UNIQUENESS IN A TWO-PHASE FREE-BOUNDARY PROBLEM

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Abstract. We investigate a two-phase free-boundary problem in heat propagation that in classical terms is formulated as follows: to find a continuous function $u(x,t)$ defined in a domain $\mathcal{D} \subset \mathbb{R}^N \times (0,T)$ which satisfies the equation

$$\Delta u + \sum a_i \, u_{x_i} - u_t = 0$$

whenever $u(x,t) \neq 0$, i.e., in the subdomains $\mathcal{D}_+ = \{(x,t) \in \mathcal{D} : u(x,t) > 0\}$ and $\mathcal{D}_- = \{(x,t) \in \mathcal{D} : u(x,t) < 0\}$. Besides, we assume that both subdomains are separated by a smooth hypersurface, the free boundary, whose normal is never time-oriented and on which the following conditions are satisfied:

$$u = 0, \quad |\nabla u^+|^2 - |\nabla u^-|^2 = 2M.$$

Here $M > 0$ is a fixed constant, and the gradients are spatial side-derivatives in the usual two-phase sense. In addition, initial data are specified, as well as either Dirichlet or Neumann data on the parabolic boundary of $\mathcal{D}$.

The problem admits *classical* solutions only for good data and for small times. To overcome this problem several generalized concepts of solution have been proposed, among them the concepts of *limit* solution and *viscosity* solution. Continuing the work done for the one-phase problem we investigate conditions under which the three concepts agree and produce a unique solution for the two-phase problem.

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1. Introduction

In this paper we investigate a *two-phase free-boundary problem* in heat propagation that in classical terms is formulated as follows: given a domain $\mathcal{D} \subset \mathbb{R}^N \times (0,T)$ and a constant $M > 0$, to find a continuous function $u(x,t)$ defined in $\mathcal{D}$ which satisfies the equation

$$\Delta u + \sum a_i\, u_{x_i} - u_t = 0 \tag{1.1}$$

in the subdomains $\mathcal{D}_+ = \{(x,t) \in \mathcal{D} : u(x,t) > 0\}$ and $\mathcal{D}_- = \{(x,t) \in \mathcal{D} : u(x,t) < 0\}$, which represent the two different phases. Besides, both subdomains must be separated by a smooth hypersurface, Γ, a so-called free boundary, whose normal is never time-oriented, and such that $\mathcal{D} = \mathcal{D}_+ \cup \mathcal{D}_- \cup \Gamma$. On Γ we have $u = 0$ and we impose the jump condition

$$|\nabla u^+|^2 - |\nabla u^-|^2 = 2M, \qquad M > 0, \tag{1.2}$$

where ∇u^+ denotes the gradient of u restricted to $\overline{\{u > 0\}}$ and ∇u^- is the gradient of $-u$ restricted to $\overline{\{u < 0\}}$. We are thus imposing a discontinuity of $|\nabla u|$ across Γ since $M \neq 0$. Finally, initial data are specified, as well as either Dirichlet or Neumann data on the parabolic boundary of $\mathcal{D}$, as we will see below. We will refer to this free-boundary problem as problem $\mathcal{P}$.

This is a model of heat propagation with change of phase. There is a corresponding *one-phase problem*, where $u \geq 0$, so that the negative domain disappears and $\mathcal{D}_-$ is replaced in the domain division by the interior of the null-set, $\mathcal{D}_0 = \{u = 0\}^\circ$. The jump condition on Γ reads then

$$|\nabla u^+|^2 = 2M. \tag{1.3}$$

This problem arises in several contexts, in particular in combustion theory and in flows in porous media, and is currently the object of active investigation. We have devoted the article [17] to investigating the questions of uniqueness of different types of solutions for the one-phase problem. We remark that the jump conditions (1.2) and (1.3) make these problems completely different from the two-phase and one-phase Stefan problems.

Let us recall that classical solutions to problem $\mathcal{P}$ in one space dimension are relatively easy to construct, but the problem is much more difficult in several space dimensions; cf. [21]. Generally, classical solutions exist only locally in time, since singularities can arise in finite time even in the one-phase problem; cf. [22].

One way of addressing the problem of existence of solutions in a more general context is the introduction of *viscosity solutions,* defined by comparison with classical solutions (see the precise definition in Section 2). Another way is to consider the problem as the limit of the equations

$$\Delta u^\varepsilon + \sum_1^N a_i u^\varepsilon_{x_i} - u^\varepsilon_t = \beta_\varepsilon(u^\varepsilon) \tag{1.4}$$

with corresponding initial and boundary conditions. We call this equation $\mathcal{P}_\varepsilon$. In the one-phase problem this regularization for small ε appears in combustion theory as the high activation energy range of the equation for equidiffusional flames, and the limit has been proposed by Zeldovich and Frank-Kamenetski [23] and produces the free-boundary problem $\mathcal{P}$ when the *reaction function* β_ε converges as $\varepsilon \to 0$ to a Dirac delta in the following scaling way:

$$\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon}); \tag{1.5}$$

see [2, 3, 13, 14]. In the two-phase problem the limit of $\mathcal{P}_\varepsilon$ has been studied in [10, 11, 20]. We call the solution of $\mathcal{P}$ obtained by such a process the *limit solution*.

Continuing the work done for the one-phase problem in [17] we investigate in this paper conditions under which the concepts of classical, viscosity and limit solution agree and produce a unique solution for the two-phase problem.

Main results. We take as spatial domain a cylinder of the form $\Omega = I \times \Sigma$ with $\Sigma \subset \mathbb{R}^{N-1}$ a smooth domain, and $I = \mathbb{R}$ (a full cylinder), $I = (0, \infty)$ or $I = (-\infty, d)$ (a semicylinder) or $I = (0, d)$, $d > 0$ (a bounded cylinder), and we put homogeneous Neumann conditions on the lateral boundary $I \times \partial\Sigma$. We require monotonicity of the initial data in the direction of the cylinder axis, but we make no requirement of monotonicity of the solution in time. In the family of problems $\mathcal{P}_\varepsilon$ we assume that the functions β_ε are defined by scaling of a single function $\beta : \mathbb{R} \to \mathbb{R}$ satisfying the following:

- β is a Lipschitz-continuous function,
- $\beta > 0$ in $(0, 1)$ and $\beta \equiv 0$ otherwise,
- $\int \beta(s)\, ds = M$.

We then define $\beta_\varepsilon(s) = 1/\varepsilon\beta(s/\varepsilon)$. The coefficients a_i of the first-order terms in the differential operator are assumed to be independent of x_1, the direction of the cylinder axis, and belong to $C^{\alpha,\frac{\alpha}{2}}(\overline{\Sigma} \times [0, T])$.

Our results can be summarized as saying that, with this type of domain, coefficients and reaction function β_ε, and under suitable assumptions on the initial and boundary data, *if a classical solution of problem $\mathcal{P}$ exists in a certain time interval, then it is at the same time the unique classical solution, the unique limit solution and also the unique viscosity solution in that time interval.* We show in particular that there is a unique limit solution independent of the choice of the function β. Moreover, we actually prove that the limit exists for any suitable approximation of the initial datum.

The present jump condition on the free boundary can be generalized in the form

$$G(|\nabla u^+|, |\nabla u^-|) = 0, \tag{1.6}$$

for a suitable function G, increasing in the first argument and decreasing in the second. Such a type of generalized jump condition appears in the study of stationary problems, which arise in hydrodynamics; cf. [7, 8, 9] and their references. The free-boundary evolution problem with jump function $G(a,b) = a - b - M$ has been studied by several authors; cf. [1], [4], [5]. Our results on uniqueness of classical and viscosity solutions apply without changes to these general jump conditions. We have made the choice $G(a,b) = a^2 - b^2 - 2M$ in this paper because it is the one obtained in the limit of the regularized problems $\mathcal{P}_\varepsilon$.

Outline of the paper. In Section 2, we give precise definitions of the classical and viscosity solutions and prove a first consistency result (Propositions 2.1 and 2.2). In Section 3, we prove that, under certain assumptions on the domain and on the initial datum, a classical solution to problem $\mathcal{P}$ is the unique classical solution and also the unique viscosity solution (Theorems 3.1 and 3.2 and Corollary 3.1). These two sections are basically adaptations of the results on the one-phase problem in [17].

The novelty of the two-phase problem begins in Section 4 where we construct the one-dimensional stationary profiles which are needed later in the analysis of the approximations of classical sub- and supersolutions. In particular they will appear as blowup profiles of solutions of the problems $\mathcal{P}_\varepsilon$ as $\varepsilon \to 0$.

In Section 5, we prove that a classical subsolution to problem $\mathcal{P}$ is the uniform limit of a family of subsolutions to problem $\mathcal{P}_\varepsilon$ and we prove the analogous result for supersolutions. The technique of the construction differs from that of the one-phase case in the choice of the approximate initial data, the profiles used in rounding the free boundary gradient discontinuity and the levels at which the pieces of solution are pasted in order to obtain the super- and subsolutions to $\mathcal{P}_\varepsilon$.

In Section 6, we show that, under assumptions similar to those in Section 3, a classical solution to problem $\mathcal{P}$ is the uniform limit of any family of solutions to problem $\mathcal{P}_\varepsilon$ (Theorems 6.1, 6.2 and 6.3).

A final section (Section 7) is devoted to discussing the technical differences between this problem and the one-phase problem, and to commenting on possible extensions and related works.

Notation. Throughout the paper N will denote the spatial dimension, $\Sigma \subset \mathbb{R}^{N-1}$ will be a bounded C^3 domain with unit exterior normal η' and $\eta = (0, \eta')$ will denote the unit exterior normal to $\mathbb{R}\times\Sigma$. In addition, the following notation will be used:

For any $x_0 \in \mathbb{R}^N$, $t_0 \in \mathbb{R}$ and $\tau > 0$, $B_\tau(x_0) := \{x \in \mathbb{R}^N : |x - x_0| < \tau\}$ and $B_\tau(x_0, t_0) := \{(x, t) \in \mathbb{R}^{N+1} : |x - x_0|^2 + |t - t_0|^2 < \tau^2\}$.

When necessary, we will denote points in $\mathbb{R}^N$ by $x = (x_1, x')$, with $x' \in \mathbb{R}^{N-1}$. Given a function v, we will denote $v^+ = \max(v, 0)$, $v^- = \max(-v, 0)$.

The symbols Δ and ∇ will denote the corresponding operators in the space variables; the symbol ∂_p applied to a domain will denote a parabolic boundary.

Let us define the Hölder spaces we are going to use. Let $m \geq 0$ be an integer and $0 < \alpha < 1$. For a space-time cylinder $Q = \Omega \times (0, T) \subset \mathbb{R}^{N+1}$, $C^{m+\alpha, \frac{m+\alpha}{2}}(Q)$ is the parabolic Hölder space denoted by $H^{m+\alpha, \frac{m+\alpha}{2}}(Q)$ in [16]. If $\mathcal{D} \subset \mathbb{R}^{N+1}$ is a general domain, then $C^{m+\alpha, \frac{m+\alpha}{2}}(\mathcal{D})$ will denote the space of functions in $C^{m+\alpha, \frac{m+\alpha}{2}}(Q)$ for every space-time cylinder $Q \subset \mathcal{D}$. If $\mathcal{D}$ is bounded, we will say that $u \in C^{m+\alpha, \frac{m+\alpha}{2}}(\overline{\mathcal{D}})$ if there exists a domain $\mathcal{D}'$ with $\overline{\mathcal{D}} \subset \mathcal{D}'$ and a function $u' \in C^{m+\alpha, \frac{m+\alpha}{2}}(\mathcal{D}')$ such that $u = u'$ in $\overline{\mathcal{D}}$. If $\mathcal{D}$ is unbounded, we will say that $u \in C^{m+\alpha, \frac{m+\alpha}{2}}(\overline{\mathcal{D}})$ if $u \in C^{m+\alpha, \frac{m+\alpha}{2}}(\overline{\mathcal{D}'})$ for every bounded domain $\mathcal{D}' \subset \mathcal{D}$. The space $C^1(\overline{\mathcal{D}})$ is defined in an analogous way.

