\equiv \bigcup_{(x_0,t_0)\in K} Q_\tau^-(x_0, t_0).
\end{aligned}$$

When necessary, we will denote points in $\mathbb{R}^N$ by $x = (x_1, x')$, with $x' \in \mathbb{R}^{N-1}$. Also, $\langle \cdot, \cdot \rangle$ will mean the usual scalar product in $\mathbb{R}^N$. Given a function v, we will denote $v^+ = \max(v, 0)$, $v^- = \max(-v, 0)$.

In addition, the symbols Δ and ∇ will denote the corresponding operators in the space variables; the symbol ∂_p will denote parabolic boundary.

Finally, we will say that a function v is in the class $Lip_{loc}(1, 1/2)$ in a domain $\mathcal{D} \subset \mathbb{R}^{N+1}$, if for every $\mathcal{D}' \subset\subset \mathcal{D}$, there exists a constant $L = L(\mathcal{D}')$ such that

$$|v(x,t) - v(y,s)| \leq L(|x - y| + |t - s|^{1/2})$$

for every $(x,t), (y,s) \in \mathcal{D}'$. If the constant L does not depend on the set $\mathcal{D}'$, we will say that $v \in Lip(1, 1/2)$ in $\mathcal{D}$.

For the existence of a limit function for a subsequence $u^{\varepsilon_{n_k}}$ we only need the weaker condition that for every compact $K \subset \mathcal{N}_\tau^-(K) \subset \mathcal{D}$,

$$\|v^\varepsilon - u^\varepsilon\|_{L^\infty(\mathcal{N}_\tau^-(K))} = \mathrm{O}(\varepsilon). \tag{1.5}$$

Then, we have (see [9])

$$\|v^\varepsilon - u^\varepsilon\|_{C^{2,1}(K)} = \mathrm{O}(\varepsilon). \tag{1.6}$$

Under this assumption, we are able to apply the results of [4] and get the uniform Lipschitz estimates needed to pass to the limit in (1.1). This is done in §2 where we also prove some technical lemmas that are used throughout the paper.

In §3 and §4 we assume that $u^\varepsilon \to 0$ in $\{u = 0\}$ fast enough. This is an essential condition that was already considered in [7]. This assumption is a natural one in applications, roughly speaking it means that the mixture temperature reaches the flame temperature only if some combustion is taking place.

We also assume that there exists $\lim_{\varepsilon\to 0}(v^\varepsilon - u^\varepsilon)/\varepsilon =: w_0$ and, as a consequence of the hypothesis that $u^\varepsilon \to 0$ in $\{u = 0\}$ fast enough, we show that necessarily $w_0 > -1$ in $\{u \equiv 0\}^\circ$. So that, in §3 and §4 we assume that for every $K \subset \mathcal{N}_\tau^-(\mathcal{K}) \subset \mathcal{D}$ compact

$$\frac{v^\varepsilon - u^\varepsilon}{\varepsilon} \to w_0 \qquad \text{uniformly in } N_\tau^-(K). \tag{1.7}$$

Thus,

$$\Big\|\frac{v^\varepsilon - u^\varepsilon}{\varepsilon} - w_0\Big\|_{C^{2,1}(K)} \to 0. \tag{1.8}$$

And, for the sake of simplicity, we assume that $w_0 > -1$ in $\mathcal{D}$.

In §3, we show that the limit function u is a solution to the free boundary problem (1.4) in a pointwise sense.

Finally, in §4 we prove that the limit function u is in fact a viscosity solution of the free boundary problem (1.4) under a nondegeneracy assumption on the limit function u. We also prove some results that give the necessary nondegeneracy of u.

Our presentation is of a local nature, so that our hypotheses are stated in terms of the solution $(u^\varepsilon, v^\varepsilon)$. As can be seen in the example treated in Corollary 4.3 it is possible to deduce our hypotheses on $(u^\varepsilon, v^\varepsilon)$ from conditions on its initial-boundary datum.

2. Uniform estimates

In this section we consider a family $u^\varepsilon, v^\varepsilon$ of solutions to (1.1) in a domain $\mathcal{D}$ which are uniformly bounded in L^∞ norm in $\mathcal{D}$. We show that the functions $u^\varepsilon, v^\varepsilon$ are locally uniformly bounded in the seminorm $Lip(1, 1/2)$. Then, we get further local uniform estimates and pass to the limit as $\varepsilon \to 0$. We also show that the limit function u is a solution to the free boundary problem (1.4) in a very weak sense. Finally, we prove an approximation lemma that will be used throughout the rest of the paper and some lemmas concerning particular limit functions. Also, we prove a proposition that justifies the hypothesis we make in the following sections.

For convenience, let us define the following function

$$w^\varepsilon(x,t) = v^\varepsilon(x,t) - u^\varepsilon(x,t), \tag{2.1}$$

then, w^ε is a caloric function with $\|w^\varepsilon\|_{C^{2,1}(K)} = \mathrm{O}(\varepsilon)$ for every compact set $K \subset \mathcal{D}$.

We begin with a proposition (which is a consequence of [4]) that gives us the uniform control on the gradients of solutions of (1.1).

Proposition 2.1. *Let* $(u^\varepsilon, v^\varepsilon)$ *be solutions of* (1.1) *such that* $\|u^\varepsilon\|_\infty \le \mathcal{A}$, $v^\varepsilon \ge 0$ *and verify* (1.5). *Let* $K \subset \mathcal{D}$ *compact and* $\tau > 0$ *such that* $\mathcal{N}_\tau^-(K) \subset \mathcal{D}$. *Then, there exists* $L = L(\tau, \mathcal{A})$ *such that*

$$|\nabla u^\varepsilon(x,t)| \le L, \quad |\nabla v^\varepsilon(x,t)| \le L.$$

Proof. Let us start by making the following observation

$$u^\varepsilon = v^\varepsilon - w^\varepsilon \ge -w^\varepsilon \ge -C\varepsilon.$$

Then, let $z^\varepsilon = \frac{1}{C+1}(u^\varepsilon + C\varepsilon)$ and we define, for $(x_0,t_0) \in K$

$$z^\varepsilon_\tau(x,t) = \frac{1}{\tau} z^\varepsilon(x_0 + \tau x, t_0 + \tau^2 t).$$

In $B_1(0) \times [-1,0]$, z^ε_τ verifies (with $B \ge \|f\|_\infty$)

$$\begin{aligned} 0 \le \Delta z^\varepsilon_\tau - \frac{\partial z^\varepsilon_\tau}{\partial t} &\le \frac{\tau}{C+1}(C\varepsilon + |u^\varepsilon|)\frac{1}{\varepsilon^2} f(\frac{u^\varepsilon}{\varepsilon}) \\ &\le B\tau\frac{1}{\varepsilon}\mathcal{X}_{[-C\varepsilon,\varepsilon]}(u^\varepsilon) = \frac{B}{\varepsilon/\tau}\mathcal{X}_{[0,\varepsilon/\tau]}(z^\varepsilon_\tau). \end{aligned}$$

On the other hand

$$|z^\varepsilon_\tau(x,t)| \le \frac{|u^\varepsilon(x,t)| + C}{\tau(1+C)} \le \frac{1}{\tau}\frac{\mathcal{A}+C}{1+C}.$$

Therefore, by Theorem 2 of [4] it follows that

$$|\nabla z^\varepsilon_\tau(x,t)| \le \bar{L} = \bar{L}(\tau, \mathcal{A}) \qquad \text{in } B_{1/2}(0) \times (-1/2, 0].$$

In particular,

$$\begin{aligned} |\nabla u^\varepsilon(x_0,t_0)| &= (C+1)|\nabla z^\varepsilon(x_0,t_0)| = (C+1)|\nabla z^\varepsilon_\tau(0,0)| \le (C+1)\bar{L}, \\ |\nabla v^\varepsilon(x_0,t_0)| &\le |\nabla u^\varepsilon(x_0,t_0)| + |\nabla w^\varepsilon(x_0,t_0)| \le (C+1)\bar{L} + C. \end{aligned}$$

The proof is finished $\square$

As is usual in parabolic theory, Lipschitz regularity in space, gives Hölder 1/2 regularity in time. We follow here ideas in [5] and [7].

Proposition 2.2. *Let $(u^\varepsilon, v^\varepsilon)$ be solutions of (1.1) such that $\|u^\varepsilon\|_\infty \le \mathcal{A}$, $v^\varepsilon \ge 0$, and verify (1.5). Let $K \subset \mathcal{D}$ compact and $\tau > 0$ such that $\mathcal{N}_\tau(K) \subset \mathcal{D}$. Then there exists $C = C(\tau, \mathcal{A})$ such that*

$$|u^\varepsilon(x, t+\Delta t) - u^\varepsilon(x,t)| \le C|\Delta t|^{1/2}, \quad |v^\varepsilon(x, t+\Delta t) - v^\varepsilon(x,t)| \le C|\Delta t|^{1/2},$$

for every $(x,t), (x, t+\Delta t) \in K$.

Proof. As in Proposition 2.1 we define $z^\varepsilon = \frac{1}{C+1}(u^\varepsilon + C\varepsilon)$ and

$$z^\varepsilon_\lambda(x,t) = \frac{1}{\lambda} z^\varepsilon(x_0 + \lambda x, t_0 + \lambda^2 t),$$

for $0 < \lambda < \tau$ and $(x_0, t_0) \in K$.

By a simple computation we get, as in Prop. 2.1

$$0 \le \Delta z^\varepsilon_\lambda - \frac{\partial z^\varepsilon_\lambda}{\partial t} \le \frac{B}{\varepsilon/\lambda} \mathcal{X}_{[0,\varepsilon/\lambda]}(z^\varepsilon_\lambda).$$

Now, $z^\varepsilon_\lambda \ge 0$, and in $\{z^\varepsilon_\lambda > 1\}$ we have

$$|\Delta z^\varepsilon_\lambda - \frac{\partial z^\varepsilon_\lambda}{\partial t}| \begin{cases} \le B & \text{if } \varepsilon/\lambda \ge 1 \\ = 0 & \text{if } \varepsilon/\lambda < 1. \end{cases}$$

Moreover, we have that

$$|\nabla z^\varepsilon_\lambda(x,t)| = \frac{1}{C+1} |\nabla u^\varepsilon(x_0 + \lambda x, t_0 + \lambda^2 t)| \le \bar{L}$$

in $B_{\tau/\lambda}(0) \times [0, \tau^2/\lambda^2]$. Then, by Proposition 2.2 of [5], we have

$$|z^\varepsilon_\lambda(0,t) - z^\varepsilon_\lambda(0,0)| \le C(\bar{L}) \qquad \forall\ 0 \le t \le \frac{1}{4N+B}$$

which, in terms of u^ε, is

$$|u^\varepsilon(x_0, t_0 + \lambda^2 t) - u^\varepsilon(x_0, t_0)| \le C(\bar{L})\lambda.$$

In particular

$$|u^\varepsilon(x_0, t_0 + \frac{\lambda^2}{4N+B}) - u^\varepsilon(x_0, t_0)| \le C(\bar{L})\lambda.$$

Let $(x_0, t_0 + \Delta t) \in K$. If $0 < \Delta t < \frac{\tau^2}{4N+B}$, we take $\lambda = \Delta t^{1/2}\sqrt{4N+B} < \tau$ to get

$$|u^\varepsilon(x_0, t_0 + \Delta t) - u^\varepsilon(x_0, t_0)| \le C(\bar{L})\sqrt{4N+B}\,\Delta t^{1/2}.$$

If $\Delta t \ge \frac{\tau^2}{4N+B}$, we have

$$|u^\varepsilon(x_0, t_0 + \Delta t) - u^\varepsilon(x_0, t_0)| \le 2\mathcal{A} \le \frac{2\mathcal{A}}{\tau}\sqrt{4N+B}\,\Delta t^{1/2}.$$

The analogous inequality for v^ε is an immediate consequence of (1.6). ☐

Remark 2.1. *Under the hypothesis of the previous propositions, we have that*

$$u^\varepsilon \in Lip_{loc}(1, 1/2).$$

Proposition 2.3. *Let* $(u^\varepsilon, v^\varepsilon)$ *be solutions of* (1.1) *such that* $\|u^\varepsilon\|_\infty \le \mathcal{A}$, $v^\varepsilon \ge 0$ *and verify* (1.5). *Then, for every sequence* $\varepsilon_n \to 0$, *there exists* $\varepsilon_{n'} \to 0$ *a subsequence and* $u \in Lip_{loc}(1, 1/2)$ *such that*

1. $u^{\varepsilon_{n'}} \to u$ *uniformly on compacts subsets of* $\mathcal{D}$.
2. $\nabla u^{\varepsilon_{n'}} \to \nabla u$ *in* L^2_{loc}.
3. $\frac{\partial u^{\varepsilon_{n'}}}{\partial t} \to \frac{\partial u}{\partial t}$ *weakly in* L^2_{loc}.
4. $\Delta u - \frac{\partial u}{\partial t} = 0$ *in* $\{u > 0\}$
5. *For every compact* $K \subset \mathcal{D}$, *exists* $C_K > 0$ *such that*
$$\left\| \frac{\partial u^\varepsilon}{\partial t} \right\|_{L^2(K)} \le C_K$$
for every $\varepsilon > 0$.

Proof. The proof is similar to lemma 3.1 of [5].

Let $K \subset \mathcal{D}$ be a compact set, and $\tau > 0$ such that $\mathcal{N}_{3\tau}(K) \subset \mathcal{D}$. Let $L = L(K)$ such that

$$|u^\varepsilon(x,t) - u^\varepsilon(y,s)| \le L\left(|x-y| + |t-s|^{1/2}\right),$$

where $(x,t), (y,s) \in \mathcal{N}_\tau(K)$.

Then, by Arzela-Ascoli's theorem, there exists $\varepsilon_{n'} \to 0$ and $u \in Lip(1, 1/2)$ in $\mathcal{N}_\tau(K)$ such that $u^{\varepsilon_{n'}} \to u$ uniformly in $\mathcal{N}_\tau(K)$. By a standard diagonal argument, (1) follows.