In addition, M will denote a positive constant that will remain fixed throughout the paper, corresponding to the free-boundary condition (1.2) or its regularizations.

Given a domain $\mathcal{D} \subset \mathbb{R}^{N+1}$, we will write

$$\mathcal{L}u := \Delta u + \sum a_i \, u_{x_i} - u_t, \qquad a_i \in L^\infty(\mathcal{D}) \cap C^{\alpha, \frac{\alpha}{2}}(\overline{\mathcal{D}}).$$

In all the results where the space domain is a cylinder, $\Omega = I \times \Sigma$ with I an interval, we also assume that the coefficients a_i are independent of x_1; that is, $a_i = a_i(x', t)$, $a_i \in C^{\alpha, \frac{\alpha}{2}}(\overline{\Sigma} \times [0, T])$. Finally, we will say that u is supercaloric if $\mathcal{L}u \leq 0$ and u is subcaloric if $\mathcal{L}u \geq 0$.

2. Preliminaries on classical and viscosity solutions

In this section we give precise definitions of the concepts of classical and viscosity solution and derive some consequences. In particular, we prove that in the situations considered in this paper a classical solution is a viscosity solution.

Definition 2.1. Let Q be a space-time domain of the form $\Omega \times (T_1, T_2)$, with $\Omega \subset \mathbb{R}^N$. Let v be a continuous function in $\overline{Q}$. Then v is called a classical subsolution (supersolution) to $\mathcal{P}$ in Q if
(i) $\mathcal{L}v \geq 0 \ (\leq 0)$ in $Q^+ := Q \cap \{v > 0\}$,
(ii) $\mathcal{L}v \geq 0 \ (\leq 0)$ in $Q^- := Q \cap \{v \leq 0\}^\circ$,
(iii) $v \in C^1(\overline{Q^+}) \cap C^1(\overline{Q^-})$, $\nabla v \in C^{\alpha,\frac{\alpha}{2}}(\overline{Q^+}) \cap C^{\alpha,\frac{\alpha}{2}}(\overline{Q^-})$,
(iv) For any $(x,t) \in \{v = 0\} \cap \partial\{v > 0\}$, we have $\nabla v^+(x,t) \neq 0$ and

$$\left(\frac{\partial v^+}{\partial \nu}\right)^2 - \left(\frac{\partial v^-}{\partial \nu}\right)^2 \geq 2M \qquad (\leq 2M), \tag{2.1}$$

where $\nu := -\frac{\nabla v^+}{|\nabla v^+|}$. We say that v is a classical solution to $\mathcal{P}$ in Q if it is both a classical subsolution and a classical supersolution to $\mathcal{P}$.

There is a subtle notation aspect in the preceding definition. According to (iii) v^+ can be continued as a C^1 function in a neighbourhood of Q^+. What we are really imposing in (iv) is that the gradient of this extended function does not vanish on $\{v = 0\} \cap \partial\{v > 0\}$, and we call this gradient ∇v^+, though it really is the (lateral) gradient of v restricted to $\overline{Q^+}$. We also have $|\nabla v^+| = -\partial v^+/\partial\nu$ as an appropriate lateral limit. Using the same conventions on the gradient of v^- we may write the jump condition as

$$|\nabla v^+|^2 - |\nabla v^-|^2 \geq 2M \qquad (\leq 2M). \tag{2.2}$$

Definition 2.2. Let $u \in C(\overline{Q})$; u is called a viscosity subsolution (supersolution) to $\mathcal{P}$ in Q if, for every space-time subcylinder $Q' \subset Q$ and for every v a bounded, classical supersolution (subsolution) to $\mathcal{P}$ in Q', with $Q' \cap \partial\{v > 0\}$ bounded,

$$u \leq v \quad (u \geq v) \quad \text{on } \partial_p Q' \text{ and}$$
$$v > 0 \quad \text{on } \overline{\{u > 0\}} \cap \partial_p Q' \quad (u > 0 \quad \text{on } \overline{\{v > 0\}} \cap \partial_p Q')$$

implies that $u \leq v$ $(u \geq v)$ in Q'.

The function u is called a viscosity solution to $\mathcal{P}$ if it is both a viscosity supersolution and a viscosity subsolution to $\mathcal{P}$.

We can now prove the consistency between both concepts of solution.

Proposition 2.1. *If u is a bounded, classical supersolution (subsolution) to $\mathcal{P}$ in Q with $Q \cap \partial\{u > 0\}$ bounded, then u is a viscosity supersolution (subsolution) to $\mathcal{P}$ in Q.*

Proof. The proof follows the lines of the proof of Proposition 2.1 in [17].

Definition 2.3. Let $\Omega \subset \mathbb{R}^N$ be a domain and let $Q = \Omega \times (0,T)$. Let Γ_N be an open C^1 subset of $\partial\Omega$ and let $\partial_N Q = \Gamma_N \times (0,T)$. We say that $u \in C(\overline{Q})$ is a viscosity solution to $\mathcal{P}$ in Q with $\frac{\partial u}{\partial \eta} = 0$ on $\partial_N Q$, if the following holds: for every space-time subcylinder $Q' \subset Q$ and for every v a bounded, classical supersolution (subsolution) to $\mathcal{P}$ in Q', with $Q' \cap \partial\{v > 0\}$ bounded, such that $\frac{\partial v}{\partial \eta} = 0$ on $\partial_p Q' \cap \partial_N Q$,

$$
\begin{aligned}
&u \le v \quad (u \ge v) \quad \text{ on } \partial_p Q' \setminus \partial_N Q \quad \text{and} \\
&v > 0 \;\text{ on }\; \overline{\{u > 0\}} \cap \overline{\partial_p Q' \setminus \partial_N Q} \quad \left(u > 0 \quad \text{ on } \overline{\{v > 0\}} \cap \overline{\partial_p Q' \setminus \partial_N Q}\right)
\end{aligned}
$$

implies that $u \le v$ $(u \ge v)$ in Q'.

Proposition 2.2. *Let $\Omega = \mathbb{R} \times \Sigma$ (respectively $(0,+\infty) \times \Sigma$, $(-\infty,d) \times \Sigma$, $(0,d) \times \Sigma$), $Q = \Omega \times (0,T)$ and $\partial_N Q = \mathbb{R} \times \partial\Sigma \times (0,T)$ (respectively $\partial_N Q = (0,+\infty) \times \partial\Sigma \times (0,T)$, $\partial_N Q = (-\infty,d) \times \partial\Sigma \times (0,T)$, $\partial_N Q = (0,d) \times \partial\Sigma \times (0,T)$).*

Let u be a bounded classical solution to $\mathcal{P}$ in Q with $Q \cap \partial\{u > 0\}$ bounded and $\frac{\partial u}{\partial \eta} = 0$ on $\partial_N Q$. Then u is a viscosity solution to $\mathcal{P}$ in Q with $\frac{\partial u}{\partial \eta} = 0$ on $\partial_N Q$.

Proof. The proof follows the lines of the proof of Proposition 2.2 in [17]. □

In the next propositions we will show that, in the situations considered in this paper, a classical solution has a bounded free boundary, and in particular, it is a viscosity solution.

Proposition 2.3. *Let $\Omega = \mathbb{R} \times \Sigma$, $Q = \Omega \times (0,T)$, $\partial_N Q = \mathbb{R} \times \partial\Sigma \times (0,T)$ and $\partial_D Q = \partial_p Q \setminus \partial_N Q$. Let u be a bounded classical solution to $\mathcal{P}$ in Q with $\frac{\partial u}{\partial \eta} = 0$ on $\partial_N Q$ and $||u||_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q})} < \infty$, such that $u\big|_{\partial_D Q}$ has a bounded, nonempty free boundary and $u_{x_1} < 0$ on $\partial_D Q$. Then $Q \cap \partial\{u > 0\}$ is bounded.*

Proof. The proof follows the lines of the proof of Proposition 2.4 in [17]. In fact, for $\mathcal{A} \ge ||a_1||$, $L = ||u||_{L^\infty(Q)}$, $c > 0$ small and $K > 0$ large, let

$$
\begin{aligned}
v_-(x,t) = {}& c\Big(1 - \exp\Big\{\frac{\alpha}{c}(x_1 + \mathcal{A}t) + \frac{\alpha^2}{c^2}t + K\frac{\alpha}{c}\Big\}\Big)^+ \\
& - \frac{\gamma}{\alpha}c\Big(1 - \exp\big\{\frac{\alpha}{c}(x_1 + \mathcal{A}t) + \frac{\alpha^2}{c^2}t + K\frac{\alpha}{c}\big\}\Big)^-
\end{aligned}
$$

where $\gamma > 0$ and $\alpha = \sqrt{2M + \gamma^2}$, and let

$$
v_+(x,t) = 2L\Big(1 - \exp\Big\{\frac{\sqrt{2M}}{2L}(x_1 - \mathcal{A}t) + \frac{2M}{4L^2}t - \frac{K\sqrt{2M}}{2L} - \log 2\Big\}\Big)^+.
$$

It holds that v_- is a bounded classical subsolution to $\mathcal{P}$ in Q and v_+ is a bounded classical supersolution to $\mathcal{P}$ in Q with $\frac{\partial v_+}{\partial \eta} = \frac{\partial v_-}{\partial \eta} = 0$ on $\partial_N Q$. In addition, $v_\pm$ have bounded free boundaries, and $v_-(x,0) \leq u(x,0) \leq v_+(x,0)$. Moreover,

$$u > 0 \quad \text{on } \overline{\{v_- > 0\}} \cap \{t = 0\}, \qquad v_+ > 0 \quad \text{on } \overline{\{u > 0\}} \cap \{t = 0\}.$$

Therefore, proceeding as in the proof of Proposition 2.4 in [17] we get

$$v_-(x,t) \leq u(x,t) \leq v_+(x,t) \quad \text{in } \; Q \cap \{t \leq T\},$$

which implies that $Q \cap \partial\{u > 0\}$ is bounded and completes the proof. $\square$

The next propositions can be proved in a way similar to Proposition 2.3 (in the proof of Proposition 2.5 we use Proposition 2.1 instead of Proposition 2.2).

Proposition 2.4. *Let $\Omega = (0,+\infty) \times \Sigma$, $Q = \Omega \times (0,T)$, $\partial_N Q = (0,+\infty) \times \partial\Sigma \times (0,T)$ and $\partial_D Q = \partial_p Q \setminus \partial_N Q$. Let u be a bounded classical solution to $\mathcal{P}$ in Q with $\frac{\partial u}{\partial \eta} = 0$ on $\partial_N Q$ and $||u||_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q})} < \infty$, such that $u\big|_{\partial_D Q}$ has a bounded, nonempty free boundary and $u_{x_1} < 0$ on $\partial_D Q$. Assume that $u(0,x',t) > 0$ for $(x',t) \in \overline{\Sigma} \times [0,T]$. Then $Q \cap \partial\{u > 0\}$ is bounded.*

An analogous result holds if we let $\Omega = (-\infty,d) \times \Sigma$ or $\Omega = (0,d) \times \Sigma$ with the corresponding sign assumptions on u on $x_1 = 0, d$.

Proposition 2.5. *Let $\Omega = (0,+\infty) \times \Sigma$, $Q = \Omega \times (0,T)$ and $\partial_D Q = \partial_p Q$. Let u be a bounded classical solution to $\mathcal{P}$ in Q with $||u||_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q})} < \infty$, such that $u\big|_{\partial_D Q}$ has a bounded, nonempty free boundary and $u_{x_1} < 0$ on $\partial_D Q$. Assume that $u(0,x',t) > 0$ for $(x',t) \in \overline{\Sigma} \times [0,T]$. Then $Q \cap \partial\{u > 0\}$ is bounded.*

The same result holds if we let instead $\Omega = \mathbb{R} \times \Sigma$ (with no assumptions on u on $\{0\} \times \overline{\Sigma} \times [0,T]$), and also if $\Omega = (-\infty,d) \times \Sigma$ or $\Omega = (0,d) \times \Sigma$ with the corresponding sign assumptions on u on $x_1 = 0, d$.

3. Uniqueness of classical and viscosity solutions

In this section we show that, under suitable assumptions, a classical solution is the unique viscosity solution to the initial and boundary value problem associated to $\mathcal{P}$ and, in particular, it is the unique classical solution. This is done in Theorems 3.1 and 3.2 and Corollary 3.1. We also show comparison.

Theorem 3.1. *Let $\Omega = (0,d) \times \Sigma$, $Q = \Omega \times (0,T)$, $\partial_N Q = (0,d) \times \partial\Sigma \times (0,T)$ and $\partial_D Q = \partial_p Q \setminus \partial_N Q$. Let u be a bounded classical solution to $\mathcal{P}$ in Q, with $\frac{\partial u}{\partial \eta} = 0$ on $\partial_N Q$, such that $u_{x_1} < 0$ on $\partial_D Q$. Assume that $u(0,x',t) > 0$*

for $(x',t) \in \overline{\Sigma} \times [0,T]$ *with* $u(0,x',t) \in C^{2,1}(\overline{\Sigma} \times [0,T])$ *and* $u(d,x',t) < 0$ *for* $(x',t) \in \overline{\Sigma} \times [0,T]$ *with* $u(d,x',t) \in C^{2,1}(\overline{\Sigma} \times [0,T])$*. Let* $v \in C(\overline{Q})$ *be a viscosity solution to* $\mathcal{P}$ *in* Q *with* $\frac{\partial v}{\partial \eta} = 0$ *on* $\partial_N Q$*. If* $v = u$ *on* $\partial_D Q$*, then* $v = u$ *in* $\overline{Q}$*.*

An analogous result holds if we let $\Omega = (0,+\infty) \times \Sigma$ *or* $\Omega = (-\infty, d) \times \Sigma$*, with the corresponding sign condition on* u *on* $x_1 = 0$ *or* $x_1 = d$*, or if* $\Omega = \mathbb{R} \times \Sigma$ *with no sign condition on* u*. In these cases we require that* $u\big|_{\partial_D Q}$ *has a bounded, nonempty free boundary and* $||u||_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q})} < \infty$*.*

Proof. Let $a > 0$; we will extend u to $Q_a = (-a, d+a) \times \Sigma \times (0,T)$ in such a way that $u \in C(\overline{Q}_a)$, $u > 0$ in $-a < x_1 < 0$, $u < 0$ in $d < x_1 < d+a$, $\mathcal{L}u \leq 0$ in $Q_a \cap \{u > 0\}$ and $\mathcal{L}u \geq 0$ in $Q_a \cap \{u < 0\}$. Let $c > 0$ be such that $u_{x_1} \leq -c$ on $\partial_D Q$. Let $F_0(x',t) \in C^{2,1}(\overline{\Sigma} \times [0,T])$ be such that $u(0,x',t) = F_0(x',t)$, and let $F_d(x',t) \in C^{2,1}(\overline{\Sigma} \times [0,T])$ be such that $u(d,x',t) = F_d(x',t)$. Finally, let us define $u(x_1,x',t)$ in $Q_a \setminus Q$ in the following way:

$$u(x_1,x',t) = F_0(x',t) - cx_1 - kx_1^2 \quad \text{for } x_1 \in (-a,0),$$
$$u(x_1,x',t) = F_d(x',t) - c(x_1 - d) + k(x_1 - d)^2 \quad \text{for } x_1 \in (d, d+a).$$

Thus, clearly if k is large enough u satisfies all the requirements. Now the result follows by proceeding in a way similar to the proof of Theorem 3.1 in [17]. □

For two classical solutions we have the following uniqueness result, a consequence of Proposition 2.2 and Theorem 3.1.

Corollary 3.1. *Let* Ω*,* Q*,* $\partial_N Q$*,* $\partial_D Q$ *and* u *be as in Theorem* 3.1*. Let* v *be a bounded classical solution to* $\mathcal{P}$ *in* Q *with* $\frac{\partial v}{\partial \eta} = 0$ *on* $\partial_N Q$*, such that* $v = u$ *on* $\partial_D Q$*. Then,* $v = u$ *in* $\overline{Q}$*.*

A comparison principle for bounded classical solutions follows from Proposition 2.2 if the free boundaries are bounded in Q and separated on $\partial_D Q$. With a monotonicity assumption on $\partial_D Q$, we get a different comparison result.

Corollary 3.2. *Let* Ω*,* Q*,* $\partial_N Q$*,* $\partial_D Q$ *and* u *as in Theorem* 3.1*. Let* v *be a bounded classical solution to* $\mathcal{P}$ *in* Q *with* $\frac{\partial v}{\partial \eta} = 0$ *on* $\partial_N Q$*, and such that* $Q \cap \partial\{v > 0\}$ *is bounded. If* $v \geq u$ *on* $\partial_D Q$*, it holds that* $v \geq u$ *in* $\overline{Q}$*.*

In the next theorem we prove the uniqueness of the viscosity solution under different assumptions from those in Theorem 3.1. As in Corollaries 3.1 and 3.2, uniqueness and comparison of classical solutions follow.

Theorem 3.2. *The results of Theorem* 3.1 *hold if we let instead* $\partial_N Q = \emptyset$ *so that* $\partial_D Q = \partial_p Q$.

Corollary 3.3. *Let* u *be as in Theorem* 3.1. *Then* u *is a decreasing function in* $\overline{Q}$ *in the direction* $e_1 = (1, 0, \dots, 0)$ *and* $u_{x_1} < 0$ *in* $\overline{Q} \cap \{u > 0\}$ *and in* $\overline{Q} \cap \{u < 0\}$. *Moreover, for every* $\varepsilon \neq 0$, *the level set* $\{u = \varepsilon\}$ *is given by* $x_1 = g_\varepsilon(x', t)$ *with* $g_\varepsilon \in C^1(\overline{\Sigma} \times [0, T])$ *and* $\nabla_{x'} g_\varepsilon \in C^{\alpha, \frac{\alpha}{2}}(\overline{\Sigma} \times [0, T])$.

The same conclusion holds under the assumptions of Theorem 3.2.

Proof. Proceeding as in Corollary 3.3 in [17], we deduce that for every $\delta > 0$ small, $u(x, t) \geq u_\delta(x, t) = u(x_1 + \delta, x', t)$ for $(x, t) \in \overline{Q}$, which implies that $u_{x_1} \leq 0$ in $\{u > 0\} \cup \{u \leq 0\}^\circ$.

Then, reasoning in a way similar to Corollary 3.3 in [17] we can see that $u_{x_1} < 0$ both in $\{u > 0\} \cap \overline{Q}$ and in $\{u \leq 0\}^\circ \cap \overline{Q}$, and thus the result follows.

4. Auxiliary two-phase profiles

This section contains the construction of the one-dimensional stationary solution of the simplified problem

$$\Delta u - u_t = \beta(u) \tag{4.1}$$

where the function β is as in Section 1 and $M = \int \beta(s)\, ds$. The results will be used in the next sections where (4.1) appears as a blowup limit. The transport term will disappear in the blowup process.

We start from the piecewise-linear solution of the free-boundary problem given in an interval $(0, R)$ by the formula: $u(s) = A - \alpha s$ for $0 \leq s \leq \frac{A}{\alpha}$, $u(s) = -A - \gamma(s - R)$ for $\frac{A}{\alpha} \leq s \leq R$, where $A, \alpha, \gamma > 0$ and $R = \frac{A}{\alpha} + \frac{A}{\gamma}$. The free-boundary condition implies the relation $\alpha^2 = 2M + \gamma^2$. We are interested in constructing sub- and supersolutions. In the first case we will replace the term $2M$ in the previous free-boundary condition by $2M + \delta_0$, in the second case, by $2M - \delta_0$.

Our aim is to construct solutions of (4.1) in the same interval joining the level $u = A$ at $s = 0$ to $u = -A$ at $s = R$ in such a way that the slope at $s = 0$ is larger than $-\alpha$ and at $s = R$ smaller than $-\gamma$ for the case of subsolutions. The inequalities are reversed for the case of supersolutions.

We start by analyzing the initial value problem.

Lemma 4.1. *For every* $L > 0$ *and* $\delta_0 > 0$ *there exists* $A(L, M, \delta_0) > 1$ *such that for every* $A \geq A(L, M, \delta_0)$, $\alpha, \gamma > 0$ *and* $\gamma \leq L$ *with* $\alpha^2 = 2M + \delta_0 + \gamma^2$ *there exists* $0 < \delta < \delta_0$ *such that the solution to*

$$\begin{cases} \psi'' = \beta(\psi) & \text{for } s > 0 \\ \psi(0) = A, & \psi'(0) = -\sqrt{\alpha^2 - \delta} \end{cases} \tag{4.2}$$

FIGURE 1. Construction of ψ, $\alpha' = \sqrt{\alpha^2 - \delta}$, $\gamma' = \sqrt{\gamma^2 + \delta_0 - \delta}$

satisfies that $\psi(R) = -A$, *where* $R = \frac{A}{\alpha} + \frac{A}{\gamma}$. *Moreover,*

$$\psi'(R) = -\sqrt{\gamma^2 + \delta_0 - \delta} < -\gamma.$$

Proof. Observe that β acts only on the range $0 < \psi < 1$ and also that we will take $A > 1$. Let $0 < \delta < \delta_0$ and $A > 1$ be fixed for the moment. Let $s_0 = \frac{A-1}{\sqrt{\alpha^2 - \delta}}$. Then the solution ψ to (4.2) satisfies $\psi(s) = A - \sqrt{\alpha^2 - \delta}\; s$ for $0 < s < s_0$ and $\psi(s_0) = 1$. Let $B(\psi) = \int_0^{\psi} \beta(\tau)\, d\tau$. Then,

$$(\psi')^2 = \alpha^2 - \delta - 2\int_{\psi(s)}^{1} \beta(\tau)\, d\tau = \gamma^2 + \delta_0 - \delta + 2B(\psi(s)),$$

so that

$$\int_{\psi(s)}^{1} \frac{d\psi}{\sqrt{\gamma^2 + \delta_0 - \delta + 2B(\psi)}} = s - s_0.$$

Let $s_1 = s_0 + \int_0^1 \frac{d\psi}{\sqrt{\gamma^2 + \delta_0 - \delta + 2B(\psi)}}$. Then $\psi(s_1) = 0$, $\psi'(s_1) = -\sqrt{\gamma^2 + \delta_0 - \delta}$ and $\psi(s) = -\sqrt{\gamma^2 + \delta_0 - \delta}\; (s - s_1)$ for $s > s_1$. Thus,

$$\psi(R) = -\sqrt{\gamma^2 + \delta_0 - \delta}(R - s_1).$$

Therefore, $\psi(R) = -A$ if and only if

$$A\Big[\frac{-1}{\sqrt{\gamma^2 + \delta_0 - \delta}} + \frac{1}{\sqrt{2M + \delta_0 + \gamma^2}} + \frac{1}{\gamma}\Big] = s_1. \tag{4.3}$$

Equation (4.3) is equivalent to

$$A\Big[\frac{1}{\sqrt{2M+\delta_0+\gamma^2}}-\frac{1}{\sqrt{2M+\delta_0-\delta+\gamma^2}}+\frac{1}{\gamma}-\frac{1}{\sqrt{\gamma^2+\delta_0-\delta}}\Big]$$
$$+\frac{1}{\sqrt{2M+\delta_0-\delta+\gamma^2}}-\int_0^1\frac{d\psi}{\sqrt{\gamma^2+\delta_0-\delta+2B(\psi)}}=0. \tag{4.4}$$

In order to solve this equation we consider the function

$$f(\delta)=A\Big[\frac{1}{\sqrt{2M+\delta_0+\gamma^2}}-\frac{1}{\sqrt{2M+\delta_0-\delta+\gamma^2}}+\frac{1}{\gamma}-\frac{1}{\sqrt{\gamma^2+\delta_0-\delta}}\Big]$$
$$+\frac{1}{\sqrt{2M+\delta_0-\delta+\gamma^2}}-\int_0^1\frac{d\psi}{\sqrt{\gamma^2+\delta_0-\delta+2B(\psi)}}.$$