Let us now find uniform bounds for $\frac{\partial u^\varepsilon}{\partial t}$ in $L^2_{\mathrm{loc}}(\mathcal{D})$. In fact, u^ε verifies

$$\Delta u^\varepsilon - \frac{\partial u^\varepsilon}{\partial t} = v^\varepsilon f_\varepsilon(u^\varepsilon).$$

Now, let $(x_0, t_0) \in K$ and let us multiply the equation by $u^\varepsilon_t \psi^2$ where $\psi \ge 0$, $\psi = \psi(x) \in C^\infty_c(B_\tau(x_0))$, $\psi \equiv 1$ in $B_{\tau/2}(x_0)$. Then, integrating by parts, we get

$$\begin{aligned}\iint_{Q_\tau(x_0,t_0)} (u^\varepsilon_t)^2 \psi^2\,dxdt + \frac{1}{2}\iint_{Q_\tau(x_0,t_0)} (|\nabla u^\varepsilon|^2)_t \psi^2\,dxdt + 2\iint_{Q_\tau(x_0,t_0)} \nabla u^\varepsilon u^\varepsilon_t \psi \nabla\psi\,dxdt \\ = -\iint_{Q_\tau(x_0,t_0)} v^\varepsilon f_\varepsilon(u^\varepsilon) u^\varepsilon_t \psi^2\,dxdt.\end{aligned}$$

Now we use Young's inequality to obtain

$$
\begin{aligned}
&\frac{1}{2}\iint_{Q_\tau(x_0,t_0)} (u_t^\varepsilon)^2\psi^2 dxdt + \frac{1}{2}\int_{B_\tau(x_0)} |\nabla u^\varepsilon(x_0, t_0+\tau^2)|^2\psi^2 dx \le \\
&\frac{1}{2}\int_{B_\tau(x_0)} |\nabla u^\varepsilon(x_0, t_0-\tau^2)|^2\psi^2 dx - \iint_{Q_\tau(x_0,t_0)} v^\varepsilon f_\varepsilon(u^\varepsilon) u_t^\varepsilon \psi^2 dxdt \\
&+ C\iint_{Q_\tau(x_0,t_0)} |\nabla u^\varepsilon|^2 |\nabla\psi|^2 dxdt.
\end{aligned}
$$

Then, by Proposition 2.1

$$
\begin{aligned}
&\int_{B_{\tau/2}(x_0)}\int_{t_0-\tau^2}^{t_0+\tau^2} (u_t^\varepsilon)^2 dxdt \le \int_{B_\tau(x_0)} |\nabla u^\varepsilon(x_0,t_0-\tau^2)|^2\psi^2 dx \\
&+ 2\left|\iint_{Q_\tau(x_0,t_0)} v^\varepsilon f_\varepsilon(u^\varepsilon) u_t^\varepsilon\psi^2 dxdt\right| + C\iint_{Q_\tau(x_0,t_0)} |\nabla u^\varepsilon|^2|\nabla\psi|^2 dxdt \\
&\le C(\tau) + 2\left|\iint_{Q_\tau(x_0,t_0)} v^\varepsilon f_\varepsilon(u^\varepsilon) u_t^\varepsilon\psi^2 dxdt\right|.
\end{aligned}
$$

Hence, it only remains to get bounds on

$$\iint_{Q_\tau} \psi^2 u_t^\varepsilon v^\varepsilon f_\varepsilon(u^\varepsilon) dxdt = I.$$

Let

$$\mathcal{G}_\varepsilon(u,x,t) = \int_0^u (w^\varepsilon(x,t)+s) f_\varepsilon(s) ds,$$

then

$$\frac{\partial}{\partial t}\left(\mathcal{G}_\varepsilon(u^\varepsilon,x,t)\right) = \frac{\partial u^\varepsilon}{\partial t} v^\varepsilon f_\varepsilon(u^\varepsilon) + \frac{\partial \mathcal{G}_\varepsilon}{\partial t}(u^\varepsilon,x,t),$$

so that we get

$$I = \iint_{Q_\tau} \psi^2 \frac{\partial}{\partial t}\left(\mathcal{G}_\varepsilon(u^\varepsilon,x,t)\right) dxdt - \iint_{Q_\tau}\psi^2 \frac{\partial \mathcal{G}_\varepsilon}{\partial t}(u^\varepsilon,x,t) dxdt = A - B.$$

Let us first get bounds on A:

$$
\begin{aligned}
A &= \int_{t_0-\tau^2}^{t_0+\tau^2}\int_{B_\tau(x_0)} \psi^2\frac{\partial}{\partial t}\left(\mathcal{G}_\varepsilon(u^\varepsilon,x,t)\right) dxdt = \int_{B_\tau(x_0)}\psi^2\left[\int_{t_0-\tau^2}^{t_0+\tau^2}\frac{\partial}{\partial t}\left(\mathcal{G}_\varepsilon(u^\varepsilon,x,t)\right) dt\right] dx \\
&= \int_{B_\tau(x_0)}\psi^2\left[\mathcal{G}_\varepsilon(u^\varepsilon(x,t_0+\tau^2),x,t_0+\tau^2) - \mathcal{G}_\varepsilon(u^\varepsilon(x,t_0-\tau^2),x,t_0-\tau^2)\right] dx.
\end{aligned}
$$

Since $u^\varepsilon \ge -C\varepsilon$, $f_\varepsilon(s) = 0$ if $s \ge \varepsilon$ and $|w^\varepsilon| = \mathrm{O}(\varepsilon)$, we have

$$|\mathcal{G}_\varepsilon(u^\varepsilon,x,t)| \le C\varepsilon\int_{-C\varepsilon}^{\varepsilon} f_\varepsilon(s)ds + \int_{-C\varepsilon}^{\varepsilon} s f_\varepsilon(s) ds \le C,$$

so that

$$|A| \le C(\tau).$$

It only remains to get bounds on B. For that purpose, let us first make the following observation:

$$\left|\frac{\partial \mathcal{G}_\varepsilon}{\partial t}(u^\varepsilon, x, t)\right| = \left|\frac{\partial w^\varepsilon}{\partial t}(x,t)\int_0^{u^\varepsilon} f_\varepsilon(s)\,ds\right| \le \frac{C}{\varepsilon}\left|\frac{\partial w^\varepsilon}{\partial t}(x,t)\right|.$$

By (1.6),

$$\left|\frac{\partial w^\varepsilon}{\partial t}\right| \le C\varepsilon \quad \text{ for } (x,t)\in \mathcal{N}_\tau(K).$$

Therefore, using the fact that $0\le\psi\le 1$, we get

$$B \le \frac{C}{\varepsilon}\iint_{Q_\tau}\left|\frac{\partial w^\varepsilon}{\partial t}(x,t)\right| dxdt \le \frac{C}{\varepsilon}|Q_\tau|\left|\frac{\partial w^\varepsilon}{\partial t}\right| \le C(K,\tau).$$

Thus,

$$\int_{B_{\tau/2}(x_0)}\int_{t_0-\tau^2}^{t_0+\tau^2}(u_t^\varepsilon)^2\,dxdt \le C,$$

with C independent of ε and $(x_0,t_0)\in K$. Now, as K is compact,

$$\iint_K (u_t^\varepsilon)^2\,dxdt \le C,$$

so that, for a subsequence, $\frac{\partial}{\partial t}u^{\varepsilon_{n'}} \to \frac{\partial}{\partial t}u$ weakly in $L^2(K)$ and by a standard diagonal argument, (3) follows.

Let us see that u is a solution of the heat equation in $\{u>0\}$. In fact, from the fact that $u^\varepsilon\to u$ uniformly on compact subsets of $\mathcal{D}$, we deduce that every point $(x_0,t_0)\in\{u>0\}$ has a neighborhood V such that $u^\varepsilon(x,t)\ge\lambda>0$ for some $\lambda>0$. Therefore, for $\varepsilon<\lambda$, $f_\varepsilon(u^\varepsilon(x,t))=0$ in V. Thus u^ε is caloric in V for every $\varepsilon<\lambda$, and then, the same fact holds for u.

Let us finally analyze the convergence of the gradients. We already know that $\|\nabla u^\varepsilon\|_{L^\infty(\mathcal{N}_\tau(K))}\le L$. So we can assume that $\nabla u^\varepsilon\to\nabla u$ weakly in $L^2(\mathcal{N}_\tau(K))$. In particular

$$\iint_{\mathcal{N}_\tau(K)}\phi|\nabla u|^2 \le \liminf_{\varepsilon\to 0}\iint_{\mathcal{N}_\tau(K)}\phi|\nabla u^\varepsilon|^2,$$

for every nonnegative $\phi\in L^\infty(\mathcal{D})$.

We follow here ideas from [2] and [7] in order to prove that we have strong convergence.

Since $\Delta u - u_t = 0$ in $\{u>0\}$, if we take $\delta>0$ and multiply this equation by $(u-\delta)^+\psi(x)$ with $\psi\in L^\infty(\mathcal{D})$ and nonnegative, we get after integration by parts in $Q_\tau(x_0,t_0)$,

$$\begin{aligned}\iint_{\{u>\delta\}}|\nabla u|^2\psi = &-\iint_{\{u>\delta\}} u\nabla u\nabla\psi + \delta\iint_{\{u>\delta\}}\nabla u\nabla\psi\\ &-\frac{1}{2}\int_{\{u>\delta\}}(u-\delta)^2(x,t_0+\tau^2)\psi(x) + \frac{1}{2}\int_{\{u>\delta\}}(u-\delta)^2(x,t_0-\tau^2)\psi(x).\end{aligned}$$

Now, letting $\delta \to 0$, we get

$$\iint_{\{u>0\}} |\nabla u|^2 \psi = -\iint_{\{u>0\}} u \nabla u \nabla \psi - \\ -\frac{1}{2}\int_{\{u>0\}} u^2(x, t_0+\tau^2)\psi(x) + \frac{1}{2}\int_{\{u>0\}} u^2(x, t_0-\tau^2)\psi(x).$$

On the other hand, since $\psi \geq 0$, $f_\varepsilon \geq 0$ and $u^\varepsilon \geq -C\varepsilon$, multiplying (1.1) by $(u^\varepsilon + C\varepsilon)\psi$ and integrating by parts we get

$$\iint_{Q_\tau(x_0,t_0)} |\nabla u^\varepsilon|^2 \psi \leq -\iint_{Q_\tau(x_0,t_0)} u^\varepsilon \nabla u^\varepsilon \nabla \psi - C\varepsilon \iint_{Q_\tau(x_0,t_0)} \nabla u^\varepsilon \nabla \psi \\ -\frac{1}{2}\int_{B_\tau(x_0)} (u^\varepsilon + C\varepsilon)^2(x, t_0+\tau^2)\psi(x) + \frac{1}{2}\int_{B_\tau(x_0)} (u^\varepsilon + C\varepsilon)^2(x, t_0-\tau^2)\psi(x).$$

Thus,

$$\limsup_{\varepsilon\to 0} \iint_{Q_\tau(x_0,t_0)} |\nabla u^\varepsilon|^2 \psi \leq \iint_{Q_\tau(x_0,t_0)} |\nabla u|^2 \psi,$$

so that

$$\|\psi^{1/2}\nabla u^\varepsilon\|_{L^2(Q_\tau(x_0,t_0))} \to \|\psi^{1/2}\nabla u\|_{L^2(Q_\tau(x_0,t_0))}.$$

Since, in addition,

$$\psi^{1/2}\nabla u^\varepsilon \to \psi^{1/2}\nabla u \text{ weakly in } L^2(Q_\tau(x_0,t_0)),$$

it follows that

$$\psi^{1/2}\nabla u^\varepsilon \to \psi^{1/2}\nabla u \text{ in } L^2(Q_\tau(x_0,t_0)).$$

Therefore, as $\psi \equiv 1$ in $B_{\tau/2}(x_0)$,

$$\nabla u^\varepsilon \to \nabla u \text{ in } L^2(Q_{\tau/2}(x_0,t_0))$$

and since K is compact, this implies that

$$\nabla u^\varepsilon \to \nabla u \text{ in } L^2(K).$$

By the same standard diagonal argument used before, the assertion of the theorem follows. □

Next we show that the limit function u is a solution of the free boundary problem in a very weak sense.

Proposition 2.4. *Let $(u^{\varepsilon_j}, v^{\varepsilon_j})$ be a family of solutions of* (1.1) *in a domain $\mathcal{D} \subseteq \mathbb{R}^{N+1}$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of $\mathcal{D}$, $v^{\varepsilon_j} \geq 0$ and verify* (1.5). *Then, there exists a locally finite measure μ supported on the free boundary $\mathcal{D} \cap \partial\{u>0\}$ such that $v^{\varepsilon_j} f_{\varepsilon_j}(u^{\varepsilon_j}) \to \mu$ weakly in $\mathcal{D}$ and therefore*

$$\Delta u - \frac{\partial u}{\partial t} = \mu \quad \text{in } \mathcal{D},$$

that is $\forall\ \phi \in C_c^\infty(\mathcal{D})$

$$\iint_{\mathcal{D}} (u\phi_t - \nabla u \nabla \phi)\,dxdt = \iint_{\mathcal{D}} \phi\ d\mu.$$

Proof. The proof follows as that of proposition 3.1 of [5] □

Now we state an approximation lemma that will be used throughout the rest of the paper.

Lemma 2.1. *Let $(u^{\varepsilon_j}, v^{\varepsilon_j})$ be a family of solutions of (1.1) in a domain $\mathcal{D} \subseteq \mathbb{R}^{N+1}$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of $\mathcal{D}$, $v^{\varepsilon_j} \geq 0$ and verify (1.5). Let $(x_0, t_0) \in \mathcal{D} \cap \partial\{u > 0\}$ and let $(x_n, t_n) \in \mathcal{D} \cap \partial\{u > 0\}$ be such that $(x_n, t_n) \to (x_0, t_0)$ as $n \to \infty$. Let $\lambda_n \to 0$, $u_{\lambda_n}(x,t) = \frac{1}{\lambda_n} u(x_n + \lambda_n x, t_n + \lambda_n^2 t)$ and $(u^{\varepsilon_j})_{\lambda_n}(x,t) = \frac{1}{\lambda_n} u^{\varepsilon_j}(x_n + \lambda_n x, t_n + \lambda_n^2 t)$. Assume that $u_{\lambda_n} \to U$ as $n \to \infty$ uniformly on compact sets of R^{N+1}. Then, there exists $j(n) \to \infty$ such that for every $j_n \geq j(n)$ there holds that $\frac{\varepsilon_{j_n}}{\lambda_n} \to 0$ and*

1. *$(u^{\varepsilon_{j_n}})_{\lambda_n} \to U$ uniformly on compact sets of R^{N+1},*
2. *$\nabla (u^{\varepsilon_{j_n}})_{\lambda_n} \to \nabla U$ in $L^2_{loc}(R^{N+1})$,*
3. *$\frac{\partial}{\partial t}(u^{\varepsilon_{j_n}})_{\lambda_n} \to \frac{\partial}{\partial t} U$ weakly in $L^2(R^{N+1})$.*
   Also, we deduce that
4. *$\nabla u_{\lambda_n} \to \nabla U$ in $L^2(R^{N+1})$,*
5. *$\frac{\partial}{\partial t} u_{\lambda_n} \to \frac{\partial}{\partial t} U$ weakly in $L^2(R^{N+1})$.*

Proof. The proof is a straightforward adaptation of lemma 3.2 of [5] □

Now we state some lemmas on special limits of solutions to P_ε that will be used throughout the paper.