We will see that there exists $A(L,M,\delta_0)>1$ such that if $A>A(L,M,\delta_0)$ it holds that $f(0)>0$ and $f(\delta_0)<0$. This will prove the lemma. In fact, since $0<\gamma\le L$,

$$\begin{aligned}
f(0)&=A\Big(\frac{1}{\gamma}-\frac{1}{\sqrt{\gamma^2+\delta_0}}\Big)+\frac{1}{\sqrt{2M+\delta_0+\gamma^2}}-\int_0^1\frac{d\psi}{\sqrt{\gamma^2+\delta_0+2B(\psi)}}\\
&\ge A\frac{\delta_0}{2(L^2+\delta_0)^{\frac32}}+\frac{1}{\sqrt{2M+\delta_0+L^2}}-\int_0^1\frac{d\psi}{\sqrt{\delta_0+2B(\psi)}}\\
&> A\frac{\delta_0}{2(L^2+\delta_0)^{\frac32}}-\frac{1}{\sqrt{\delta_0}}>0
\end{aligned}$$

if $A\ge A_0(L,\delta_0)$. On the other hand, if $A\ge A_1(L,\delta_0,M)$, then

$$\begin{aligned}
f(\delta_0)&=A\Big(\frac{1}{\sqrt{2M+\delta_0+\gamma^2}}-\frac{1}{\sqrt{2M+\gamma^2}}\Big)+\frac{1}{\sqrt{2M+\gamma^2}}-\int_0^1\frac{d\psi}{\sqrt{\gamma^2+2B(\psi)}}\\
&\le -A\frac{\delta_0}{2(2M+\delta_0+L^2)^{\frac32}}+\frac{1}{\sqrt{2M}}<0.
\end{aligned}$$

Corollary 4.1. *Let $L>0$ and $\delta_0>0$. Let $A(L,M,\delta_0)>1$ as in Lemma 4.1. Then, for every $A\ge A(L,M,\delta_0)$, $\alpha,\gamma>0$ and $\gamma\le L$ with $\alpha^2=2M+\delta_0+\gamma^2$, there exists a unique solution to*

$$\begin{cases}\varphi''=\beta(\varphi) & \text{for } 0<s<R\\ \varphi(0)=A, \quad \varphi(R)=-A\end{cases} \tag{4.5}$$

with $R=\frac{A}{\alpha}+\frac{A}{\gamma}$. This solution satisfies that there exists $0<\delta<\delta_0$ such that

$$|\varphi'(0)|=\sqrt{\alpha^2-\delta} \quad \text{and} \quad |\varphi'(R)|=\sqrt{\gamma^2+\delta_0-\delta}. \tag{4.6}$$

Proof. Let $A(L, M, \delta_0)$ be as in Lemma 4.1. Let $0 < \delta < \delta_0$ be such that the solution ψ^δ to (4.2) satisfies that $\psi^\delta(R) = -A$. Then, ψ^δ satisfies (4.6). So that, it only remains to prove that this is the only solution to (4.5). In fact, let ψ_b be the solution to

$$\begin{cases} \psi'' = \beta(\psi) & \text{for } s > 0 \\ \psi(0) = A, & \psi'(0) = -b. \end{cases} \tag{4.7}$$

If $b \leq \sqrt{2M}$, it holds that $\psi_b \geq 0$ (see [17], Lemma 4.1) so that $\psi_b(R) \neq -A$. If $b > \sqrt{2M}$, it holds that

$$\psi_b(R) = -\sqrt{b^2 - 2M}\ \Big(R - \frac{A-1}{b} - \int_0^1 \frac{d\psi}{\sqrt{2B(\psi) - 2M + b^2}}\Big).$$

From this formula, it is easy to see that $\psi_{b_1}(R) < \psi_{b_2}(R)$ if $b_1 > b_2$, so that ψ^δ is the unique solution to (4.5), and the corollary is proved. □

We turn now to the case of supersolutions.

Lemma 4.2. *For every $L > 0$ and $\delta_0 > 0$ there exists $A(L, M, \delta_0) > 1$ such that for every $A \geq A(L, M, \delta_0)$, $\alpha > 0$ and $\sqrt{\delta_0} \leq \gamma \leq L$ with $\alpha^2 = 2M - \delta_0 + \gamma^2$, there exists $0 < \delta < \delta_0$ such that the solution to*

$$\begin{cases} \psi'' = \beta(\psi) & \textit{for } s > 0 \\ \psi(0) = A, & \psi'(0) = -\sqrt{\alpha^2 + \delta} \end{cases} \tag{4.8}$$

satisfies that $\psi(R) = -A$ where $R = \frac{A}{\alpha} + \frac{A}{\gamma}$. Moreover,

$$\psi'(R) = -\sqrt{\gamma^2 - \delta_0 + \delta} > -\gamma.$$

Proof. The proof follows as that of Lemma 4.1.

Corollary 4.2. *Let $L > 0$ and $\delta_0 > 0$. Let $A(L, M, \delta_0) > 1$ as in Lemma 4.2. Then, for every $A \geq A(L, M, \delta_0)$, $\alpha > 0$ and $\sqrt{\delta_0} \leq \gamma \leq L$ with $\alpha^2 = 2M - \delta_0 + \gamma^2$, there exists a unique solution to* (4.5) *with $R = \frac{A}{\alpha} + \frac{A}{\gamma}$. This solution satisfies that there exists $0 < \delta < \delta_0$ such that*

$$|\varphi'(0)| = \sqrt{\alpha^2 + \delta} \quad \textit{and} \quad |\varphi'(R)| = \sqrt{\gamma^2 - \delta_0 + \delta}. \tag{4.9}$$

Proof. The proof follows as that of Corollary 4.1. □

Let us now make precise the relation between the solution to (4.7) and that of (4.5). For $b > \sqrt{2M}$, let us call $\psi(b, s)$ the solution to (4.7). Let

$$R(b) = \frac{A-1}{b} + \int_0^1 \frac{d\psi}{\sqrt{2B(\psi) - 2M + b^2}} + \frac{A}{\sqrt{b^2 - 2M}}.$$

Then, $\psi(b,R(b))=-A$. On the other hand, let

$$s(b)=\frac{A-1}{b}+\int_0^1\frac{d\psi}{\sqrt{2B(\psi)-2M+b^2}};$$

then $\psi(b,s(b))=0$. Observe that $R(b)$ and $s(b)$ are C^∞ functions in $(\sqrt{2M},\infty)$, and $\frac{\partial R}{\partial b}<0$. Now, let $b(R)$ be the inverse of $R(b)$, so that $b\in C^\infty(0,+\infty)$. Finally, let $\varphi(R,s)=\psi(b(R),s)$. That is, $\varphi(R,s)$ is the solution to (4.5). The following holds:

Proposition 4.1. *φ is locally Lipschitz continuous in $\{(R,s):R>0\,,0\le s\le R\}$. Moreover, for every $R_1>0$ there exists $r_1>0$ such that $\varphi\in C^\infty(\{(R,s)\,:\,R\ge R_1\,,\,0\le s\le r_1\}\cup\{(R,s)\,:\,R\ge R_1\,,\,R-r_1\le s\le R\})$.*

Proof. The Lipschitz continuity of φ as a function of (R,s) follows immediately from that of ψ as a function of its initial datum b and the variable s. On the other hand,

$$\psi(b,s)=A-bs \qquad \text{for } 0\le s\le (A-1)/b$$
$$\psi(b,s)=-\sqrt{b^2-2M}\,(s-s(b)) \quad \text{for } s\ge s(b).$$

Therefore, if $b\le b_1$ it follows that there exists $r_1>0$ such that

$$\psi(b,s)=A-bs \qquad \text{for } 0\le s\le r_1$$
$$\psi(b,s)=-\sqrt{b^2-2M}\,(s-s(b)) \quad \text{for } R(b)-r_1\le s\le R(b).$$

Thus, for every $R_1>0$ if we let $b_1=b(R_1)$, we see that the result on the C^∞ regularity of φ follows. □

Now we prove a characterization of global solutions to (4.1).

Lemma 4.3. *Let $\mathcal{R}_\eta=\{(x,t):0<x_1<R,-\infty<t\le\eta\}$, $A=A(L,M,\delta_0)>1$ as in Corollary 4.1 and $U\in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\mathcal{R}}_\eta)$ be such that*

$$\begin{cases}\Delta U-U_t=\beta(U) & \text{in } \mathcal{R}_\eta\\ U=A & \text{in } \{x_1=0\}\\ U=-A & \text{in } \{x_1=R\}\\ -A\le U\le A & \text{in } \overline{\mathcal{R}}_\eta\end{cases} \tag{4.10}$$

with $R=\frac{A}{\alpha}+\frac{A}{\gamma}$, $\alpha^2-\gamma^2=2M+\delta_0$, $\alpha,\gamma>0$, $\gamma\le L$ and $\delta_0>0$. There exists $0<\delta<\delta_0$ such that $U(x,t)=\psi^\delta(x_1)$ where ψ^δ is the solution to (4.2). Thus,

$$|\nabla U|_{x_1=0}=\sqrt{\alpha^2-\delta}\quad,\quad |\nabla U|_{x_1=R}=\sqrt{\gamma^2+\delta_0-\delta}. \tag{4.11}$$

Proof. Let V be the solution to

$$\begin{cases} \Delta V - V_t = \beta(V) & \text{for } 0 < x_1 < R\ ,\ t > 0 \\ V = A & \text{for } x_1 = 0\ ,\ t > 0 \\ V = -A & \text{for } x_1 = R\ ,\ t > 0 \\ V = -A & \text{for } t = 0, \end{cases}$$

and let W be the solution to

$$\begin{cases} \Delta W - W_t = \beta(W) & \text{for } 0 < x_1 < R\ ,\ t > 0 \\ W = A & \text{for } x_1 = 0\ ,\ t > 0 \\ W = -A & \text{for } x_1 = R\ ,\ t > 0 \\ W = A & \text{for } t = 0. \end{cases}$$

Then, $V = V(x_1, t)$ and $W = W(x_1, t)$. Let $V_k(x,t) = V(x, t+k)$, $W_k(x,t) = W(x, t+k)$ for $t \geq -k$. Then,

$$V_k(x,t) \leq U(x,t) \leq W_k(x,t) \qquad \text{for } 0 < x_1 < R\ ,\ -k \leq t \leq \eta.$$

Since $V_k(x,t) \to \psi^-(x)$ and $W_k(x,t) \to \psi^+(x)$, $k \to \infty$ where $\psi^- = \psi^-(x_1)$ and $\psi^+ = \psi^+(x_1)$ are solutions to (4.5), and (4.5) has a unique solution ψ (by Corollary 4.1), it holds that $U(x,t) \equiv \psi(x_1)$. In particular, by Corollary 4.1 there exists $0 < \delta < \delta_0$ such that $|\nabla U|_{x_1=0} = |\psi'(0)| = \sqrt{\alpha^2 - \delta}$ and $|\nabla U|_{x_1=R} = |\psi'(R)| = \sqrt{\gamma^2 + \delta_0 - \delta}$, so that the lemma is proved. □

Analogously,

Lemma 4.4. *Let $\mathcal{R}_\eta$ and U be as in Lemma 4.3 with $R = \frac{A}{\alpha} + \frac{A}{\gamma}$, $\alpha^2 - \gamma^2 = 2M - \delta_0$, $\alpha > 0$, $\sqrt{\delta_0} \leq \gamma \leq L$ and $A = A(L, M, \delta_0)$ as in Corollary 4.2. There exists $0 < \delta < \delta_0$ such that*

$$|\nabla U|_{x_1=0} = \sqrt{\alpha^2 + \delta}, \qquad |\nabla U|_{x_1=R} = \sqrt{\gamma^2 - \delta_0 + \delta}. \tag{4.12}$$

Proof. The proof follows as that of Lemma 4.3. Here we use Corollary 4.2 instead of Corollary 4.1.

5. Approximation results

In this section we prove that, under certain assumptions, a classical subsolution to problem $\mathcal{P}$ is the uniform limit of a family of subsolutions to problem $\mathcal{P}_\varepsilon$ (Theorem 5.1). We prove the analogous result for supersolutions (Theorem 5.2).

Throughout this section we will assume that $\Omega = \mathbb{R} \times \Sigma$ is a full cylinder (respectively $\Omega = (0, +\infty) \times \Sigma$ and $\Omega = (-\infty, d) \times \Sigma$ is a semicylinder or $\Omega = (0, d) \times \Sigma$ is a bounded cylinder). We define $Q = \Omega \times (0, T)$, and we let $\partial_N Q = \mathbb{R} \times \partial\Sigma \times (0, T)$ (respectively $\partial_N Q = (0, +\infty) \times \partial\Sigma \times (0, T)$ and

$\partial_N Q = (-\infty, d) \times \partial\Sigma \times (0,T)$ or $\partial_N Q = (0,d) \times \partial\Sigma \times (0,T)$). In addition, w will be a function satisfying the following list of conditions:

(i) For every $A > 0$, there exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$, $\{w > A\varepsilon\}$ is given by $x_1 < p_\varepsilon(x',t)$ and $\{w < -A\varepsilon\}$ is given by $x_1 > q_\varepsilon(x',t)$, with $p_\varepsilon, q_\varepsilon \in C^1(\overline{\Sigma} \times [0,T])$ and $\nabla_{x'} p_\varepsilon, \nabla_{x'} q_\varepsilon \in C^{\alpha,\frac{\alpha}{2}}(\overline{\Sigma} \times [0,T])$. Moreover, $||p_\varepsilon(x',0)||_{C^{1+\alpha}(\overline{\Sigma})} \le C$ and $||q_\varepsilon(x',0)||_{C^{1+\alpha}(\overline{\Sigma})} \le C$ for ε small.

(ii) $|\nabla w^-(x_0,t_0)| > 0$ for every $(x_0,t_0) \in \overline{Q \cap \partial\{w>0\}}$.

(iii) In case $\Omega = (0,+\infty) \times \Sigma$, we assume that $w(0,x',t) > 0$ for $(x',t) \in \overline{\Sigma} \times [0,T]$.

(iv) In case $\Omega = (-\infty,d) \times \Sigma$, we assume that $w(d,x',t) < 0$ for $(x',t) \in \overline{\Sigma} \times [0,T]$.

(v) In case $\Omega = (0,d) \times \Sigma$, we assume that $w(0,x',t) > 0$ for $(x',t) \in \overline{\Sigma} \times [0,T]$ and that $w(d,x',t) < 0$ for $(x',t) \in \overline{\Sigma} \times [0,T]$.

We call this list of conditions **(H1)**.

Theorem 5.1. *Let w be a classical subsolution to $\mathcal{P}$ in Q, with $\frac{\partial w}{\partial \eta} = 0$ on $\partial_N Q$, satisfying* **(H1)**. *Assume, in addition, that there exists $\delta_0 > 0$ such that*

$$|\nabla w^+|^2 - |\nabla w^-|^2 = 2M + \delta_0 \qquad \text{on } Q \cap \partial\{w>0\}.$$

Let $A = A(L,M,\delta_0) > 1$ be the constant in Lemma 4.1, where $L > 0$ is such that $|\nabla w| \le L$ in a neighborhood of the free boundary $\overline{Q \cap \partial\{w>0\}}$.

Then, there exists a family $v^\varepsilon \in C(\overline{Q})$, with $\nabla v^\varepsilon \in L^2_{loc}(\overline{Q})$, of weak subsolutions to $\mathcal{P}_\varepsilon$ in Q, with $\frac{\partial v^\varepsilon}{\partial \eta} = 0$ on $\partial_N Q$, such that, as $\varepsilon \to 0$, $v^\varepsilon \to w$ uniformly in $\overline{Q}$.

Moreover, $v^\varepsilon = w$ in $\{|w| \ge A\varepsilon\}$ and $\nabla v^\varepsilon \in C(\{|w| \le A\varepsilon\} \cap \{t > 0\})$.

Proof. Step I. Construction of the family v^ε. For every $\varepsilon > 0$ small, we define the domain $\mathcal{D}^\varepsilon$ in the following way: $\mathcal{D}^\varepsilon = \{(x,t) \in Q : p_\varepsilon(x',t) < x_1 < q_\varepsilon(x',t)\}$. Let w^ε be the solution to $\mathcal{P}_\varepsilon$ in $\mathcal{D}^\varepsilon$ with boundary data

$$w^\varepsilon(x,t) = \begin{cases} A\varepsilon & \text{on } x_1 = p_\varepsilon(x',t), \\ -A\varepsilon & \text{on } x_1 = q_\varepsilon(x',t), \end{cases}$$

$$\frac{\partial w^\varepsilon}{\partial \eta} = 0 \text{ on } \partial_N \mathcal{D}^\varepsilon := \partial \mathcal{D}^\varepsilon \cap \partial_N Q,$$

and initial data

$$w_0^\varepsilon(x) \qquad \text{on } \partial\mathcal{D}^\varepsilon \cap \{t = 0\}.$$

In a first stage we make an additional assumption. We need some notation: For every $x \in \Omega$, we let $r(x) \in \partial\{w(x,0) > 0\}$ be defined as

$r(x) = (p_0(x'), x')$ where $\{x_1 = p_0(x')\,,\, x' \in \Sigma\} = \Omega \cap \partial\{w(x,0) > 0\}$, and $p_0 \in C^{1+\alpha}(\overline{\Sigma})$, so that $r \in C^{1+\alpha}(\overline{\Omega}; \overline{\Omega})$ and

$$r(x) \to x_0 \quad \text{if } x_0 \in \Omega \cap \partial\{w(x,0) > 0\} \quad \text{and } x \to x_0.$$

Then, we assume that

$$|\nabla w_0^{\pm}(r(x))| = F^{\pm}(x') \in C^{1+\alpha}(\overline{\Sigma}) \quad \text{with } \frac{\partial}{\partial \eta'} F^{\pm} = 0 \text{ on } \partial\Sigma. \tag{5.1}$$

In order to construct the approximate initial function w_0^ε we smooth out w_0 near its free boundary by means of the profile $\varphi = \varphi(R; s)$, with $0 \le s \le R$, a solution to (4.5). This is, we let

$$\begin{aligned} w_0^\varepsilon(x) = \varepsilon\varphi\Big(&\frac{A}{|\nabla w_0^+(r(x))|} + \frac{A}{|\nabla w_0^-(r(x))|}; \frac{A}{|\nabla w_0^+(r(x))|} \\ &- \frac{w_0^+(x)}{\varepsilon|\nabla w_0^+(r(x))|} + \frac{w_0^-(x)}{\varepsilon|\nabla w_0^-(r(x))|}\Big), \end{aligned} \tag{5.2}$$

where $w_0(x) = w(x,0)$. Clearly, $w_0^\varepsilon \in C^\alpha(\overline{\mathcal{D}}^\varepsilon \cap \{t = 0\})$.

For the existence and regularity of the solution w^ε of the problem thus stated we refer to Theorem 4.1 in [19], where it is shown that there exists a unique solution $w^\varepsilon \in C(\overline{\mathcal{D}}^\varepsilon)$ with $\nabla w^\varepsilon \in C(\overline{\mathcal{D}}^\varepsilon \cap \{t > 0\}) \cap L^2(\mathcal{D}^\varepsilon)$.

Finally, we define the family v^ε as follows:

$$v^\varepsilon = \begin{cases} w & \text{in } \{|w| \ge A\varepsilon\}, \\ w^\varepsilon & \text{in } \mathcal{D}^\varepsilon. \end{cases}$$

On the other hand we can see that, if $t_\varepsilon/\varepsilon$ is small enough, w_0^ε is $C^{1+\alpha}$ in a neighborhood of each point x_ε such that $w(x_\varepsilon, t_\varepsilon) = \pm A\varepsilon$.

In fact, let us write $w_0^\varepsilon(x) = \varepsilon\varphi(R(x), s_\varepsilon(x))$. Let L be such that $|\nabla w|, |w_t| \le L$ in a neighborhood of $\overline{Q \cap \partial\{w > 0\}}$, and let $R_1 = \frac{2A}{L}$. Now let $r_1 > 0$ be the constant in Proposition 4.1 and let $(x_\varepsilon, t_\varepsilon)$ be such that $w(x_\varepsilon, t_\varepsilon) = A\varepsilon$. Then,

$$w_0(x) = w_0(x) - w(x_\varepsilon, t_\varepsilon) + A\varepsilon \ge A\varepsilon - L(|x - x_\varepsilon| + t_\varepsilon) \ge \frac{A}{2}\varepsilon > 0$$

if $|x - x_\varepsilon| \le \mu_1\varepsilon$ and $\frac{t_\varepsilon}{\varepsilon} \le \mu_1$, where μ_1 is small enough.

Therefore, for $|x - x_\varepsilon| \le \mu_1\varepsilon$ and $\frac{t_\varepsilon}{\varepsilon} \le \mu_1$,

$$s_\varepsilon(x) = \frac{A\varepsilon - w_0(x)}{\varepsilon|\nabla w_0^+(r(x))|} \le \frac{L(|x - x_\varepsilon| + t_\varepsilon)}{\varepsilon\sqrt{2M}} \le r_1$$

if $|x - x_\varepsilon| \le \mu_0\varepsilon$ and $\frac{t_\varepsilon}{\varepsilon} \le \mu_0$, where $\mu_0 \le \mu_1$.

Analogously, let $R(x)=\frac{A}{|\nabla w_0^+(r(x))|}+\frac{A}{|\nabla w_0^-(r(x))|}$. If $w(x_\varepsilon,t_\varepsilon)=-A\varepsilon$, there exists $\mu_2>0$ such that $w_0(x)<0$ when $|x-x_\varepsilon|\le\mu_2\varepsilon$ and $\frac{t_\varepsilon}{\varepsilon}\le\mu_2$. Therefore, if $|x-x_\varepsilon|\le\mu_0\varepsilon$, $\frac{t_\varepsilon}{\varepsilon}\le\mu_0$ and μ_0 is small enough,

$$s_\varepsilon(x)=R(x)-\frac{-A\varepsilon-w_0(x)}{\varepsilon|\nabla w_0^-(r(x))|}\ge R(x)-\frac{L(|x-x_\varepsilon|+t_\varepsilon)}{\varepsilon\gamma_0}\ge R(x)-r_1$$

where $\gamma_0>0$ is such that $|\nabla w_0^-|\ge\gamma_0$.

So, since $R(x)\ge R_1$, it holds that $w_0^\varepsilon\in C^{1+\alpha}(\overline{\Omega\cap B_{\mu_0\varepsilon}(x_\varepsilon)})$ if $w(x_\varepsilon,t_\varepsilon)=\pm A\varepsilon$ and $\frac{t_\varepsilon}{\varepsilon}\le\mu_0$.

Step II. *Passage to the limit.* Since $|\varphi(R;s)|\le A$ for $0\le s\le R$, it follows that $|w^\varepsilon(x,0)|\le A\varepsilon$ in $\overline{\mathcal{D}}^\varepsilon\cap\{t=0\}$. Applying the comparison principle for solutions of $\mathcal{P}_\varepsilon$ we deduce that $|w^\varepsilon|\le A\varepsilon$ in $\mathcal{D}^\varepsilon$. Hence,

$$\sup_{\overline{Q}}|v^\varepsilon-w|\le 2A\varepsilon,$$

and therefore the convergence of the family v^ε follows.

Step III. Let us show that there exists $\varepsilon_0>0$ such that the functions v^ε are subsolutions to $\mathcal{P}_\varepsilon$ for $\varepsilon<\varepsilon_0$.

If $|v^\varepsilon|>A\varepsilon$, then $v^\varepsilon=w$, which by hypothesis is subcaloric. Since $\beta_\varepsilon(s)=0$ when $s>\varepsilon$ or $s\le 0$, it follows that the v^ε are subsolutions to $\mathcal{P}_\varepsilon$ here.