Lemma 2.2. *Let $(u^{\varepsilon_j}, v^{\varepsilon_j})$ be a solution to (1.1) in a domain $\mathcal{D} \subset \mathbb{R}^{N+1}$ such that $v^{\varepsilon_j} \geq 0$, and verify (1.7) in $\mathcal{D}$ with $w_0 =$ constant. Let $(x_0, t_0) \in \mathcal{D}$ and assume that u^{ε_j} converges to $u = \alpha(x - x_0)_1^+$ uniformly on compact subsets of $\mathcal{D}$, with $\alpha \in \mathbb{R}$ and $\varepsilon_j \to 0$. Then,*

$$0 \leq \alpha \leq \sqrt{2M}. \tag{2.2}$$

where $M = \int_{-w_0}^1 (s + w_0) f(s)\, ds$.

Proof. The proof is an adaptation of Proposition 5.2 of [5].

Without loss of generality we may assume that $(x_0, t_0) = (0, 0)$.

First we see that necessarily $\alpha \geq 0$ since u is subcaloric in $\mathcal{D}$ and $u(0,0) = 0$. If $\alpha = 0$ there is nothing to prove. So let us assume that $\alpha > 0$.

Let $\psi \in C_c^\infty(\mathcal{D})$. Multiplying (1.1) by $u^\varepsilon_{x_1}\psi$ and integrating by parts we get

$$\begin{aligned} \iint_{\mathcal{D}} u_t^{\varepsilon_j} u_{x_1}^{\varepsilon_j} \psi = & \frac{1}{2} \iint_{\mathcal{D}} |\nabla u^{\varepsilon_j}|^2 \psi_{x_1} - \iint_{\mathcal{D}} u_{x_1}^{\varepsilon_j} \nabla u^{\varepsilon_j} \nabla \psi \\ & \iint_{\mathcal{D}} \mathcal{B}_{\varepsilon_j}(u^{\varepsilon_j}, x, t) \psi_{x_1} + \iint_{\mathcal{D}} w_{x_1}^{\varepsilon_j} \Big(\int_{-w_0}^{u^{\varepsilon_j}} f_{\varepsilon_j}(s) ds \Big) \psi, \end{aligned} \tag{2.3}$$

where $\mathcal{B}_\varepsilon(u, x, t) = \int_{-w_0\varepsilon}^u (s + w^\varepsilon) f_\varepsilon(s) ds$.

In order to pass to the limit in (2.3) we observe that, by Proposition 2.3

$$(u^{\varepsilon_j})_t \to 0 \qquad \text{weakly in } L^2_{loc}(\mathcal{D}),$$

$$\nabla u^{\varepsilon_j} \to \alpha \mathcal{X}_{\{x_1>0\}} e_1 \qquad \text{in } L^2_{loc}(\mathcal{D}).$$

On the other hand,

$$\frac{\nabla w^{\varepsilon_j}}{\varepsilon_j} \to 0 \qquad \text{uniformly on compact subsets of } \mathcal{D}.$$

Therefore, in order to pass to the limit in (2.3) we only need to analyze the limit of $\mathcal{B}_{\varepsilon_j}(u^{\varepsilon_j}, x, t)$. On one hand, it is easy to see that

$$\mathcal{B}_{\varepsilon_j}(u^{\varepsilon_j}(x,t), x, t) \to M \tag{2.4}$$

for every (x, t) such that $x_1 > 0$. In fact,

$$\mathcal{B}_{\varepsilon_j}(u^{\varepsilon_j}, x, t) = \int_{-w_0}^{\frac{u^{\varepsilon_j}}{\varepsilon_j}} \Big(s + \frac{w^{\varepsilon_j}}{\varepsilon_j}\Big) f(s)\, ds = \int_{-w_0}^{1} \Big(s + \frac{w^{\varepsilon_j}}{\varepsilon_j}\Big) f(s)\, ds$$

if j is large enough. Since $|\mathcal{B}_{\varepsilon_j}(u^{\varepsilon_j}, x, t)| \le C$ there holds that (2.4) holds in $L^1_{loc}(\{x_1 \ge 0\}$.

On the other hand, there exists $\bar{M}(x,t) \in L^\infty(\mathcal{D})$ such that $\mathcal{B}_{\varepsilon_j}(u^{\varepsilon_j}, x, t) \to \bar{M}(x,t)$ weakly in $L^2_{loc}(\mathcal{D})$. Clearly, $\bar{M}(x,t) = M$ in $\{x_1 > 0\}$. Let us see that $\bar{M}(x,t) = \bar{M}(t)$ in $\{x_1 < 0\}$. In fact,

$$\begin{aligned}
\nabla(\mathcal{B}_{\varepsilon_j}(u^{\varepsilon_j}(x,t), x, t)) &= \frac{\partial \mathcal{B}_{\varepsilon_j}}{\partial u}(u^{\varepsilon_j}, x, t) \nabla u^{\varepsilon_j} + \nabla \mathcal{B}_{\varepsilon_j}(u^{\varepsilon_j}, x, t) \\
&= (u^{\varepsilon_j} + w^{\varepsilon_j}) f_{\varepsilon_j}(u^{\varepsilon_j}) \nabla u^{\varepsilon_j} + \nabla w^{\varepsilon_j} \int_{-w_0 \varepsilon_j}^{u^{\varepsilon_j}} f_{\varepsilon_j}(s)\, ds \\
&= v^{\varepsilon_j} f_{\varepsilon_j}(u^{\varepsilon_j}) \nabla u^{\varepsilon_j} + \frac{\nabla w^{\varepsilon_j}}{\varepsilon_j} \int_{-w_0}^{\frac{u^{\varepsilon_j}}{\varepsilon_j}} f(s)\, ds.
\end{aligned}$$

Since $v^{\varepsilon_j} f_{\varepsilon_j}(u^{\varepsilon_j}) \to 0$ in $L^1_{loc}(\{x_1 < 0\})$, ∇u^{ε_j} is uniformly bounded in $L^\infty(\mathcal{D}')$ if $\mathcal{D}' \subset\subset \mathcal{D}$ and $\frac{\nabla w^{\varepsilon_j}}{\varepsilon_j} \to 0$ uniformly on compact subsets of $\mathcal{D}$, there holds that

$$\nabla(\mathcal{B}_{\varepsilon_j}(u^{\varepsilon_j}(x,t), x, t)) \to 0 \qquad \text{in } L^1_{loc}(\{x_1 < 0\}).$$

So that, passing to the limit in (2.3) we get

$$\frac{\alpha^2}{2} \iint_{\{x_1>0\}} \psi_{x_1} = M \iint_{\{x_1>0\}} \psi_{x_1} + \int_{\{x_1<0\}} \bar{M}(t) \psi_{x_1}.$$

Thus, integrating in the variable x_1 we get

$$\int_{\{x_1=0\}} \Big(\frac{\alpha^2}{2} - M + \bar{M}(t)\Big) \psi = 0.$$

Since ψ is arbitrary, we conclude that

$$\frac{\alpha^2}{2} - M + \bar{M}(t) = 0.$$

Finally, we notice that $\bar{M}(t) \geq 0$. In fact,

$$\begin{aligned}\mathcal{B}_{\varepsilon_j}(u^{\varepsilon_j}, x, t) &= \int_{-\frac{w^{\varepsilon_j}}{\varepsilon_j}}^{\frac{u^{\varepsilon_j}}{\varepsilon_j}} \Big(s + \frac{w^{\varepsilon_j}}{\varepsilon_j}\Big) f(s) ds + \int_{-w_0}^{-\frac{w^{\varepsilon_j}}{\varepsilon_j}} \Big(s + \frac{w^{\varepsilon_j}}{\varepsilon_j}\Big) f(s) ds \\ &\geq \int_{-w_0}^{-\frac{w^{\varepsilon_j}}{\varepsilon_j}} \Big(s + \frac{w^{\varepsilon_j}}{\varepsilon_j}\Big) f(s) ds \to 0\end{aligned}$$

since $\frac{w^{\varepsilon_j}}{\varepsilon_j} \to w_0$ uniformly on compact subsets of $\mathcal{D}$.

Thus,

$$\alpha = \sqrt{2(M - \bar{M}(t))} \leq \sqrt{2M}$$

and the proof is complete. □

Lemma 2.3. *Let $(u^{\varepsilon_j}, v^{\varepsilon_j})$ be a solution to (1.1) in a domain $\mathcal{D} \subset \mathbb{R}^{N+1}$ such that $v^{\varepsilon_j} \geq 0$ and verify (1.7) with $w_0 =$ constant in $\mathcal{D}$. Let $(x_0, t_0) \in \mathcal{D}$ and assume that u^{ε_j} converges to $u = \alpha(x - x_0)_1^+ + \bar{\alpha}(x - x_0)_1^-$ uniformly on compact subsets of $\mathcal{D}$, with $\alpha, \bar{\alpha} > 0$ and $\varepsilon_j \to 0$. Then,*

$$\bar{\alpha} = \alpha \leq \sqrt{2M} \tag{2.5}$$

where $M = \int_{-w_0}^{1} (s + w_0) f(s)\, ds$.

Proof. We argue in a similar way as in Proposition 5.3 of [5].

We will denote $Q_r = Q_r(0,0)$. Without loss of generality we will assume that $(x_0, t_0) = (0,0)$ and that $Q_2 \subset\subset \mathcal{D}$.

As before, u^ε satisfies

$$\begin{aligned}\iint_{\mathcal{D}} u_t^\varepsilon u_{x_1}^\varepsilon \psi =& \frac{1}{2} \iint_{\mathcal{D}} |\nabla u^\varepsilon|^2 \psi_{x_1} - \iint_{\mathcal{D}} u_{x_1}^\varepsilon \nabla u^\varepsilon \nabla \psi \\ &+ \iint_{\mathcal{D}} \mathcal{B}_\varepsilon(u^\varepsilon, x, t) \psi_{x_1} + \iint_{\mathcal{D}} w_{x_1}^\varepsilon \left(\int_0^{u^\varepsilon} f_\varepsilon(s) ds \right) \psi.\end{aligned}$$

We want to pass to the limit. By Proposition 2.3 and the fact that u^{ε_j} converge to $\alpha x_1^+ + \bar{\alpha} x_1^-$ we have that

$$\begin{aligned} u_t^{\varepsilon_j} &\to 0 \qquad \text{weakly in } L^2_{loc}(\mathcal{D}), \\ \nabla u^{\varepsilon_j} &\to \alpha \mathcal{X}_{\{x_1 > 0\}} e_1 - \bar{\alpha} \mathcal{X}_{\{x_1 < 0\}} e_1 \quad \text{in } L^2_{loc}(\mathcal{D}).\end{aligned}$$

Clearly, as $\alpha, \bar{\alpha} > 0$, $\mathcal{B}(u^{\varepsilon_j}, x, t) \to M$ in $L^1_{loc}(\mathcal{D})$.

So, passing to the limit in the latter equation for the subsequence ε_j, we get

$$-\frac{\alpha^2}{2} \iint_{\{x_1 > 0\}} \psi_{x_1} - \frac{\overline{\alpha}^2}{2} \iint_{\{x_1 < 0\}} \psi_{x_1} + M \iint \psi_{x_1} = 0.$$

Integrating in the x_1 variable, we conclude that

$$\alpha = \bar{\alpha}.$$

Next, we will assume that $\alpha > \sqrt{2M}$ and arrive at a contradiction.

First, let us consider z^{ε_j}, defined in Q_2, the solution to

$$\Delta z^{\varepsilon_j} - z_t^{\varepsilon_j} = \Big(\beta_{\varepsilon_j}(z^{\varepsilon_j}) + W_{\varepsilon_j} f_{\varepsilon_j}(z^{\varepsilon_j})\Big)\,\rho_{\varepsilon_j}(z^{\varepsilon_j}/\varepsilon_j) \quad \text{in } Q_2 \tag{2.6}$$

with boundary conditions

$$z^{\varepsilon_j} = u - b^{\varepsilon_j} \quad \text{on } \partial_p Q_2$$

where $\beta_\varepsilon(s) = s f_\varepsilon(s)$, $W_\varepsilon = \sup_{Q_2} w^\varepsilon$, $b^{\varepsilon_j} = \sup_{Q_2} |u^{\varepsilon_j} - u|$ and ρ_{ε_j} is a smooth cutoff function with support in $[-(w_0 + 2C_{\varepsilon_j}), 2]$ and $\rho_{\varepsilon_j} \equiv 1$ in $[-(w_0 + C_{\varepsilon_j}), 1]$ (Here $C_{\varepsilon_j} \to 0^+$ is such that $\Big|w^{\varepsilon_j}/\varepsilon_j - w_0\Big| \le C_{\varepsilon_j}$ in Q_2 so that $u^{\varepsilon_j}/\varepsilon_j \ge -(w_0 + C_{\varepsilon_j})$ in Q_2).

Observe that $z^{\varepsilon_j}(x_1, x', t) = z^{\varepsilon_j}(-x_1, x', t)$ in Q_2.

It is easy to see that the proofs of Propositions 2.1 and 2.2 can be adapted to z^{ε_j} so that, for a subsequence, that we still call ε_j, there holds that $z^{\varepsilon_j} \to z$ uniformly on compact sets of Q_2. We will show that $z = u$.