If $|v^\varepsilon|<A\varepsilon$, then we are in $\mathcal{D}^\varepsilon$, and therefore, by construction, the v^ε are solutions to $\mathcal{P}_\varepsilon$. That is, the v^ε's are continuous functions, and they are piecewise subsolutions to $\mathcal{P}_\varepsilon$. In order to see that the v^ε are globally subsolutions to $\mathcal{P}_\varepsilon$, it suffices to see that the jumps of the gradients (which occur at smooth surfaces) have the right sign.

To this effect, we will show that there exists $\varepsilon_0>0$ such that

$$|\nabla w^\varepsilon|\le|\nabla w|\quad\text{on }\{w=A\varepsilon\},\ \text{for }\varepsilon<\varepsilon_0,\quad\text{and}\tag{5.3}$$

$$|\nabla w^\varepsilon|\ge|\nabla w|\quad\text{on }\{w=-A\varepsilon\},\ \text{for }\varepsilon<\varepsilon_0.\tag{5.4}$$

Case I. If (5.3) does not hold, then, for every $j\in\mathbb{N}$, there exist $\varepsilon_j>0$ and $(x_{\varepsilon_j},t_{\varepsilon_j})\in Q$, with $\varepsilon_j\to 0$ and $(x_{\varepsilon_j},t_{\varepsilon_j})\to(x_0,t_0)\in\partial\{w>0\}$, such that

$$w^{\varepsilon_j}(x_{\varepsilon_j},t_{\varepsilon_j})=A\varepsilon_j\quad\text{and}\quad|\nabla w^{\varepsilon_j}(x_{\varepsilon_j},t_{\varepsilon_j})|>|\nabla w(x_{\varepsilon_j},t_{\varepsilon_j})|.\tag{5.5}$$

From now on we will drop the subscript j when referring to the sequences defined above and $\varepsilon\to 0$ will mean $j\to\infty$.

Since on the lateral boundary we have the Neumann data $\frac{\partial w^\varepsilon}{\partial\eta}=0$, we will use a reflection argument and assume that the points $(x_\varepsilon,t_\varepsilon)$ are far from the lateral boundary (with a different equation).

In fact, if $(x_0,t_0)\in\mathbb{R}\times\partial\Sigma\times[0,T]$ we apply Proposition A.1 in [17] and deduce that there exists a change of variables $y=h(x)$ such that $h(x_0)=0$ and such that the function

$$u^\varepsilon(y,t)=\begin{cases} w^\varepsilon(x,t) & \text{for } y_N\ge 0\\ u^\varepsilon(y_1,\dots,y_{N-1},-y_N,t) & \text{for } y_N<0\end{cases}$$

is a weak solution to

$$\sum_{i,j}\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial u^\varepsilon}{\partial y_j}\right)+\sum_i b_i(y,t)\frac{\partial u^\varepsilon}{\partial y_i}-u^\varepsilon_t=\beta_\varepsilon(u^\varepsilon)\quad\text{in } \{|u^\varepsilon|<A\varepsilon\}$$

for y in a neighborhood $\mathcal{N}$ of the origin and $t\in[0,T]$. Here $a_{ij}\in W^{1,\infty}(\mathcal{N})$, $b_i\in L^\infty(\mathcal{N}\times[0,T])$.

We choose the variables in such a way that $\nabla h_1(x_0)=-\frac{\nabla w^+(x_0,t_0)}{|\nabla w^+(x_0,t_0)|}=-\frac{\nabla w^-(x_0,t_0)}{|\nabla w^-(x_0,t_0)|}$, $\nabla h_i(x_0)\cdot\nabla h_j(x_0)=\delta_{ij}$ and $a_{ij}(0)=\delta_{ij}$. We will sometimes denote $y=(y_1,y')$. And we denote $y_\varepsilon=h(x_\varepsilon)$. We point out that the change of variables, the neighborhood $\mathcal{N}$ and the coefficients in the equation depend only on the domain Σ.

If, on the other hand, $(x_0,t_0)\in\Omega\times[0,T]$ we change the origin and perform a rotation in the space variables, and we are in a situation similar to the one above.

In any case, since $\nabla h_1(x_0)=-\frac{\nabla w^+(x_0,t_0)}{|\nabla w^+(x_0,t_0)|}$, $\nabla h_i(x_0)\cdot\nabla h_j(x_0)=\delta_{ij}$, $\{w^\varepsilon=A\varepsilon\}=\{w=A\varepsilon\}$ and $\{w^\varepsilon=-A\varepsilon\}=\{w=-A\varepsilon\}$, there exist a family f_ε and a family g_ε of smooth functions such that, in a neighborhood of $(y_\varepsilon,t_\varepsilon)$,

$$\begin{aligned}
&\{u^\varepsilon=A\varepsilon\}=\{(y,t): y_1-y_{\varepsilon 1}=f_\varepsilon(y'-{y_\varepsilon}',t-t_\varepsilon)\},\\
&\{u^\varepsilon=-A\varepsilon\}=\{(y,t): y_1-y_{\varepsilon 1}=g_\varepsilon(y'-{y_\varepsilon}',t-t_\varepsilon)\},\\
&\{|u^\varepsilon|<A\varepsilon\}=\{(y,t): f_\varepsilon(y'-{y_\varepsilon}',t-t_\varepsilon)<y_1-y_{\varepsilon 1}<g_\varepsilon(y'-{y_\varepsilon}',t-t_\varepsilon)\},
\end{aligned}\tag{5.6}$$

where it holds that $f_\varepsilon(0,0)=0$, $|\nabla_{y'}f_\varepsilon(0,0)|\to 0$, $|\nabla_{y'}g_\varepsilon(0,0)|\to 0$ and $\varepsilon\to 0$. We can assume that (5.6) holds in $\big(B_\rho(y_\varepsilon)\times(t_\varepsilon-\rho^2,t_\varepsilon+\rho^2)\big)\cap\{0\le t\le T\}$ for some $\rho>0$. Let us now define

$$\begin{aligned}
&\bar u^\varepsilon(y,t)=\frac{1}{\varepsilon}u^\varepsilon(y_\varepsilon+\varepsilon y,t_\varepsilon+\varepsilon^2 t),\\
&\bar f_\varepsilon(y',t)=\frac{1}{\varepsilon}f_\varepsilon(\varepsilon y',\varepsilon^2 t),\quad \bar g_\varepsilon(y',t)=\frac{1}{\varepsilon}g_\varepsilon(\varepsilon y',\varepsilon^2 t),
\end{aligned}$$

and let $\tau_\varepsilon = \frac{t_\varepsilon}{\varepsilon^2}$, $\xi_\varepsilon = \frac{T-t_\varepsilon}{\varepsilon^2}$. We have, for a subsequence, $\tau_\varepsilon \to \tau$ and $\xi_\varepsilon \to \xi$, where $0 \le \tau, \xi \le +\infty$ and τ and ξ cannot be both finite. We now let

$$\mathcal{A}_\varepsilon = \left\{ (y,t) : |y| < \frac{\rho}{\varepsilon}\,,\ -\min(\tau_\varepsilon, \frac{\rho^2}{\varepsilon^2}) < t < \min(\xi_\varepsilon, \frac{\rho^2}{\varepsilon^2}) \right\}.$$

Then, the functions $\bar u^\varepsilon$ are weak solutions to

$$\begin{aligned}
\sum_{i,j} \frac{\partial}{\partial y_i}\big(a^\varepsilon_{ij}\frac{\partial \bar u^\varepsilon}{\partial y_j}\big) + \sum_i b^\varepsilon_i \frac{\partial \bar u^\varepsilon}{\partial y_i} - \bar u^\varepsilon_t &= \beta(\bar u^\varepsilon) \text{ in } \{\bar f_\varepsilon(y',t) < y_1 < \bar g_\varepsilon(y',t)\} \cap \mathcal{A}_\varepsilon,\\
\bar u^\varepsilon = A \qquad & \text{on } \{y_1 = \bar f_\varepsilon(y',t)\} \cap \mathcal{A}_\varepsilon,\\
\bar u^\varepsilon = -A \qquad & \text{on } \{y_1 = \bar g_\varepsilon(y',t)\} \cap \mathcal{A}_\varepsilon,\\
|\bar u^\varepsilon| \le A \qquad & \text{in } \{\bar f_\varepsilon(y',t) \le y_1 \le \bar g_\varepsilon(y',t)\} \cap \overline{\mathcal{A}_\varepsilon},
\end{aligned} \tag{5.7}$$

where $a^\varepsilon_{ij}(y) = a_{ij}(y_\varepsilon + \varepsilon y)$, $b^\varepsilon_i(y,t) = \varepsilon b_i(y_\varepsilon + \varepsilon y, t_\varepsilon + \varepsilon^2 t)$.

Note that $\bar f_\varepsilon(y',t) \to 0$ uniformly for (y',t) in compact subsets of $\mathbb{R}^{N-1} \times \mathbb{R}$. Let us see that $\bar g_\varepsilon(y',t) \to R$ uniformly for (y',t) in compact subsets of $\mathbb{R}^{N-1} \times \mathbb{R}$, where $R = \frac{A}{|\nabla w^+(x_0,t_0)|} + \frac{A}{|\nabla w^-(x_0,t_0)|}$.

In fact, it suffices to prove that $\bar g_\varepsilon(0,0) \to R$. Let $u(y,t) = w(x,t)$. Let $d_1 > 0$ be such that $u(y_{\varepsilon 1} + d_1, y'_\varepsilon, t_\varepsilon) = 0$ and $d_2 > 0$ be such that $u(y_{\varepsilon 1} + d_1 + d_2, y'_\varepsilon, t_\varepsilon) = -A\varepsilon$. Then,

$$-A\varepsilon = \int_0^{d_1} u_{y_1}(s + y_{\varepsilon 1}, y'_\varepsilon, t_\varepsilon)\, ds = u_{y_1}(\tilde s + y_{\varepsilon 1}, y'_\varepsilon, t_\varepsilon)\, d_1,$$

and

$$-A\varepsilon = \int_{d_1}^{d_1+d_2} u_{y_1}(s + y_{\varepsilon 1}, y'_\varepsilon, t_\varepsilon)\, ds = u_{y_1}(\tilde{\tilde s} + y_{\varepsilon 1}, y'_\varepsilon, t_\varepsilon)\, d_2.$$

So,

$$\frac{g_\varepsilon(0,0)}{\varepsilon} = \frac{d_1+d_2}{\varepsilon} = \frac{A}{|u_{y_1}(\tilde s + y_{\varepsilon 1}, y'_\varepsilon, t_\varepsilon)|} + \frac{A}{|u_{y_1}(\tilde{\tilde s} + y_{\varepsilon 1}, y'_\varepsilon, t_\varepsilon)|}.$$

Therefore,

$$\begin{aligned}
\bar g_\varepsilon(0,0) = \frac{g_\varepsilon(0,0)}{\varepsilon} &\to \frac{A}{|u^+_{y_1}(0,0)|} + \frac{A}{|u^-_{y_1}(0,0)|}\\
&= \frac{A}{|\nabla w^+(x_0,t_0)|} + \frac{A}{|\nabla w^-(x_0,t_0)|} = R.
\end{aligned}$$

So we are under the hypotheses of a compactness result which is precisely stated at the end of this proof as Lemma 5.1. According to this result there exists a function $\bar{u}$ such that, for a subsequence,

$$\begin{aligned}
&\bar{u} \in C^{2+\alpha,1+\frac{\alpha}{2}}\big(\{0 \le y_1 \le R,\ -\tau < t < \xi\}\big),\\
&\bar{u}^\varepsilon \to \bar{u} \qquad \text{uniformly on compact subsets of } \{0 < y_1 < R,\ -\tau < t < \xi\},\\
&\Delta\bar{u} - \bar{u}_t = \beta(\bar{u}) \quad \text{in } \{0 < y_1 < R,\ -\tau < t < \xi\},\\
&\bar{u} = A \qquad \text{on } \{y_1 = 0,\ -\tau < t < \xi\},\\
&\bar{u} = -A \qquad \text{on } \{y_1 = R,\ -\tau < t < \xi\},\\
&|\bar{u}| \le A \qquad \text{in } \{0 \le y_1 \le R,\ -\tau < t < \xi\}.
\end{aligned}$$

We will divide the remainder of the proof into two cases, depending on whether $\tau = +\infty$ or $\tau < +\infty$.