First,

$$\begin{aligned}\Delta u^{\varepsilon_j} - u_t^{\varepsilon_j} &= v^{\varepsilon_j} f_{\varepsilon_j}(u^{\varepsilon_j}) = (u^{\varepsilon_j} + w^{\varepsilon_j}) f_{\varepsilon_j}(u^{\varepsilon_j}) \le \beta_{\varepsilon_j}(u^{\varepsilon_j}) + W_{\varepsilon_j} f_{\varepsilon_j}(u^{\varepsilon_j}) \\ &= \Big(\beta_{\varepsilon_j}(u^{\varepsilon_j}) + W_{\varepsilon_j} f_{\varepsilon_j}(u^{\varepsilon_j})\Big)\rho_{\varepsilon_j}(u^{\varepsilon_j}/\varepsilon_j) \qquad \text{in } Q_2.\end{aligned}$$

From the fact that $z^{\varepsilon_j} \le u^{\varepsilon_j}$ on $\partial_p Q_2$, we deduce that $z^{\varepsilon_j} \le u^{\varepsilon_j}$ in Q_2 and therefore $z \le u$.

In order to see that $u \le z$, we consider $a^{\varepsilon_j} \in C^2(\mathbb{R})$ such that

$$\begin{aligned}&a_{ss}^{\varepsilon_j} = \Big(\beta(a^{\varepsilon_j}) + \tfrac{W_{\varepsilon_j}}{\varepsilon_j} f(a^{\varepsilon_j})\Big)\,\rho_{\varepsilon_j}(a^{\varepsilon_j}),\ \ s \in \mathbb{R} \\ &a^{\varepsilon_j}(0) = 1,\ \ a_s^{\varepsilon_j}(0) = \alpha\end{aligned}$$

Integrating the equation we get, for every $s \in \mathbb{R}$, that

$$0 < \gamma - \kappa_{\varepsilon_j} \le a_s^{\varepsilon_j}(s) \le \alpha$$

where $\frac{1}{2}\gamma^2 \equiv \frac{1}{2}\alpha^2 - M > 0$ and $\kappa_{\varepsilon_j} \to 0$ when $j \to \infty$.

It follows that there exists $\overline{s}_{\varepsilon_j} < 0$ such that

$$a^{\varepsilon_j}(s) = \begin{cases} 1 + \alpha s & s \ge 0 \\ (\gamma - \kappa_{\varepsilon_j})(s - \overline{s}_{\varepsilon_j}) & s \le \overline{s}_{\varepsilon_j}\end{cases}$$

and it is easy to see that $\overline{s}_{\varepsilon_j}$ are uniformly bounded by below and moreover, there exists $\overline{s} < 0$ such that $\overline{s}_{\varepsilon_j} \to \overline{s}$.

Now let

$$\tilde{a}^{\varepsilon_j}(x) = \varepsilon_j a^{\varepsilon_j}\Big(\frac{x_1}{\varepsilon_j} - \frac{b^{\varepsilon_j}}{(\gamma - \kappa_{\varepsilon_j})\varepsilon_j} + \overline{s}_{\varepsilon_j}\Big).$$

Using that $\tilde{a}^{\varepsilon_j}(0,x',t) = -b^{\varepsilon_j}$ and the bounds on $a_s^{\varepsilon_j}$, we deduce that

$$\tilde{a}^{\varepsilon_j} \le u - b^{\varepsilon_j} \text{ in } Q_2.$$

Now, since $\tilde{a}^{\varepsilon_j} \le z^{\varepsilon_j}$ on $\partial_p Q_2$, and $\tilde{a}^{\varepsilon_j}$ is a one dimensional stationary solution to (2.6), we have that $\tilde{a}^{\varepsilon_j} \le z^{\varepsilon_j}$ in Q_2. Since $\tilde{a}^{\varepsilon_j} \to u$ uniformly on compact subsets of $\{x_1 > 0\}$, we deduce that $u \le z$ in $Q_2 \cap \{x_1 > 0\}$.

Finally, we notice that $z^{\varepsilon_j}(x_1, x', t) = z^{\varepsilon_j}(-x_1, x', t)$, so we conclude that $u \le z$ in Q_2.

Now, the proof of the Lemma follows as in [5] where it is shown, for the case in which $w^\varepsilon \equiv 0$, that if $z^{\varepsilon_j} \to \alpha x_1^+ + \alpha x_1^-$ uniformly on compact subsets of Q_2, where z^{ε_j} are solutions to (2.6) which are symmetric in the x_1 variable, and $\alpha > 0$, there holds that $\alpha \le \sqrt{2M}$. So that we arrive at a contradiction since we have assumed that $\alpha > \sqrt{2M}$. Here we use again that $\frac{w^{\varepsilon_j}}{\varepsilon_j} \to w_0$ and $\liminf \mathcal{B}_{\varepsilon_j}(u^{\varepsilon_j}, x, t) \ge 0$. The proof is finished. □

Lemma 2.4. *Let $(u^{\varepsilon_j}, v^{\varepsilon_j})$ be a solution to (1.1) in a domain $\mathcal{D}_j$ such that $v^{\varepsilon_j} \ge 0$ and satisfies (1.7) in $\mathcal{D}_j$ with $w_0 =$ constant. Here $\mathcal{D}_j$ is such that $\mathcal{D}_j \subset \mathcal{D}_{j+1}$ and $\cup_j \mathcal{D}_j = \mathbb{R}^{N+1}$. Let us assume that $u^{\varepsilon_j} \to U$ uniformly on compact subsets of $\mathbb{R}^{N+1}$ as $j \to \infty$ and $\varepsilon_j \to 0$, with $U \ge 0$, $U \in Lip(1, 1/2)$ and $\partial\{U > 0\} \ne \emptyset$. Then,*

$$|\nabla U| \le \sqrt{2M} \qquad \text{in } \mathbb{R}^{N+1} \tag{2.7}$$

with $M = \int_{-w_0}^1 (s + w_0) f(s)\, ds$.

Proof. The proof is similar to that of Theorem 6.2 in [5]. Here we use Lemmas 2.2 and 2.3 instead of Propositions 5.2 and 5.3 in [5]. □

3. The free boundary condition

In this section, we find the free boundary condition for the limit problem and we show that the limit function u is a solution to the free boundary problem (1.4) in a pointwise sense, under the assumption that the free boundary admits an inward spatial normal in a parabolic measure theoretic sense (Definition 3.1).

Throughout this section we will assume that (1.7) holds and that for every $K \subset \{u \equiv 0\}^\circ$ compact there exists $0 < \eta < 1$ and $\varepsilon_0 > 0$ such that, for $\varepsilon < \varepsilon_0$

$$\frac{u^\varepsilon}{\varepsilon} \le \eta \qquad \text{in } K. \tag{3.1}$$

This assumption is a natural one in applications, roughly speaking it means that the mixture temperature reaches the flame temperature only if some combustion is taking place.

As a consequence there holds that

$$w_0 = \lim_{\varepsilon \to 0} \frac{v^\varepsilon - u^\varepsilon}{\varepsilon} \ge -\limsup_{\varepsilon \to 0} \frac{u^\varepsilon}{\varepsilon} \ge -\eta > -1 \quad \text{in } K.$$

So that, for the sake of simplicity we will assume from now on that $w_0 > -1$ in $\mathcal{D}$.

We start this section with a lemma that is the essential ingredient in the subsequent proofs.

Lemma 3.1. *Let $(u^{\varepsilon_k}, v^{\varepsilon_k})$ be a solution to (1.1) in a domain $\mathcal{D} \subset \mathbb{R}^{N+1}$ such that $v^{\varepsilon_k} \geq 0$ and (1.7) and (3.1) are satisfied with $w_0 > -1$. Let $u = \lim u^{\varepsilon_k}$, with $\varepsilon_k \to 0$, and $\mathcal{B}_{\varepsilon_k}(u, x, t) = \int_{-w_0\varepsilon_k}^{u} \big(s + w^{\varepsilon_k}\big) f_{\varepsilon_k}(s)ds$. Then,*

$$\mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k}, x, t) \to M(x,t)\mathcal{X}_{\{u>0\}}, \quad \textit{in } L^1_{loc}(\mathcal{D}).$$

where $M(x,t) = \int_{-w_0(x,t)}^{1} (s + w_0(x,t)) f(s)ds$.

Proof. First, let us observe that

$$\begin{aligned}\int_{-w_0\varepsilon}^{\varepsilon} (w^\varepsilon + s) f_\varepsilon(s) ds &= \int_{-w_0\varepsilon}^{\varepsilon} (w^\varepsilon + s)\frac{1}{\varepsilon^2} f\left(\frac{s}{\varepsilon}\right) ds \\ &= \int_{-w_0}^{1} \Big(\frac{w^\varepsilon}{\varepsilon} + s\Big) f(s) ds.\end{aligned}$$

Therefore,

$$\lim_{\varepsilon_k \to 0} \int_{-w_0\varepsilon_k}^{\varepsilon_k} (w^{\varepsilon_k} + s) f_{\varepsilon_k}(s) ds = M(x,t).$$

Let us now see that $\mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k}, x, t) \to M(x,t)$ uniformly on compact subsets of $\{u > 0\}$.

Let $K \subset\subset \{u > 0\}$, then there exists $\lambda > 0$ and ε_0 such that $u^{\varepsilon_k}(x,t) > \lambda \quad \forall \varepsilon_k < \varepsilon_0$, $(x,t) \in K$. Thus, we have

$$\begin{aligned}\lim_{k\to\infty} \mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k}(x,t), x, t) &= \lim_{k\to\infty} \int_{-w_0\varepsilon_k}^{u^{\varepsilon_k}(x,t)} (w^\varepsilon + s) f_{\varepsilon_k}(s) ds \\ &= \lim_{k\to\infty} \int_{-w_0\varepsilon_k}^{\varepsilon_k} (w^\varepsilon + s) f_{\varepsilon_k}(s) ds = M(x,t).\end{aligned}$$

Since $|\mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k}, x, t)| \leq C$ on every compact subset of $\mathcal{D}$, there holds, for a subsequence that we still call ε_k that

$$\mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k}, x, t) \to \bar{M}(x,t) \qquad \text{weakly in } L^2_{loc}(\mathcal{D}).$$

Clearly, $\bar{M}(x,t) = M(x,t)$ in $\{u > 0\}$. Let us see that $\bar{M}(x,t) = 0$ in $\{u \equiv 0\}^\circ$. In fact, let K be a compact subset of $\{u \equiv 0\}^\circ$. For every $\varepsilon_1, \varepsilon_2 > 0$ there holds that, for k large enough

$$\begin{aligned}
&|\{(x,t)\in K\,/\,\varepsilon_1<\mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k},x,t)<M(x,t)-\varepsilon_2\}|\\
&\quad\le|\{(x,t)\in K\,/\,\frac{u^{\varepsilon_k}}{\varepsilon_k}(x,t)>-w_0(x,t)\,,\ \frac{\varepsilon_1}{2}<\int_{-w_0}^{\frac{u^{\varepsilon_k}}{\varepsilon_k}}(s+w_0)f(s)ds<M-\frac{\varepsilon_2}{2}\}|\\
&\quad\le|\{(x,t)\in K\,/\,-w_0(x,t)+\mu\le\frac{u^{\varepsilon_k}}{\varepsilon_k}\le 1-\mu\}|\\
&\quad\le|\{(x,t)\in K\,/\,\frac{v^{\varepsilon_k}}{\varepsilon_k}\ge\frac{\mu}{2}\,,\ \frac{u^{\varepsilon_k}}{\varepsilon_k}\le 1-\mu\}|\\
&\quad\le|\{(x,t)\in K\,/\,v^{\varepsilon_k}f_{\varepsilon_k}(u^{\varepsilon_k})\ge\frac{C_\mu}{2\varepsilon_k}\}|.
\end{aligned}$$

Since $v^{\varepsilon_k}f_{\varepsilon_k}(u^{\varepsilon_k})\to 0$ as measures in K and $v^{\varepsilon_k}\ge 0$, $f_{\varepsilon_k}\ge 0$ there holds that

$$v^{\varepsilon_k}f_{\varepsilon_k}(u^{\varepsilon_k})\to 0 \qquad \text{in } L^1(K).$$

Therefore,

$$|\{(x,t)\in K\,/\,\varepsilon_1<\mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k},x,t)<M(x,t)-\varepsilon_2\}|\to 0.$$

On the other hand, let $1>\eta>\sup_K(-w_0)$ be the constant in (3.1) in K, there holds that

$$\begin{aligned}
\mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k},x,t)&=\int_{-\frac{w^{\varepsilon_k}}{\varepsilon_k}}^{\frac{u^{\varepsilon_k}}{\varepsilon_k}}\Big(s+\frac{w^{\varepsilon_k}}{\varepsilon_k}\Big)f(s)ds+\int_{-w_0}^{-\frac{w^{\varepsilon_k}}{\varepsilon_k}}\Big(s+\frac{w^{\varepsilon_k}}{\varepsilon_k}\Big)f(s)ds\\
&\le\int_{-\frac{w^{\varepsilon_k}}{\varepsilon_k}}^{\eta}\Big(s+\frac{w^{\varepsilon_k}}{\varepsilon_k}\Big)f(s)ds+\int_{-w_0}^{-\frac{w^{\varepsilon_k}}{\varepsilon_k}}\Big(s+\frac{w^{\varepsilon_k}}{\varepsilon_k}\Big)f(s)ds\\
&=\int_{-w_0}^{\eta}\Big(s+\frac{w^{\varepsilon_k}}{\varepsilon_k}\Big)f(s)ds\to\int_{-w_0}^{\eta}(s+w_0)f(s)ds<M(x,t)
\end{aligned}$$

since $-\frac{w^{\varepsilon_k}}{\varepsilon_k}\le\frac{u^{\varepsilon_k}}{\varepsilon_k}\le\eta$ in K. Therefore,

$$\limsup\mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k},x,t)\le\int_{-w_0}^{\eta}(s+w_0)f(s)ds<M(x,t).$$

So that, for $\varepsilon_2>0$ small we get

$$|\{(x,t)\in K\,/\,\varepsilon_1<\mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k},x,t)\}|=|\{(x,t)\in K\,/\,\varepsilon_1<\mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k},x,t)<M-\varepsilon_2\}|\to 0.$$

Let us now see that $\bar M(x,t)=0$ in K. As in Lemma 2.2 we see that $\bar M(x,t)\ge 0$. Now assume that for some $\varepsilon_1>0$ we have $|\{\bar M(x,t)>\varepsilon_1\}|>0$. Then, there exists m such that $|\{\bar M(x,t)>\varepsilon_1+\frac{1}{m}\}|:=|A_m|>0$.

Now,

$$\int_{A_m}\mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k},x,t)\to\int_{A_m}\bar M(x,t)>\Big(\varepsilon_1+\frac{1}{m}\Big)|A_m|$$

but,

$$\int_{A_m} \mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k}, x, t) = \int_{A_m \cap \{\mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k}, x,t) > \varepsilon_1\}} \mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k}, x, t) + \int_{A_m \cap \{\mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k}, x,t) \le \varepsilon_1\}} \mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k}, x, t).$$

Since the first term in the right hand side goes to zero and the second is bounded by $\varepsilon_1 |A_m|$, we get a contradiction.

The proof is finished. □

Let us give the definition of regular point.

Definition 3.1. *We say that ν is the interior unit spatial normal to the free boundary $\partial\{u > 0\}$ at a point $(x_0, t_0) \in \partial\{u > 0\}$ in the parabolic measure theoretic sense, if $\nu \in \mathbb{R}^N$, $|\nu| = 1$ and*

$$\lim_{r \to 0} \frac{1}{r^{N+2}} \iint_{Q_r(x_0,t_0)} |\mathcal{X}_{\{u>0\}} - \mathcal{X}_{\{(x,t)/\ \langle x - x_0, \nu\rangle > 0\}}| dx dt = 0.$$

Definition 3.2. *We say that (x_0, t_0) is a regular point of $\partial\{u > 0\}$ if there exists an interior unit spatial normal to $\partial\{u > 0\}$ at (x_0, t_0) in the parabolic measure theoretic sense.*

We can now prove the main result of this section.

Theorem 3.1. *Let $(u^{\varepsilon_j}, v^{\varepsilon_j})$ be a family of uniformly bounded solutions of (1.1) in a domain $\mathcal{D} \subset \mathbb{R}^{N+1}$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subset of $\mathcal{D}$, $v^{\varepsilon_j} \ge 0$ and verify (1.7) and (3.1), with $w_0 > -1$. If (x_0, t_0) is a regular point of $\mathcal{D} \cap \partial\{u > 0\}$, then u has the asymptotic development*

$$u(x, t) = \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0| + |t - t_0|^{1/2})$$

with $\alpha = \sqrt{2M(x_0, t_0)}$, where $M = \int_{-w_0}^{1} (s + w_0) f(s)\, ds$. Here ν is the interior unit spatial normal to the free boundary at (x_0, t_0) in the parabolic measure theoretic sense.

Proof. We assume, without loss of generality, that $(x_0, t_0) = (0, 0)$ and $\nu = e_1 = (1, 0, ..., 0)$.

Let $\psi \in C_c^\infty(\mathcal{D})$. We proceed as in Lemma 2.2. Let us multiply the equation for u^ε by $u^\varepsilon_{x_1}\psi$ and integrate by parts. We have

$$\begin{aligned}\iint_{\mathcal{D}} u^\varepsilon_t u^\varepsilon_{x_1} \psi =& \frac{1}{2} \iint_{\mathcal{D}} |\nabla u^\varepsilon|^2 \psi_{x_1} - \iint_{\mathcal{D}} u^\varepsilon_{x_1} \nabla u^\varepsilon \nabla \psi \\ &+ \iint_{\mathcal{D}} \mathcal{B}_\varepsilon(u^\varepsilon, x, t) \psi_{x_1} + \iint_{\mathcal{D}} \left(\frac{w^\varepsilon}{\varepsilon} - w_0\right) f(-w_0)(w_0)_{x_1} \psi \\ &+ \iint_{\mathcal{D}} \frac{w^\varepsilon_{x_1}}{\varepsilon} \left(\int_{-w_0}^{\frac{u^\varepsilon}{\varepsilon}} f(s) ds\right) \psi\end{aligned}$$

Since

$$\mathcal{B}_{\varepsilon_j}(u^{\varepsilon_j},x,t)=\int_{-w_0}^{\frac{u^{\varepsilon_j}}{\varepsilon_j}}(s+w_0)f(s)ds+\int_{-w_0}^{\frac{u^{\varepsilon_j}}{\varepsilon_j}}\Big(\frac{w^{\varepsilon_j}}{\varepsilon_j}-w_0\Big)f(s)ds,$$

and $\mathcal{B}_{\varepsilon_j}(u^{\varepsilon_j},x,t)\to 0$ weakly in $L^1(K)$ for every $K\subset\{u\equiv 0\}^\circ$ compact, there holds that

$$F_{\varepsilon_j}(x,t):=\int_{-w_0}^{\frac{u^{\varepsilon_j}}{\varepsilon_j}}(s+w_0)f(s)ds\to 0\qquad\text{weakly in } L^1(K).$$

Since F_{ε_j} is nonnegative there holds that

$$F_{\varepsilon_j}\to 0\qquad\text{in } L^1(K).$$

So that, for a subsequence that we still call ε_j there holds that

$$F_{\varepsilon_j}\to 0\qquad a.e.\ K.$$

Thus,

$$\frac{u^{\varepsilon_j}}{\varepsilon_j}\to -w_0\qquad a.e.\ K.$$

Therefore,

$$\int_{-w_0}^{\frac{u^{\varepsilon_j}}{\varepsilon_j}}f(s)ds\to\Big(\int_{-w_0}^{1}f(s)ds\Big)\mathcal{X}_{\{u>0\}}\qquad a.e.\ \mathcal{D}. \tag{3.2}$$

By using Proposition 2.3, Lemma 3.1 and (3.2) we can pass to the limit (for the sequence $\varepsilon_j\to 0$) in the latter equation and get

(3.3)

$$\begin{aligned}\iint_{\mathcal{D}}u_tu_{x_1}\psi&=\frac12\iint_{\mathcal{D}}|\nabla u|^2\psi_{x_1}-\iint_{\mathcal{D}}u_{x_1}\nabla u\nabla\psi+\iint_{\{u>0\}}M(x,t)\psi_{x_1}\\&\quad+\iint_{\{u>0\}}(w_0)_{x_1}\Big(\int_{-w_0}^{1}f(s)ds\Big)\psi\end{aligned}$$

for every $\psi\in C_c^\infty(\mathcal{D})$.

Now, let $\psi^\lambda(x,t)=\lambda\psi(\frac{x-x_0}{\lambda},\frac{t-t_0}{\lambda^2})$. Replacing ψ by ψ^λ in (3.3) and changing variables, we get for $u_\lambda(x,t)=\frac1\lambda u(x_0+\lambda x,t_0+\lambda^2t)$,

$$\begin{aligned}\iint(u_\lambda)_t(u_\lambda)_{x_1}\psi&=\frac12\iint|\nabla u_\lambda|^2\psi_{x_1}-\iint(u_\lambda)_{x_1}\nabla u_\lambda\nabla\psi\\&\quad+\iint_{\{u_\lambda>0\}}M(\lambda x,\lambda^2t)\psi_{x_1}+\iint_{\{u>0\}}(w_0)_{x_1}\Big(\int_{-w_0}^{1}f(s)ds\Big)\psi^\lambda.\end{aligned}\tag{3.4}$$

Let $r>0$ be such that $Q_r(x_0,t_0)\subset\subset\mathcal{D}$. We have that $u_\lambda\in Lip(1,1/2)$ in $Q_{r/\lambda}(0,0)$ uniformly in λ, and $u_\lambda(0,0)=0$. Therefore, for every $\lambda_n\to 0$, there exists a subsequence $\lambda_{n'}\to 0$ and a function $U\in Lip(1,1/2)$ in $\mathbb{R}^{N+1}$ such that $u_{\lambda_{n'}}\to U$ uniformly on compact sets of $\mathbb{R}^{N+1}$.

By our assumption on (x_0,t_0), we can easily see that for every $k>0$

$$|\{u_\lambda>0\}\cap\{x_1<0\}\cap Q_k(0,0)|\to 0\text{ as }\lambda\to 0, \tag{3.5}$$

and

$$|\{u_\lambda = 0\} \cap \{x_1 > 0\} \cap Q_k(0,0)| \to 0 \text{ as } \lambda \to 0. \tag{3.6}$$

Now, using lemma 2.1 and the fact that $\psi^\lambda \to 0$ uniformly in $\mathcal{D}$ and $\operatorname{supp}\psi^\lambda \subset \operatorname{supp}\psi$, we can pass to the limit in (3.4) and get

$$\begin{aligned} &\iint_{\{x_1>0\}} U_t U_{x_1}\psi = \\ &\frac{1}{2}\iint_{\{x_1>0\}} |\nabla U|^2 \psi_{x_1} - \iint_{\{x_1>0\}} U_{x_1}\nabla U \nabla\psi + M(0,0)\iint_{\{x_1>0\}} \psi_{x_1}. \end{aligned} \tag{3.7}$$

Our aim is to prove that $U = \alpha x_1^+$. First, by (3.5) and (3.6), we deduce that $U = 0$ in $\{x_1 < 0\}$. On the other hand, U is a solution to the heat equation in $\{U > 0\} \subset \{x_1 > 0\}$. By Corollary A.1 in [5], for every $\bar{x}' \in \mathbb{R}^{N-1}$, $\bar{t} \in \mathbb{R}$ there exists $\alpha \geq 0$ such that

$$U(x,t) = \alpha x_1^+ + o(|(x_1, x') - (0, \bar{x}')| + |t - \bar{t}|^{1/2}) \ \text{ in } \{x_1 > 0\} \cap \{t < \bar{t}\}.$$

Replacing the test function ψ by $\Phi^\lambda(x,t) = \lambda\Phi(\frac{x_1}{\lambda}, \frac{x'-\bar{x}'}{\lambda}, \frac{t-\bar{t}}{\lambda^2})$ with $\Phi \in C_c^\infty(\{t < 0\})$ and proceeding as above we get

$$-\frac{\alpha^2}{2}\iint_{\{x_1>0\}} \Phi_{x_1} + M(0,0)\iint_{\{x_1>0\}} \Phi_{x_1} = 0. \tag{3.8}$$

In order to pass to the limit for a sequence $\lambda_n \to 0$ we have used Lemma 2.1. (See [6], Theorem 3.1 for the details).

Thus, $\alpha = \sqrt{2M(0,0)}$.

In order to see that $U = \alpha x_1^+$ we use Lemma 2.4. In fact, by Lemma 2.1 there exists a sequence $j_n \to \infty$ such that

$$u^{\delta_n} := \frac{1}{\lambda_n} u^{\varepsilon_{j_n}}(\lambda_n x, \lambda_n^2 t) \to U(x,t)$$

uniformly on compact subsets of $\mathbb{R}^{N+1}$. We recall that $(u^{\delta_n}, v^{\delta_n})$ is a solution to (1.1) with ε replaced by δ_n. Moreover,

$$\frac{w^{\delta_n}}{\delta_n} = \frac{w^{\varepsilon_{j_n}}(\lambda_n x, \lambda_n^2 t)}{\varepsilon_{j_n}} \to w_0(0,0)$$

uniformly on compact sets of $\mathbb{R}^{N+1}$.

Moreover, $U \geq 0$ and $\partial\{U > 0\} \neq \emptyset$. By Lemma 2.4 we have that $|\nabla U| \leq \alpha(= \sqrt{2M(0,0)}$. Since $U \equiv 0$ in $\{x_1 = 0\}$ we deduce that

$$U \leq \alpha x_1 \qquad \text{in } \{x_1 > 0\}.$$

By Hopf's Principle, we deduce that

$$U = \alpha x_1 \qquad \text{in } \{x_1 > 0\}.$$

The theorem is proved. $\square$

Remark 3.1. *It is clear from the proof that the result is still true if we replace condition* (3.1) *by the following property:* $\frac{u^{\varepsilon_j}}{\varepsilon_j} \to -w_0$ *a.e.* $\{u \equiv 0\}^\circ$.

4. Viscosity solutions

In this section we prove that, under suitable assumptions, the limit function u is a viscosity solution of the free boundary problem (1.4).

For the sake of completeness, we state here the definition of viscosity solution that was introduced in [6] for the two phase case of this problem when $w_0 = 0$.

Definition 4.1. *Let Q be a cylinder in $\mathbb{R}^N \times (0,T)$ and let $v \in C(\overline{Q})$. Then v is called a classical subsolution (supersolution) of* (1.4) *in Q if $v \geq 0$ and*

1. $\Delta v - v_t \geq 0 \ (\leq 0)$ *in* $\Omega^+ \equiv Q \cap \{v > 0\}$.
2. $v \in C^1(\overline{\Omega^+})$.
3. *For any* $(x,t) \in \partial\Omega^+ \cap Q$, $\nabla v(x,t) \neq 0$, *and*
$$|\nabla v(x,t)| \geq \sqrt{2M} \quad (\leq \sqrt{2M}).$$

We say that v is a classical solution in Q if it is both a classical subsolution and a classical supersolution.

Definition 4.2. *Let u be a continuous nonnegative function in Q; u is called a viscosity subsolution (supersolution) of* (1.4) *in Q if, for every subcylinder $Q' \subset\subset Q$ and for every classical supersolution (subsolution) v in Q',*

$$\begin{array}{ll} u \leq v \text{ on } \partial_p Q' & (u \geq v \text{ on } \partial_p Q') \text{ and} \\ v > 0 \text{ on } \overline{\{u > 0\}} \cap \partial_p Q' & (u > 0 \text{ on } \overline{\{v > 0\}} \cap \partial_p Q') \end{array}$$

implies that $u \leq v$ $(u \geq v)$ in Q'.

The function u is called a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

Definition 4.3. *Let u be a continuous nonnegative function in $\mathcal{D}$ and let $(x_0,t_0) \in \partial\{u > 0\} \cap \mathcal{D}$. We say that (x_0,t_0) is a regular point from the nonpositive side, if there exists a regular nonnegative function v in $\mathcal{D}$ such that $v > u$ in $\{u > 0\}$ for $t < t_0$ and $v(x_0,t_0) = u(x_0,t_0)$.*

Finally we need the following definition on nondegeneracy.