So, assume first that $\tau = +\infty$. In this case, Lemma 5.1 also gives

$$|\nabla\bar{u}^\varepsilon(0,0)| \to |\nabla\bar{u}(0,0)|.$$

On the other hand, $\bar{u}$ is under the hypotheses of Lemma 4.3, and therefore there exists $\delta > 0$ such that

$$|\nabla\bar{u}| = \sqrt{|\nabla w^+(x_0,t_0)|^2 - \delta} \quad \text{on } \{y_1 = 0\},$$

which yields

$$|\nabla\bar{u}^\varepsilon(0,0)| \le \sqrt{|\nabla w^+(x_0,t_0)|^2 - \delta/2},$$

for ε small. But this gives

$$|\nabla w^\varepsilon(x_\varepsilon,t_\varepsilon)| < |\nabla w(x_\varepsilon,t_\varepsilon)|,$$

for ε small. This contradicts (5.5) and completes the proof in case $\tau = +\infty$.

Assume now that $\tau < +\infty$. (In this case $\xi = +\infty$.) It holds that $\bar{u}^\varepsilon(y,-\tau_\varepsilon) = \frac{1}{\varepsilon}u^\varepsilon(y_\varepsilon + \varepsilon y, 0)$; then,

$$\begin{aligned}
\bar{u}^\varepsilon(y,-\tau_\varepsilon) = \varphi\Big(&\frac{A}{|\nabla w_0^+(r(h^{-1}(y_\varepsilon+\varepsilon y)))|} + \frac{A}{|\nabla w_0^-(r(h^{-1}(y_\varepsilon+\varepsilon y)))|};\\
&\frac{A}{|\nabla w_0^+(r(h^{-1}(y_\varepsilon+\varepsilon y)))|} - \frac{w_0^+(h^{-1}(y_\varepsilon+\varepsilon y))}{\varepsilon|\nabla w_0^+(r(h^{-1}(y_\varepsilon+\varepsilon y)))|}\\
&+ \frac{w_0^-(h^{-1}(y_\varepsilon+\varepsilon y))}{\varepsilon|\nabla w_0^-(r(h^{-1}(y_\varepsilon+\varepsilon y)))|}\Big)
\end{aligned} \tag{5.8}$$

when $x_0 \in \mathbb{R}\times\Sigma$. When $x_0 \in \mathbb{R}\times\partial\Sigma$, (5.8) holds for $(y_\varepsilon + \varepsilon y)_N \ge 0$ and we obtain $\bar{u}^\varepsilon(y,-\tau_\varepsilon)$ for $(y_\varepsilon + \varepsilon y)_N < 0$, recalling that

$$u^\varepsilon(y,0) = u^\varepsilon(y_1,\dots,y_{N-1},-y_N,0) \qquad \text{for } y_N < 0.$$

We want to apply here the result of Lemma 5.1 corresponding to $\tau < +\infty$. In fact, we can see that there exist $C, \mu_0 > 0$ such that $||\bar{u}^\varepsilon(y,-\tau_\varepsilon)||_{C^{1+\alpha}(\overline{B}_{\mu_0}(0))}$

$\leq C$. In case $x_0 \in \mathbb{R}\times\Sigma$ we use the fact that $\frac{t_\varepsilon}{\varepsilon} \to 0$ (and therefore $t_0 = 0$) when $\tau < +\infty$, so that $w_0^\varepsilon \in C^{1+\alpha}(\overline{\Omega \cap B_{\mu_0\varepsilon}(x_\varepsilon)})$. In case that $x_0 \in \mathbb{R}\times\partial\Sigma$ we argue in a similar way and we also use that $\frac{\partial u^\varepsilon}{\partial y_N}(y,0) = 0$ on $\{y_N = 0\}$.

For a proof of this last statement let us recall that, on $\{y_N = 0\}$,

$$\begin{aligned}
\frac{\partial}{\partial y_N} w_0(h^{-1}(y)) &= \frac{\partial w_0}{\partial \eta}(x) = 0\\
\frac{\partial}{\partial y_N} |\nabla w_0^\pm(r(h^{-1}(y)))| &= \frac{\partial F^\pm}{\partial \eta'}(x') = 0
\end{aligned}$$

for $x = h^{-1}(y)$. Now Lemma 5.1 gives, for a subsequence,

$$\begin{aligned}
&\bar u \in C^{\alpha,\frac{\alpha}{2}}\big(\{0 \le y_1 \le R,\, t \ge -\tau\}\big),\\
&\bar u^\varepsilon(y,-\tau_\varepsilon) \to \bar u(y,-\tau) \quad \text{uniformly on compact subsets of } \{0 < y_1 < R\}.
\end{aligned}$$

Let us observe that for every $\mu > 0$ and $\chi > 0$ there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, if $|y| < \mu$,

$$\begin{aligned}
y_1 &< \frac{A}{|\nabla w_0^+(x_0)|} - \chi \quad \text{implies that } w_0(h^{-1}(y_\varepsilon + \varepsilon y)) > 0,\\
y_1 &> \frac{A}{|\nabla w_0^+(x_0)|} + \chi \quad \text{implies that } w_0(h^{-1}(y_\varepsilon + \varepsilon y)) < 0.
\end{aligned}$$

Therefore, using the fact that $\nabla h_1(x_0) = -\frac{\nabla w^+(x_0,t_0)}{|\nabla w^+(x_0,t_0)|} = -\frac{\nabla w^-(x_0,t_0)}{|\nabla w^-(x_0,t_0)|}$ and $\nabla h_i(x_0)\cdot\nabla h_j(x_0) = \delta_{ij}$, and the fact that $h^{-1}(y_\varepsilon + \varepsilon y) \to x_0$ and $\frac{t_\varepsilon}{\varepsilon} \to 0$ we get, if $y_1 < \frac{A}{|\nabla w_0^+(x_0)|}$,

$$\begin{aligned}
A - \frac{w_0(h^{-1}(y_\varepsilon + \varepsilon y))}{\varepsilon} &= \frac{w_0^+(h^{-1}(y_\varepsilon)) - w_0^+(h^{-1}(y_\varepsilon + \varepsilon y))}{\varepsilon}\\
&+ \frac{w^+(h^{-1}(y_\varepsilon), t_\varepsilon) - w_0^+(h^{-1}(y_\varepsilon))}{\varepsilon} = -\nabla w_0^+(x_\varepsilon) D h^{-1}(y_\varepsilon)\, y + o(1)\\
&\to |\nabla w_0^+(x_0)|\, y_1.
\end{aligned}$$

So that, since $r(x) \to x_0$ if $x_0 \in \partial\{w_0 > 0\}$ is such that $x \to x_0$,

$$\bar u^\varepsilon(y,-\tau_\varepsilon) \to \varphi(R; y_1), \quad (\varepsilon \to 0) \quad \text{in } \Big\{0 < y_1 < \frac{A}{|\nabla w_0^+(x_0)|}\Big\}.$$

On the other hand, if $R > y_1 > \frac{A}{|\nabla w_0^+(x_0)|}$,

$$\begin{aligned}
&\frac{A}{|\nabla w_0^+(r(h^{-1}(y_\varepsilon+\varepsilon y)))|} + \frac{w_0^-(h^{-1}(y_\varepsilon+\varepsilon y))}{\varepsilon|\nabla w_0^-(r(h^{-1}(y_\varepsilon+\varepsilon y)))|}\\
&= \frac{A}{|\nabla w_0^+(r(h^{-1}(y_\varepsilon+\varepsilon y)))|} + \frac{A}{|\nabla w_0^-(r(h^{-1}(y_\varepsilon+\varepsilon y)))|}\\
&- \Big(\frac{w_0(h^{-1}(y_\varepsilon+\varepsilon y))}{\varepsilon} + A\Big)\frac{1}{|\nabla w_0^-(r(h^{-1}(y_\varepsilon+\varepsilon y)))|}\\
&\to R - \frac{|\nabla w_0^-(x_0)|}{|\nabla w_0^-(x_0)|}(R-y_1) = y_1.
\end{aligned}$$

In fact,

$$\begin{aligned}
\frac{w_0(h^{-1}(y_\varepsilon+\varepsilon y))}{\varepsilon} + A &= \frac{w_0(h^{-1}(y_\varepsilon+\varepsilon y)) - w_0(h^{-1}(y_\varepsilon+(1+\theta_\varepsilon)\varepsilon y))}{\varepsilon}\\
&+ \frac{w_0(h^{-1}(y_\varepsilon+(1+\theta_\varepsilon)\varepsilon y)) - w(h^{-1}(y_\varepsilon+(1+\theta_\varepsilon)\varepsilon y), t_\varepsilon)}{\varepsilon}
\end{aligned}$$

where θ_ε is such that $w(h^{-1}(y_\varepsilon + (1+\theta_\varepsilon)\varepsilon y), t_\varepsilon) = -A\varepsilon$. So that, since $(1+\theta_\varepsilon)y_1 \to R$ as $\varepsilon \to 0$,

$$\begin{aligned}
&\frac{w_0(h^{-1}(y_\varepsilon+\varepsilon y))}{\varepsilon} + A\\
&= -\nabla w_0(h^{-1}(y_\varepsilon))Dh^{-1}(y_\varepsilon)\theta_\varepsilon y + o(1) \to |\nabla w_0^-(x_0)|\,(R-y_1).
\end{aligned}$$

Thus, we get that $\bar u(y,-\tau) = \varphi(R;y_1)$.

Since the function $\varphi(R;y_1)$ is a stationary solution to equation (4.1), and $\bar u = \varphi$ on the parabolic boundary of the domain $\{0 < y_1 < R,\, t > -\tau\}$, we conclude that $\bar u(y,t) = \varphi(R;y_1)$ in $\{0 \le y_1 \le R,\, t \ge -\tau\}$. It follows that there exists $\delta > 0$ such that

$$|\nabla \bar u| = \sqrt{|\nabla w_0^+(x_0)|^2 - \delta} \quad \text{on } \{y_1 = 0, t \ge -\tau\}.$$

But Lemma 5.1 gives $|\nabla \bar u^\varepsilon(0,0)| \to |\nabla \bar u(0,0)|$, so that

$$|\nabla w^\varepsilon(x_\varepsilon,t_\varepsilon)| \le \sqrt{|\nabla w_0^+(x_0)|^2 - \delta/2} \le \sqrt{|\nabla w(x_\varepsilon,t_\varepsilon)|^2 - \delta/4},$$

for ε small. This contradicts (5.5) and completes the proof that (5.3) holds in case $\tau < +\infty$.

Case II. Assume now that (5.4) does not hold. Then, For every $j \in \mathbb{N}$ there exists $\varepsilon_j > 0$ and $(x_{\varepsilon_j}, t_{\varepsilon_j}) \in Q$ with $\varepsilon_j \to 0$ and $(x_{\varepsilon_j}, t_{\varepsilon_j}) \to (x_0,t_0) \in \partial\{w^+ > 0\}$ such that

$$w^{\varepsilon_j}(x_{\varepsilon_j}, t_{\varepsilon_j}) = -A\varepsilon_j \qquad \text{and} \qquad |\nabla w^{\varepsilon_j}(x_{\varepsilon_j}, t_{\varepsilon_j})| < |\nabla w(x_{\varepsilon_j}, t_{\varepsilon_j})|. \tag{5.9}$$

We proceed as before, but this time it holds that $g_\varepsilon(0,0) = 0$. Then, we define

$$\bar{u}^\varepsilon(y,t) = \frac{1}{\varepsilon} u^\varepsilon(y_{\varepsilon 1} + \varepsilon(y_1 - R), y'_\varepsilon + \varepsilon y', t_\varepsilon + \varepsilon^2 t)$$
$$\bar{f}_\varepsilon(y',t) = R + \frac{1}{\varepsilon} f_\varepsilon(\varepsilon y', \varepsilon^2 t), \quad \bar{g}_\varepsilon(y',t) = R + \frac{1}{\varepsilon} g_\varepsilon(\varepsilon y', \varepsilon^2 t)$$

and we let $\tau_\varepsilon, \xi_\varepsilon, \tau$ and ξ as before.