Definition 4.4. *Let u be a continuous nonnegative function in $\mathcal{D}$. Let $(x_0,t_0) \in \mathcal{D}$ be such that $u(x_0,t_0) = 0$. We say that u does not degenerate at (x_0,t_0) if there exist $r_0 > 0$ and $C > 0$ such that*

$$\sup_{\partial_p Q_r^-(x_0,t_0)} u \geq C\, r \qquad \text{for } 0 < r \leq r_0.$$

We now prove that, under suitable assumptions on the limit function u, there holds that u is a viscosity solution to the free boundary problem.

Theorem 4.1. *Let $u = \lim u^{\varepsilon_k}$, where $(u^{\varepsilon_k}, v^{\varepsilon_k})$ are uniformly bounded solutions to* (1.1) *with $v^{\varepsilon_k} \geq 0$, satisfying* (1.7) *in $\mathcal{D}$, with $w_0 > -1$, and such that u^{ε_k} either satisfies* (3.1) *of $u_t^{\varepsilon_k} \leq 0$ in $\mathcal{D}$.*

If u^+ does not degenerate at every point of the free boundary which is regular from the nonpositive side, then u is a viscosity solution of (1.4).

Proof. By Proposition 2.3 and Lemma 2.1, Theorem 4.1 of [6] can be stated for our system, thus u is a viscosity supersolution.

In order to see that it is a viscosity subsolution, let v be a classical supersolution such that

$$u \leq v \text{ in } \partial_p Q \text{ and } v > 0 \text{ in } \overline{\{u > 0\}} \cap \partial_p Q$$

we want to see that $u \leq v$ in Q.

If not, we define

$$t_0 = \sup\{0 < s < T : \ v > 0 \text{ in } \overline{\{u > 0\}} \cap Q \cap \{0 \leq t < s\}\}$$

From the definition of t_0, it follows that $t_0 > 0$ and, from our hypotheses we deduce that $v \geq u$ in $Q \cap \{0 \leq t < t_0\}$. In addition, there exists a sequence $(x(s), t(s)) \to (x_0, t_0) \in \overline{Q}$ such that $v(x(s), t(s)) = 0$, $(x(s), t(s)) \in \overline{\{u > 0\}} \cap Q$. Clearly, $u(x_0, t_0) = v(x_0, t_0) = 0$ and $(x_0, t_0) \in \partial\{u > 0\} \cap Q$. If $(x_0, t_0) \in \{v = 0\}^\circ$ then, for τ small we have $u \leq v = 0$ in $B_\tau(x_0, t_0) \cap \{t < t_0\}$ and therefore, $u \equiv 0$ there, which contradicts our hypothesis. Thus

$$v \geq u \text{ in } Q \cap \{0 \leq t \leq t_0\},$$

$$(x_0, t_0) \in \partial\{u > 0\} \cap \partial\{v > 0\} \cap Q.$$

We may assume, without loss of generality, that $(x_0, t_0) = (0, 0)$ and $Q_1(0, 0) = Q_1 \subset Q$ (consider instead of u the function $\frac{1}{\lambda_0} u(x_0 + \lambda_0 x, t_0 + \lambda_0^2 t)$ for certain $\lambda_0 > 0$ small, and analogously with v). Let us take

$$v_\lambda(x, t) = \frac{1}{\lambda} v(\lambda x, \lambda^2 t), \quad u_\lambda(x, t) = \frac{1}{\lambda} u(\lambda x, \lambda^2 t).$$

It is easy to see that there exists a sequence $\lambda_n \to 0$ and functions u_0, v_0 such that $v_\lambda \to v_0$, $u_\lambda \to u_0$.

Since v is regular, we have that $v_0(x, t) = \beta x_1^+$ with $0 \leq \beta \leq \sqrt{2M(0, 0)}$ (for some system of coordinates).

Let us see that also $u_0(x, t) = \alpha x_1^+$ for some $\alpha \geq 0$,

We may think that in Q_1, $\partial\{v > 0\}$ is the graph of some function $\psi(x', t) = x_1$, $x = (x_1, x')$ with $\psi \in Lip(1, 1/2)$, where $\psi(0, 0) = 0$ and $\{v > 0\} = \{x_1 > \psi(x', t)\}$.

Hence, we have that

$$|\psi(x',t)| \le C\left(|x'| + |t|^{1/2}\right).$$

Let $\mathcal{R} = \left\{(x,t) \in Q_1 : \ x_1 < -C\left(|x'| + |t|^{1/2}\right)\right\}$. Then $\mathcal{R} \cap \{v > 0\} = \emptyset$ and let w be the caloric function in $\mathcal{O} = Q_1^- \setminus \mathcal{R}$ with $w = 0$ in $\partial_p \mathcal{R}$ and $w = L \ge \|u\|_\infty$ in the rest of $\partial_p \mathcal{O}$.

Since u is globally subcaloric and $u \le w$ on $\partial_p \mathcal{O}$, then $u < w$ in $\mathcal{O}$.

Now, $w - u$ is supercaloric in $\mathcal{O}$, $w - u > 0$ in the interior and $w - u = 0$ at $(0,0)$, then, by lemma A.1 of [5], we have that $w - u = \delta x_1^+ + o(|x| + |t|^{1/2})$ and, since by the same lemma, w has an asymptotic development at $(0,0)$,

$$u(x,t) = \alpha x_1^+ + o(|x| + |t|^{1/2}), \quad \text{with } \alpha \ge 0.$$

Since by hypothesis u^+ does not degenerate, there follows that $\alpha > 0$.

On the other hand, since v is regular, v admits an asymptotic development at the origin in the form $v(x,t) = \beta x_1^+ + o(|x| + |t|^{1/2})$. Clearly, $\beta \ge \alpha$.

Now, let h be the caloric function in $\tilde{\mathcal{O}} := Q_1^- \cap \{v > 0\} \cap \{-\mu < t < 0\}$ for some small $\mu > 0$, with $h = v - u$ on $\partial_p \tilde{\mathcal{O}}$. And, let g be the caloric function in $\tilde{\mathcal{O}}$ with $g = v$ on $\partial_p \tilde{\mathcal{O}}$. Then, $h = g = 0$ in $Q_1^- \cap \partial\{v > 0\} \cap \{-\mu < t < 0\}$ and $h > 0$, $g > 0$ in $\tilde{\mathcal{O}}$.

Therefore, by [1], there exists $\sigma > 0$ such that $h \ge \sigma g$ in $Q_{1/2}^- \cap \{v > 0\} \cap \{-\frac{\mu}{2} < t < 0\}$.

Since u is subcaloric in Q_1^- and $u \le v$ in Q_1^- we deduce that $v - u \ge \sigma u$ in $Q_{1/2}^- \cap \{v > 0\} \cap \{-\frac{\mu}{2} < t < 0\}$. In particular $\beta - \alpha \ge \sigma\alpha > 0$.

The theorem will be finished if we show that $\alpha = \sqrt{2M(0,0)}$.

<u>Case 1:</u> u^{ε_k} verifies (3.1).

As in Theorem 3.1, we obtain

$$\begin{aligned}\iint_{\mathcal{D}} u_t u_{x_1} \psi = \frac{1}{2} \iint_{\mathcal{D}} |\nabla u|^2 \psi_{x_1} - \iint_{\mathcal{D}} u_{x_1} \nabla u \nabla \psi \\ + \iint_{\mathcal{D} \cap \{u > 0\}} M(x,t) \psi_{x_1} + \iint_{\mathcal{D} \cap \{u > 0\}} (w_0)_{x_1} \left(\int_{-w_0}^1 f(s)\,ds\right) \psi\end{aligned}$$

for every test function ψ. Then, taking $\psi^\lambda(x,t) = \lambda \psi(\frac{x}{\lambda}, \frac{t}{\lambda^2})$ and changing variables, we get

$$\begin{aligned}\iint_{\mathcal{D}} (u_\lambda)_t (u_\lambda)_{x_1} \psi = \frac{1}{2} \iint_{\mathcal{D}} |\nabla u_\lambda|^2 \psi_{x_1} - \iint_{\mathcal{D}} (u_\lambda)_{x_1} \nabla u_\lambda \nabla \psi \\ + \iint_{\mathcal{D} \cap \{u_\lambda > 0\}} M(\lambda x, \lambda^2 t) \psi_{x_1} + \iint_{\mathcal{D} \cap \{u > 0\}} (w_0)_{x_1} \left(\int_{-w_0}^1 f(s)\,ds\right) \psi^\lambda\end{aligned}$$

By Lemma 2.1, we get (for some sequence $\lambda_k \to 0$)

$$0 = -\frac{1}{2}\alpha^2 \iint_{\mathcal{D}\cap\{x_1>0\}} \psi_{x_1} + \lim_{k\to\infty} \iint_{\mathcal{D}\cap\{u_{\lambda_k}>0\}} M(\lambda_k x, \lambda_k^2 t)\psi_{x_1}.$$

We want to check that $\mathcal{X}_{\{u_{\lambda_k}>0\}} \to \mathcal{X}_{\{x_1>0\}}$ a.e. or, equivalently,

1. $\{x_1 > 0\} \subset \cup_{n=1}^{\infty} \cap_{k\geq n} \{u_{\lambda_k} > 0\} = \liminf\{u_{\lambda_k} > 0\}$ a.e.
2. $\cap_{n=1}^{\infty} \cup_{k\geq n} \{u_{\lambda_k} > 0\} = \limsup\{u_{\lambda_k} > 0\} \subset \{x_1 > 0\}$ a.e.

Let us see (1). If $x_1 > 0$, we get that $\alpha x_1 > 0$ and since $u_{\lambda_k}(x,t) \to \alpha x_1$ it follows that $u_{\lambda_k}(x,t) > 0 \quad \forall k \geq k_0$.

Let us see (2). If exists $k_j \to \infty$ with $u_{\lambda_{k_j}}(x,t) > 0$ then it must be $x_1 \geq 0$, because if $x_1 < 0$, we have that $v_{\lambda_{k_j}}(x,t) = 0$ for $j \geq j_0$ (because as v is regular, $\{v_{\lambda_k} > 0\} \to \{x_1 > 0\}$). Since $u_{\lambda_{k_j}} \leq v_{\lambda_{k_j}}$ we get a contradiction.

Therefore,

$$0 = -\frac{1}{2}\alpha^2 \iint_{\mathcal{D}\cap\{x_1>0\}} \psi_{x_1} + M(0,0) \iint_{\mathcal{D}\cap\{x_1>0\}} \psi_{x_1}.$$

So that,

$$0 = \int_{\mathcal{D}\cap\{x_1=0\}} \left(\frac{1}{2}\alpha^2 - M(0,0)\right) \psi dx' dt.$$

Since ψ is arbitrary, $\frac{1}{2}\alpha^2 = M(0,0)$, so that,

$$\alpha = \sqrt{2M(0,0)}$$

and the proof is finished

<u>Case 2:</u> $u_t^{\varepsilon_k} \leq 0$

We already now that, if we consider $u_\lambda(x,t) = \frac{1}{\lambda}u(\lambda x, \lambda^2 t)$, then it follows that

$$u_\lambda(x,t) \to u_0(x,t) \equiv \alpha x_1^+$$

uniformly on compact subsets of $\mathbb{R}^{N+1}$.

As before

$$\iint_{\mathcal{D}} u_t^{\varepsilon_k} u_{x_1}^{\varepsilon_k} \psi = \frac{1}{2} \iint_{\mathcal{D}} |\nabla u^{\varepsilon_k}|^2 \psi_{x_1} - \iint_{\mathcal{D}} u_{x_1}^{\varepsilon_k} \nabla u^{\varepsilon_k} \nabla \psi + \iint_{\mathcal{D}} \mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k}, x, t)\psi_{x_1}$$
$$+ \iint_{\mathcal{D}} w_{x_1}^{\varepsilon_k} \left(\int_{-w_0\varepsilon_k}^{u^{\varepsilon_k}} f_{\varepsilon_k}(s) ds\right) \psi + \iint_{\mathcal{D}} (w_0)_{x_1} \Big(\frac{w^{\varepsilon_k}}{\varepsilon_k} - w_0\Big) f(-w_0)\psi$$

Now, as in the previous case, if we consider first $\psi^\lambda(x,t) = \lambda\psi(\frac{x}{\lambda},\frac{t}{\lambda^2})$ and change variables, we obtain

$$\begin{aligned}\iint (u_\lambda^{\varepsilon_k})_t (u_\lambda^{\varepsilon_k})_{x_1}\psi &= \frac12 \iint |\nabla u_\lambda^{\varepsilon_k}|^2\psi_{x_1} - \iint (u_\lambda^{\varepsilon_k})_{x_1}\nabla u_\lambda^{\varepsilon_k}\nabla\psi \\ &+ \iint \mathcal{B}^\lambda_{\varepsilon_k/\lambda}(u_\lambda^{\varepsilon_k},x,t)\psi_{x_1} + \iint_{\mathcal{D}} \frac{(w^{\varepsilon_k})_{x_1}}{\varepsilon_k}\left(\int_{-w_0}^{\frac{u_k^\varepsilon}{\varepsilon_k}} f(s)\,ds\right)\psi^\lambda \\ &+ \iint_{\mathcal{D}} (w_0)_{x_1}\Big(\frac{w^{\varepsilon_k}}{\varepsilon_k} - w_0\Big) f(-w_0)\psi^\lambda\end{aligned} \tag{4.1}$$

where $\mathcal{B}^\lambda_\varepsilon(u,x,t) = \int_{-w_0(\lambda x,\lambda^2 t)\varepsilon}^u (s + w^\varepsilon(x,t)) f_\varepsilon(s)\,ds$. We want to pass to the limit as both ε_k and λ go to zero.

Using Lemma 2.1, we see that for every sequence $\lambda_n \to 0$ there exists a sequence $k_n \to \infty$ such that $\delta_n := \varepsilon_{k_n}/\lambda_n \to 0$ and $u^{\delta_n} := (u^{\varepsilon_{k_n}})_{\lambda_n} \to u_0$ uniformly on compact sets of $\mathbb{R}^{N+1}$. By Proposition 2.3 we see that we can pass to the limit in the first three terms of (4.1) (with $\varepsilon = \varepsilon_{k_n}$ and $\lambda = \lambda_n$).

Let us study the limit of $\mathcal{B}^{\lambda_n}_{\delta_n}(u^{\delta_n}(x,t),x,t)$.

It is easy to see that in $\{x_1 > 0\}$, $\mathcal{B}^{\lambda_n}_{\delta_n}(u^{\delta_n}(x,t),x,t) \to M(0,0)$ uniformly on compact sets. Now, let $K \subset \{x_1 < 0\}$ be compact. We will show that

$$\nabla(\mathcal{B}^{\lambda_n}_{\delta_n}(u^{\delta_n}(x,t),x,t)) \to 0 \quad \text{in } L^1(K)$$

In fact,

$$\begin{aligned}\nabla(\mathcal{B}^{\lambda_n}_{\delta_n}(u^{\delta_n}(x,t),x,t)) &= v^{\delta_n} f_{\delta_n}(u^{\delta_n})\nabla u^{\delta_n} \\ &+ \lambda_n \nabla w_0(\lambda_n x,\lambda_n^2 t)\Big(\frac{w^{\delta_n}}{\delta_n}(x,t) - w_0(\lambda_n x,\lambda_n^2 t)\Big) f(-w_0(\lambda_n x,\lambda_n^2 t)) \\ &+ \frac{\nabla w^{\delta_n}}{\delta_n}\int_{-w_0(\lambda_n x,\lambda_n^2 t)}^{\frac{u^{\delta_n}}{\delta_n}} f(s)\,ds\end{aligned}$$

Since $v^{\delta_n} f_{\delta_n}(u^{\delta_n}) \to 0$ as measures in K and is nonnegative, we deduce that the convergence takes place in $L^1(K)$. On the other hand, ∇u^{δ_n} is uniformly bounded. Therefore, the first term goes to zero in $L^1(K)$.

In order to see that the second and third terms go to zero uniformly in K we only need to observe that

$$\frac{u^{\delta_n}}{\delta_n}(x,t) = \frac{u^{\varepsilon_{k_n}}}{\varepsilon_{k_n}}(\lambda_n x,\lambda_n^2 t)$$

and a similar formula holds for $\frac{w^{\delta_n}}{\delta_n}$. So that

$$\Big|\frac{w^{\delta_n}}{\delta_n}(x,t)-w_0(\lambda_n x,\lambda_n^2 t)\Big|\to 0 \quad \text{uniformly on compact sets of } \mathbb{R}^{N+1},$$
$$\frac{u^{\delta_n}}{\delta_n}\geq -\frac{w^{\delta_n}}{\delta_n}\geq -C,$$
$$\frac{|\nabla w^{\delta_n}|}{\delta_n}(x,t)=\lambda_n\frac{|\nabla w^{\varepsilon_{k_n}}|}{\varepsilon_{k_n}}(\lambda_n x,\lambda_n^2 t)\to 0 \qquad \text{uniformly on compact sets of } \mathbb{R}^{N+1}.$$

On the other hand, $|\mathcal{B}_{\delta_n}^{\lambda_n}(u^{\delta_n}(x,t),x,t)|\leq C_K$, so that we have,

$$\mathcal{B}_{\delta_n}^{\lambda_n}(u^{\delta_n}(x,t),x,t)\to \bar M(t) \qquad \text{weakly in } L^2(K).$$

Let us now see that, actually, the convergence takes place in $L^1(K)$.

There holds that

$$\frac{\partial}{\partial t}\Big(\mathcal{B}_{\delta_n}^{\lambda_n}(u^{\delta_n}(x,t),x,t)\Big)=v^{\delta_n}f_{\delta_n}(u^{\delta_n})(u^{\delta_n})_t+\frac{\partial}{\partial t}\mathcal{B}_{\delta_n}^{\lambda_n}(u^{\delta_n},x,t)$$
$$\leq \frac{\partial}{\partial t}\mathcal{B}_{\delta_n}^{\lambda_n}(u^{\delta_n},x,t)\leq C_K \quad \text{in } K.$$

On the other hand, for every $(x_0,t_0)\in K$, and $Q_\tau(x_0,t_0)\subset\{x_1<0\}$

$$\iint_{Q_\tau(x_0,t_0)}\frac{\partial}{\partial t}\Big(\mathcal{B}_{\delta_n}^{\lambda_n}(u^{\delta_n}(x,t),x,t)\Big)=\int_{B_\tau(x_0)}\mathcal{B}_{\delta_n}^{\lambda_n}(u^{\delta_n}(x,t_0+\tau^2),x,(t_0+\tau^2))\,dx$$
$$-\int_{B_\tau(x_0)}\mathcal{B}_{\delta_n}^{\lambda_n}(u^{\delta_n}(x,t_0-\tau^2),x,(t_0-\tau^2))\,dx$$
$$\geq\ -C_\tau$$

since $|\mathcal{B}_{\delta_n}^{\lambda_n}(u^{\delta_n}(x,t),x,t)|\leq C_K$ for every compact set K.

Therefore there exists $C_K>0$ such that $\|\mathcal{B}_{\delta_n}^{\lambda_n}(u^{\delta_n}(x,t),x,t)\|_{W^{1,1}(K)}\leq C_K$. Hence the convergence takes place in $L^1(K)$ (for a subsequence).

Now arguing as in Lemma 3.1, we get that $\bar M(t)=0$ or $\bar M(t)=M(0,0)$.

We can now take the limit in (4.1) for the sequences ε_{k_n} and λ_n and we obtain

$$0=-\frac{1}{2}\alpha^2\iint_{\mathcal{D}\cap\{x_1>0\}}\psi_{x_1}+M(0,0)\iint_{\mathcal{D}\cap\{x_1>0\}}\psi_{x_1}+\iint_{\mathcal{D}\cap\{x_1<0\}}\bar M(t)\psi_{x_1}.$$

So that,

$$0=\int_{\mathcal{D}\cap\{x_1=0\}}\Big(\frac{1}{2}\alpha^2-M(0,0)-\bar M(t)\Big)\,\psi\,dx'dt.$$

Since ψ is arbitrary we get $\frac{1}{2}\alpha^2=M(0,0)-\bar M(t)$. So that, in particular, $\bar M(t)$ is constant and then we have that $\bar M(t)\equiv 0$ or $\bar M(t)\equiv M(0,0)$. Since $\alpha>0$ we deduce that $M(t)\equiv 0$ and

$$\alpha=\sqrt{2M(0,0)}.$$

The proof is finished. ☐

Now we prove a proposition that says that, under suitable assumptions, u^+ does not degenerate on the free boundary. The proof is similar to theorem 6.3 in [5], where the nondegeneracy of u^+ was proved in the strictly two phase case. Here we assume, instead of (3.1) the somewhat stronger condition that for every $K \subset \mathcal{D}$ compact, there exist $0 < \eta < 1$ and $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \le \varepsilon_0$

$$\frac{u^\varepsilon}{\varepsilon} \le \eta \qquad \text{in } K \cap \{u \equiv 0\}^\circ. \tag{4.2}$$

Proposition 4.1. *Let $u = \lim u^{\varepsilon_k}$, where $(u^{\varepsilon_k}, v^{\varepsilon_k})$ are uniformly bounded solutions to (1.1) satisfying (1.7) with $w_0 > -1$, such that $v^{\varepsilon_k} \ge 0$ and the functions u^{ε_k} satisfy (4.2). Let $(x_0, t_0) \in \partial\{u > 0\}$.*

Let us assume that there exists $\nu \in \mathbb{R}^N$, with $|\nu| = 1$ such that

$$\liminf_{r\to 0^+} \frac{|\{u > 0\} \cap \{\langle x - x_0, \nu\rangle > 0\} \cap Q_r^-(x_0, t_0)|}{|Q_r^-(x_0, t_0)|} > \alpha_1$$

and

$$\liminf_{r\to 0^+} \frac{|\{u = 0\}^\circ \cap \{\langle x - x_0, \nu\rangle < 0\} \cap Q_r^-(x_0, t_0)|}{|Q_r^-(x_0, t_0)|} > \alpha_2$$

with $\alpha_1 + \alpha_2 > \frac{1}{2}$, then there exists a constant $C > 0$ and $r_0 > 0$ which depends on N and f such that, if $0 < r \le r_0$,

$$\sup_{\partial_p Q_r^-(x_0,t_0)} u \ge Cr.$$

Proof. Without loss of generality, we may assume that $(x_0, t_0) = (0, 0)$ and that $\nu = e_1 = (1, 0, ..., 0)$.

We will note $Q_r^- = Q_r^-(0, 0)$ and

$$(u^\varepsilon)_r(x, t) = \frac{1}{r} u^\varepsilon(rx, r^2 t), \quad (v^\varepsilon)_r(x, t) = \frac{1}{r} v^\varepsilon(rx, r^2 t), \quad u_r(x, t) = \frac{1}{r} u(rx, r^2 t).$$

Let us see that there exists $r_0 > 0$ and a constant c such that if $r < r_0$ and $\varepsilon < \varepsilon_0 = \varepsilon_0(r)$, then

$$\iint_{Q_1^-} (v^\varepsilon)_r f_{\varepsilon/r}((u^\varepsilon)_r) dx \ge c.$$

Following [5], there exists $\gamma > 0$ small such that, for some $\lambda > 0$,

$$\frac{|\{u_r > \gamma\} \cap \{x_1 > 0\} \cap Q_1^-|}{|Q_1^-|} + \frac{|\{u_r = 0\}^\circ \cap \{x_1 < 0\} \cap Q_1^-|}{|Q_1^-|} \ge \frac{1}{2} + \lambda$$

Let us now define

$$A_r = \{u_r > \gamma\} \cap \{x_1 > 0\} \cap Q_1^-, \quad B_r = \{u_r = 0\}^\circ \cap \{x_1 < 0\} \cap Q_1^-$$
$$\text{and } -B_r = \{(x_1, x', t)/\ (-x_1, x', t) \in B_r\}.$$

Then, we have

$$|A_r \cap (-B_r)| \ge \lambda |Q_1^-| = \tilde{\lambda}.$$

Once again, following [5] we have for $0<\rho<1$ fixed, that there exists $0<x_1^r<1$ such that

$$|\Lambda_r| = |\{(x',t)/\ (x_1^r,x',t)\in A_r\cap(-B_r)\}| > \rho\tilde\lambda$$

Let $\eta>0$ be the constant in (4.2) in $Q_1(0,0)$, let $0<\delta'<\delta$, $0<b<b'<1$ be such that

$$\eta < -w_0(0,0)+\delta < b.$$

Let $\kappa>0$ be such that

$$f(s)>\kappa>0 \quad \text{for } s\le b'.$$

Then, for $(x',t)\in\Lambda_r$ we have

$$\frac{1}{\varepsilon/r}(u^\varepsilon)_r(x_1^r,x',t) > \frac{\gamma}{2(\varepsilon/r)} > b, \quad \frac{1}{\varepsilon/r}(u^\varepsilon)_r(-x_1^r,x',t) < -w_0(0,0)+\delta$$

if $\varepsilon_k<\varepsilon_1=\varepsilon_1(r)$ is small. So that, for every $(x',t)\in\Lambda_r$ there exists $\tilde x_1^r\in(-1,1)$ such that $-w_0(0,0)+\delta\le\frac{1}{\varepsilon/r}(u^\varepsilon)_r(\tilde x_1^r,x',t)\le b$.

Now, by the uniform Lipschitz regularity of $(u^\varepsilon)_r$ and $(v^\varepsilon)_r$, and (1.7), we have that for $\varepsilon\le\varepsilon_0(\le\varepsilon_1)$ and $r\le r_0$,

$$\frac{(u^\varepsilon)_r}{\varepsilon/r}(x_1,x',t)\le b' \quad\text{and}\quad \frac{(v^\varepsilon)_r}{\varepsilon/r}(x_1,x',t)\ge\delta' \quad \text{if } |x_1-\tilde x_1^r|<C\frac{\varepsilon}{r}$$

where C depends on δ,δ',b,b', on the Lipschitz constant of u^ε and v^ε in Q_1^- and r_0 depends only w_0.

Finally we have

$$\begin{aligned}\iint_{Q_1^-}(v^\varepsilon)_r f_{\varepsilon/r}((u^\varepsilon)_r) &= \iint_{Q_1^-}\frac{(v^\varepsilon)_r}{\varepsilon/r}\frac{1}{\varepsilon/r}f\Big(\frac{(u^\varepsilon)_r}{\varepsilon/r}\Big)\\ &\ge \delta'\frac{\kappa}{\varepsilon/r}\left|\left\{(x,t)\in Q_1^-/\ \frac{(v^\varepsilon)_r}{\varepsilon/r}\ge\delta' \text{ and } f\Big(\frac{(u^\varepsilon)_r}{\varepsilon/r}\Big)\ge\kappa\right\}\right|\\ &\ge \delta'\frac{\kappa}{\varepsilon/r}|\Lambda_r|2C\frac{\varepsilon}{r}\ge 2C\delta'\kappa\rho\tilde\delta\equiv c.\end{aligned}$$

The rest of the proof follows as in [5] □

Remark 4.1. *Proposition 4.1 remains true if we change the hypothesis that u^{ε_k} satisfies (4.2) by*

$$\frac{u^{\varepsilon_k}}{\varepsilon_k}\to -w_0 \qquad a.e.\ \ \{u\equiv 0\}^\circ. \tag{4.3}$$

In fact, as in the proof of Prop. 4.1 we consider for each $0<r<1$ the sets A_r and B_r. So that, for some $0<\lambda<1$

$$|A_r\cap(-B_r)|\ge\lambda|Q_1^-|.$$

Since $B_r\subset\{u_r\equiv 0\}^\circ$, there holds that

$$\frac{(u^\varepsilon)_r}{\varepsilon/r}(-x_1,x',t)\to -w_0(-rx_1,rx',r^2t) \qquad a.e.\ \ A_r\cap(-B_r).$$

Let $0<\mu<1$. There exists $C_r \subset \Big(A_r \cap (-B_r)\Big)$ such that $|C_r| = \mu|A_r\cap(-B_r)|$ and

$$\frac{(u^\varepsilon)_r}{\varepsilon/r}(-x_1,x',t) \to -w_0(-rx_1,rx',r^2t) \quad \textit{uniformly in } C_r.$$

Let $\delta>0$. There exists $\varepsilon_1=\varepsilon_1(r)$ such that

$$\frac{(u^\varepsilon)_r}{\varepsilon/r}(-x_1,x',t) \le -w_0(-rx_1,rx',r^2t)+\frac{\delta}{2} \le -w_0(0,0)+\delta \quad \textit{in } C_r$$

if $\varepsilon<\varepsilon_1$ and $r<r_0=r_0(\delta)$. Now, the proof follows as in Prop. 4.1 by taking $\tilde\lambda=\mu\lambda|Q_1^-|$ and

$$\Lambda_r := \{(x',t)\,/\,(x_1^r,x',t)\in C_r\}.$$

Remark 4.2. *Proposition 4.1 remains true if we change condition* (4.2) *by condition* (3.1). *In fact, as in the proof of Theorem 3.1 we see that condition* (3.1) *implies that*

$$\mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k},x,t)\to 0 \qquad L^1_{loc}(\{u\equiv 0\}^\circ).$$

As in Theorem 3.1 we deduce that u^{ε_k} satisfies (4.3).

Using Remark 4.1, Remark 4.2 and Theorem 4.1 we get the following Corollaries.

Corollary 4.1. *Let $u=\lim u^{\varepsilon_k}$ where $(u^{\varepsilon_k},v^{\varepsilon_k})$ are unifomly bounded solutions to* (1.1) *in a domain $\mathcal{D}\subset\mathbb{R}^{N+1}$ with $v^{\varepsilon_k}\ge 0$, which verify* (1.7) *with $w_0>-1$ and such that u^{ε_k} satisfies* (3.1). *If the free boundary $\mathcal{D}\cap\partial\{u>0\}$ is given by $x_1=g(x',t)$ with $g\in Lip(1,1/2)$, then, u is a viscosity solution of the free boundary problem* (1.4).

Corollary 4.2. *Let $u=\lim u^{\varepsilon_k}$ where $(u^{\varepsilon_k},v^{\varepsilon_k})$ are unifomly bounded solutions to* (1.1) *in a domain $\mathcal{D}\subset\mathbb{R}^{N+1}$ with $v^{\varepsilon_k}\ge 0$, which verify* (1.7) *with $w_0>-1$ and such that u^{ε_k} satisfies* (4.3) *and $u_t^{\varepsilon_k}\le 0$. If, for every $(x_0,t_0)\in\mathcal{D}\cap\partial\{u>0\}$, $\{x\in\mathbb{R}^N\,/\,(x,t_0)\in\mathcal{D}\cap\{u>0\}\}$ is given by $x_1>\Phi(x')$ with Φ, Lipschitz continuous then, u is a viscosity solution of the free boundary problem* (1.4).

Proof. We only need to see that u does not degenerate at points of the free boundary which are regular from the zero side. Let (x_0,t_0) be any such point. We see that we can apply Remark 4.1 at that point. In fact, since $u_t^{\varepsilon_k}\le 0$, u is decreasing in time. Therefore,

$$\{(x,t)\,/\,x_1>\Phi(x)\,,\,t\le t_0\}\subset\{u>0\}$$

and the parabolic density of this set is positive. □

In particular, Corollary 4.2 can be applied to solutions of (1.1) with u_0^ε constructed as in [7] and v_0^ε a small perturbation of u_0^ε.

Corollary 4.3. *Let $u_0\in C(\mathbb{R}^N)\cap C^2(\overline{\{u_0>0\}})$ be such that $||u_0||_{C^2(\overline{\{u_0>0\}})}<\infty$, $\Delta u_0\le 0$ and $(u_0)_{x_1}-\lambda|\nabla u_0|\ge 0$ in $\{u_0>0\}$ with $\lambda>0$. Assume, moreover that $0<a_2\le|\nabla u_0|\le a_1<\sqrt{2M_0}$ in a neighborhood of the free boundary: $\{x\in\{u_0>0\}\,/\,dist(x,\{u_0=$*

$0\}) \le \gamma\}$, and $M_0 = \int_0^1 sf(s)$. Then, there exists a sequence $(u_0^\varepsilon, v_0^\varepsilon) \in \left(C^1(\mathbb{R}^N)\right)^2$ with $u_0^\varepsilon \to u_0$ uniformly in $\mathbb{R}^N$ (so that u_0^ε are uniformly bounded) and, moreover, they satisfy

$$\begin{aligned}
&1)\ \Delta u_0^\varepsilon - v_0^\varepsilon f_\varepsilon(u_0^\varepsilon) \le 0\\
&2)\ (u_0^\varepsilon)_{x_1} - \lambda|\nabla u_0^\varepsilon| \ge 0\\
&3)\ \frac{v_0^\varepsilon - u_0^\varepsilon}{\varepsilon} \to w_0 \ \ \textit{uniformly on compact sets, with} \quad w_0 > -1.
\end{aligned} \tag{4.4}$$

$w_0 \in \mathbb{R}$ is any constant such that $w_0 \ge -\eta$ with $\eta > 0$ small enough.

Let $(u^\varepsilon, v^\varepsilon)$ be the solution to (1.1) with initial datum $(u_0^\varepsilon, v_0^\varepsilon)$ (so that, in particular, u^ε and v^ε are uniformly bounded). For every sequence $\varepsilon_j \to 0$ there exists a subsequence ε_{j_k} such that there exists

$$u = \lim_{k\to\infty} u^{\varepsilon_{j_k}}$$

and u is a viscosity solution to the free boundary problem (1.4).

Proof. Let u_0^ε be the approximations constructed in [7]. The approximations are constructed in the following way. First we extend u_0 to a neighborhood of $\{u_0 > 0\}$: $S := \{x \in \mathbb{R}^N \,/\, \mathrm{dist}\,(x, \{u_0 > 0\}) \le \gamma\}$ in such a way that $||u_0||_{C^2(S)} < \infty$. For ε small enough we define

$$u_0^\varepsilon(x) = \varepsilon F\left(\frac{1}{\sqrt{2M_0}}\Big(1 - \frac{u_0(x)}{\varepsilon}\Big)\right) \qquad \text{in } \{-C\varepsilon \le u_0 \le \varepsilon\}.$$

where $F \in C^2(\mathbb{R})$ is such that

$$F'' \le (1+\delta)Ff(F) + \alpha F', \quad F(0) = 1, \quad F'(0) = -\sqrt{2M_0}.$$

Here $\delta > 0$, $\alpha > 0$ are such that F has a strict minimum at a finite point $\bar{s}$ such that $\bar{s}\sqrt{2M_0} > 1$. ($\bar{s} \to +\infty$ as $\delta \to 0$), and F is decreasing for $s < \bar{s}$.

The constant C is taken as $C = \bar{s}\sqrt{2M_0} - 1$.

We define

$$\begin{aligned}
u_0^\varepsilon &= u_0 && \text{in } \{u_0 > \varepsilon\}\\
u_0^\varepsilon &= \varepsilon F(\bar{s}) && \text{in } \mathbb{R}^N \setminus \{u_0 > -C\varepsilon\}.
\end{aligned}$$

As in [7], we see that $u_0^\varepsilon \in C^1(\mathbb{R}^N)$.

Let $w_0 \in \mathbb{R}$ be such that $w_0 \ge -\eta > -F(\bar{s})$ with $\eta > 0$ to be fixed later and let

$$v_0^\varepsilon = u_0^\varepsilon + \varepsilon w_0.$$

Then, $v_0^\varepsilon \geq 0$. It is immediate to verify that (4.4) 1) is satisfied in $\{u_0 > \varepsilon\}$ and $\mathbb{R}^N \setminus \{u_0 > -C\varepsilon\}$. Let us see that it is satisfied in $\{-C\varepsilon \leq u_0 \leq \varepsilon\}$. In fact,

$$\begin{aligned}\Delta u_0^\varepsilon - v_0^\varepsilon f_\varepsilon(u_0^\varepsilon) &= \frac{1}{2M_0\varepsilon}F''|\nabla u_0|^2 - \frac{1}{\sqrt{2M_0}}F'\Delta u_0 - \frac{1}{\varepsilon}Ff(F) - \frac{w_0}{\varepsilon}f(F)\\ &\leq \frac{1+\delta}{2M_0\varepsilon}Ff(F)|\nabla u_0|^2 + \frac{\alpha}{2M_0\varepsilon}F'|\nabla u_0|^2 - \frac{a}{\sqrt{2M_0}}F' - \frac{1}{\varepsilon}Ff(F) - \frac{w_0}{\varepsilon}f(F)\end{aligned}$$

where $a > 0$ is such that $|\Delta u_0| \leq a$.

Let $0 < \mu < 1$ be such that $a_1 \leq (1-\mu)^{1/2}A\sqrt{2M_0}$ with $0 < A < 1$, and let δ in the definition of F be such that $(1+\delta)A^2 \leq 1$. Then, if ε is small enough so that $\alpha a_2^2/\sqrt{2M_0} > a\varepsilon$ there holds that

$$\begin{aligned}\Delta u_0^\varepsilon - v_0^\varepsilon f_\varepsilon(u_0^\varepsilon) &\leq \frac{1}{\varepsilon}\left[[(1+\delta)(1-\mu)A^2-1]Ff(F) + \Big(\frac{\alpha a_2^2}{2M_0} - \frac{a\varepsilon}{\sqrt{2M_0}}\Big)F' - w_0 f(F)\right]\\ &\leq \frac{1}{\varepsilon}[-\mu F - w_0]f(F) \leq \frac{1}{\varepsilon}[-\mu F(\bar s) - w_0]f(F) \leq 0\end{aligned}$$

if $\eta = \mu F(\bar s)$.

Clearly, (4.4) 3) holds. Let us see that (4.4) 2) also holds. We only need to verify this property in the set $\{-C\varepsilon < u_0 < \varepsilon\}$ and this is clear from the fact that

$$\nabla u_0^\varepsilon = -\frac{1}{\sqrt{2M_0}}F'\left(\frac{1}{\sqrt{2M_0}}\Big(1 - \frac{u_0}{\varepsilon}\Big)\right)\nabla u_0.$$

Now, by the results of Section 2, for every sequence $\varepsilon_j \to 0$ there exists a subsequence and a continuous function u such that $u^{\varepsilon_{j_k}} \to u$ uniformly on compact subsets of $\mathbb{R}^N \times (0,\infty)$.

On the other hand, u_t^ε is a solution to the following equation

$$\Delta U - U_t = \beta_\varepsilon'(u)U.$$

Here $\beta_\varepsilon(s) = sf_\varepsilon(s)$. Since, for ε small enough $u_t^\varepsilon(x,0) \leq 0$ we conclude that

$$u_t^\varepsilon \leq 0 \qquad \text{in } \mathbb{R}^N \times (0,\infty). \tag{4.5}$$

In a similar way we see that $u_{x_1} - \lambda u_{x_i} \geq 0$ for every i. So that

$$u_{x_1}^\varepsilon - \frac{\lambda}{N}|\nabla u^\varepsilon| \geq 0 \qquad \text{in } \mathbb{R}^N \times (0,\infty). \tag{4.6}$$

Clearly (4.5) and (4.6) imply that

$$u_t \leq 0 \qquad \text{and} \qquad u_{x_1} - \frac{\lambda}{N}|\nabla u| \geq 0 \qquad \text{in } \{u > 0\}.$$

In particular, the free boundary is Lipschitz in space.

So that, in order to apply Corollary 4.2 we only need to verify that u^{ε_k} satisfies (4.3). On one hand, given $K \subset \{u_0 \equiv 0\}^\circ$ compact, there exists ε_0 such that for $\varepsilon < \varepsilon_0$

$$\mathcal{B}_\varepsilon(u_0^\varepsilon, x, 0) = \int_{-w_0}^{\frac{u_0^\varepsilon}{\varepsilon}(x)} (s + w_0) f(s) = \int_{-w_0}^{F(\bar{s})} (s + w_0) f(s).$$

On the other hand,

$$\frac{\partial}{\partial t}\Big(\mathcal{B}_\varepsilon(u^\varepsilon, x, t)\Big) = v^\varepsilon f_\varepsilon(u^\varepsilon) u_t^\varepsilon \leq 0.$$

Therefore,

$$\mathcal{B}_\varepsilon(u^\varepsilon, x, t) \leq \int_{-w_0}^{F(\bar{s})} (s + w_0) f(s) \qquad \text{for } x \text{ in } K\,,\ t > 0.$$

As in the proof of Theorem 4.1 we see that, since $u_t^\varepsilon \leq 0$, there holds that $\mathcal{B}_\varepsilon(u^\varepsilon, x, t) \to \bar{M}(x,t)$ in $L^1_{loc}(\{u \equiv 0\}^\circ)$ and, for almost every (x,t) we either have $\bar{M}(x,t) = 0$ or $\bar{M}(x,t) = M = \int_{-w_0}^1 (s + w_0) f(s)$. Since

$$\nabla\Big(B_\varepsilon(u^\varepsilon, x, t)\Big) = v^\varepsilon f_\varepsilon(u^\varepsilon) \nabla u^\varepsilon \to 0 \quad \text{in } L^1_{loc}(\{u \equiv 0\}^\circ)$$

there holds that $\bar{M}(x,t) = \bar{M}(t)$ in $\{u \equiv 0\}^\circ$. Therefore,

$$\bar{M}(t) \leq \int_{-w_0}^{F(\bar{s})} (s + w_0) f(s) \quad \textit{a.e.} \ \ \{u \equiv 0\}^\circ.$$

Since $F(\bar{s}) < 1$, there holds that $\bar{M}(t) \equiv 0$.

Thus, for every sequence $\varepsilon_k \to 0$

$$\int_{-w_0}^{\frac{u^{\varepsilon_k}}{\varepsilon_k}} (s + w_0) f(s) \to 0 \qquad \textit{a.e.} \ \ \{u \equiv 0\}^\circ$$

and we deduce that u^{ε_k} satisfies (4.3). ☐

Combining the regularity results for viscosity solutions of [8], Corollary 4.1 and Corollary 4.2 we have the following regularity result for limit functions.

Corollary 4.4. *Let u as in Corollary 4.1 or Corollary 4.2. If, moreover, the free boundary $\mathcal{D} \cap \partial\{u > 0\}$ is given by $x_1 = g(x', t)$ with g Lipschitz continuous, then, u is a classical solution of the free boundary problem* (1.4).

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