So, the functions $\bar{u}^\varepsilon$ are weak solutions to (5.7) where $a^\varepsilon_{ij}(y) = a_{ij}(y_{\varepsilon 1} + \varepsilon(y_1 - R), y'_\varepsilon + \varepsilon y')$ and $b^\varepsilon_i(y,t) = \varepsilon b_i(y_{\varepsilon 1} + \varepsilon(y_1 - R), y'_\varepsilon + \varepsilon y', t_\varepsilon + \varepsilon^2 t)$.

Here $R = \frac{A}{|\nabla w^+(x_0,t_0)|} + \frac{A}{|\nabla w^-(x_0,t_0)|}$. Therefore, $\bar{g}_\varepsilon(y',t') \to R$ as $\varepsilon \to 0$ since $\frac{1}{\varepsilon} g_\varepsilon(0,0) = 0$, $|\nabla \bar{g}_\varepsilon(0,0)| \to 0$. On the other hand, proceeding as we did with g_ε in Case I, we see that $\bar{f}_\varepsilon(y',t) \to 0$ uniformly on compact subsets of $\mathbb{R}^{N-1} \times \mathbb{R}$. Now the proof follows exactly as in Case I. Here we use the estimates of Lemmas 4.3 and 5.1 on the boundary $y_1 = R$ and the fact that $\varphi(R, y_1) = \lim_{\varepsilon \to 0} \bar{u}^\varepsilon(y, -\tau_\varepsilon)$ satisfies on $y_1 = R$ that $\varphi' = -\sqrt{\gamma^2 + \delta_0 - \delta}$ with $\gamma = |\nabla w^-(x_0,t_0)|$. So the proof is finished when (5.1) holds.

Eliminating the extra regularity assumption on $|\nabla w_0^\pm|$. Assume now that (5.1) does not hold. Only small changes are needed in the above proof to overcome the lack of differentiability of $|\nabla w_0^\pm|$ in the definition of w_0^ε. This is done as follows: We can construct sequences $F_\varepsilon^\pm$ of functions which are $C^{1+\alpha}$ on $\overline{\Sigma}$ such that, with the notation of (5.1), $F_\varepsilon^\pm \to F^\pm$ uniformly in $\overline{\Sigma}$ as $\varepsilon \to 0$, and

$$\frac{\partial F_\varepsilon^\pm}{\partial \eta'} = 0 \quad \text{on } \partial\Sigma.$$

In fact, we cover a δ neighborhood of the boundary of Σ with a finite number of sets which can be seen as images of sets of the form $\mathcal{N} \times [0,\delta]$, where $\mathcal{N}$ is a ball in $\mathbb{R}^{N-2}$ of radius δ. On each of these sets we construct an approximation of $F^\pm$ with zero normal derivative in the following way. First we make a convolution of the function $F^\pm(h^{-1}(y)')$ (already extended to $\{|y_N| < 2\delta\}$ in a symmetric way) with a kernel $\phi_\varepsilon(y') = \varepsilon^{-\{N-1\}}\phi(y'/\varepsilon)$ where ϕ is a smooth function which is symmetric in the y_N variable. The approximate function is then obtained by going back to the original variables. On the other hand, far from the boundary we perform a standard regularization.

In a similar way, we can construct a partition of unity associated to these neighborhoods such that the functions of the partition with support intersecting the boundary of Σ have zero normal derivative. In this case, the convolution of the characteristic function of $2\mathcal{N} \times [-2\delta, 2\delta]$ is made with $\phi_\delta(y') = \delta^{-\{N-1\}}\phi(y'/\delta)$. This ends the construction of the functions $F_\varepsilon^\pm$.

With this construction we take as initial datum for $\varepsilon > 0$ small

$$w_0^\varepsilon(x) = \varepsilon\varphi\Big(\frac{A}{F_\varepsilon^+(x')} + \frac{A}{F_\varepsilon^-(x')}; \frac{A}{F_\varepsilon^+(x')} - \frac{w_0^+(x)}{\varepsilon F_\varepsilon^+(x')} + \frac{w_0^-(x)}{\varepsilon F_\varepsilon^-(x')}\Big)$$

instead of (5.2). From this point the proof follows as before. □

Now we state the compactness result that was used above.

Lemma 5.1. *Let ε_j, ξ_{ε_j} and τ_{ε_j} be sequences such that $\varepsilon_j > 0$, $\varepsilon_j \to 0$, $\xi_{\varepsilon_j} > 0$, $\xi_{\varepsilon_j} \to \xi$, with $0 \le \xi \le +\infty$, $\tau_{\varepsilon_j} > 0$, $\tau_{\varepsilon_j} \to \tau$ with $0 \le \tau \le +\infty$ and such that $\tau < +\infty$ implies that $\xi = +\infty$. Let $\rho > 0$ and $\mathcal{A}_{\varepsilon_j} = \big\{(x,t) : |x| < \frac{\rho}{\varepsilon_j},\ -\min(\tau_{\varepsilon_j}, \frac{\rho^2}{{\varepsilon_j}^2}) < t < \min(\xi_{\varepsilon_j}, \frac{\rho^2}{{\varepsilon_j}^2})\big\}$. Let the $\overline{u}^{\varepsilon_j}$ be weak solutions to*

$$\sum_{i,k} \frac{\partial}{\partial x_i}\big(a_{ik}^{\varepsilon_j}\frac{\partial \overline{u}^{\varepsilon_j}}{\partial x_k}\big) + \sum_i b_i^{\varepsilon_j}\frac{\partial \overline{u}^{\varepsilon_j}}{\partial x_i} - \overline{u}_t^{\varepsilon_j} = \beta(\overline{u}^{\varepsilon_j})$$

in $\{\bar f_{\varepsilon_j}(x',t) < x_1 < \bar g_{\varepsilon_j}(x',t)\} \cap \mathcal{A}_{\varepsilon_j}$,

$$\begin{aligned} &\overline{u}^{\varepsilon_j} = A && \textit{on } \{x_1 = \bar f_{\varepsilon_j}(x',t)\} \cap \mathcal{A}_{\varepsilon_j},\\ &\overline{u}^{\varepsilon_j} = -A && \textit{on } \{x_1 = \bar g_{\varepsilon_j}(x',t)\} \cap \mathcal{A}_{\varepsilon_j},\\ &|\overline{u}^{\varepsilon_j}| \le A && \textit{in } \{\bar f_{\varepsilon_j}(x',t) < x_1 < \bar g_{\varepsilon_j}(x',t)\} \cap \mathcal{A}_{\varepsilon_j}, \end{aligned}$$

with $\overline{u}^{\varepsilon_j} \in C(\{\bar f_{\varepsilon_j}(x',t) \le x_1 \le \bar g_{\varepsilon_j}(x',t)\} \cap \overline{\mathcal{A}_{\varepsilon_j}})$, and $\nabla\overline{u}^{\varepsilon_j} \in L^2$. Here $a_{ik}^{\varepsilon_j} \to \delta_{ik}$ and $b_i^{\varepsilon_j} \to 0$ uniformly on compact sets of $\mathbb{R}^N$ and of $\mathbb{R}^N \times (-\tau,\xi)$ respectively, and $\bar f_{\varepsilon_j}$ and $\bar g_{\varepsilon_j}$ are continuous functions such that $\bar f_{\varepsilon_j} \to 0$ and $\bar g_{\varepsilon_j} \to R$ uniformly on compact subsets of $\mathbb{R}^{N-1} \times (-\tau,\xi)$. Moreover, we assume that $||\bar f_{\varepsilon_j}||_{C^1(K)} + ||\nabla_{x'}\bar f_{\varepsilon_j}||_{C^{\alpha,\frac{\alpha}{2}}(K)}$ and $||\bar g_{\varepsilon_j}||_{C^1(K)} + ||\nabla_{x'}\bar g_{\varepsilon_j}||_{C^{\alpha,\frac{\alpha}{2}}(K)}$ are uniformly bounded for every compact set $K \subset \mathbb{R}^{N-1} \times (-\tau,\xi)$. In addition we assume that $||b_i^{\varepsilon_j}||_{L^\infty}$ and $||a_{ik}^{\varepsilon_j}||_{W^{1,\infty}}$ are uniformly bounded. Moreover, $a_{ik}^{\varepsilon_j}$ are uniformly parabolic with constant independent of ε_j.

Then, there exists a function $\bar u$ such that, for a subsequence,

$$\begin{aligned} &\bar u \in C^{2+\alpha,1+\frac{\alpha}{2}}\big(\{0 \le x_1 \le R,\ -\tau < t < \xi\}\big),\\ &\overline{u}^{\varepsilon_j} \to \bar u && \textit{uniformly on compact subsets of } \{0 < x_1 < R,\ -\tau < t < \xi\},\\ &\Delta\bar u - \bar u_t = \beta(\bar u) && \textit{in } \{0 < x_1 < R,\ -\tau < t < \xi\},\\ &\bar u = A && \textit{on } \{x_1 = 0,\ -\tau < t < \xi\},\\ &\bar u = -A && \textit{on } \{x_1 = R,\ -\tau < t < \xi\},\\ &|\bar u| \le A && \textit{in } \{0 \le x_1 \le R,\ -\tau < t < \xi\}. \end{aligned}$$

If $\xi < +\infty$, we require in addition that

$$||\bar f_{\varepsilon_j}(x', t+\xi_{\varepsilon_j} - \xi)||_{C^1(K)} + ||\nabla_{x'}\bar f_{\varepsilon_j}(x', t+\xi_{\varepsilon_j}-\xi)||_{C^{\alpha,\frac{\alpha}{2}}(K)} \quad \textit{and}$$

$$||\bar g_{\varepsilon_j}(x',t+\xi_{\varepsilon_j}-\xi)||_{C^1(K)}+||\nabla_{x'}\bar g_{\varepsilon_j}(x',t+\xi_{\varepsilon_j}-\xi)||_{C^{\alpha,\frac{\alpha}{2}}(K)}$$

be uniformly bounded for every compact set $K\subset\mathbb{R}^{N-1}\times(-\infty,\xi]$, *and we deduce that* $\overline{u}\in C^{2+\alpha,1+\frac{\alpha}{2}}(\{0\le x_1\le R,\ t\le\xi\})$. *If* $\tau<+\infty$, *we let*

$$\mathcal{B}_{\varepsilon_j}=\big\{x:|x'|<\frac{\rho}{\varepsilon_j},\ \bar f_{\varepsilon_j}(x',-\tau_{\varepsilon_j})<x_1<\bar g_{\varepsilon_j}(x',-\tau_{\varepsilon_j})\big\},$$

and we require, in addition, that for every $\mu>0$

$$||u^{\varepsilon_j}(x,-\tau_{\varepsilon_j})||_{C^\alpha(\overline{\mathcal{B}}_{\varepsilon_j}\cap B_\mu(0))}\le C$$

and that there exists $\mu_0>0$ *such that (with the notation* $x=(x_1,x')$*),*

$$||\overline{u}^{\varepsilon_j}(x,-\tau_{\varepsilon_j})||_{C^{1+\alpha}(\overline{\mathcal{B}}_{\varepsilon_j}\cap B_{\mu_0}(0,0))}\le C;\quad ||\overline{u}^{\varepsilon_j}(x,-\tau_{\varepsilon_j})||_{C^{1+\alpha}(\overline{\mathcal{B}}_{\varepsilon_j}\cap B_{\mu_0}(R,0))}\le C.$$

Moreover, we assume that

$$||\bar f_{\varepsilon_j}(x',t-\tau_{\varepsilon_j}+\tau)||_{C^1(K)}+||\nabla_{x'}\bar f_{\varepsilon_j}(x',t-\tau_{\varepsilon_j}+\tau)||_{C^{\alpha,\frac{\alpha}{2}}(K)}\quad\text{and}$$

$$||\bar g_{\varepsilon_j}(x',t-\tau_{\varepsilon_j}+\tau)||_{C^1(K)}+||\nabla_{x'}\bar g_{\varepsilon_j}(x',t-\tau_{\varepsilon_j}+\tau)||_{C^{\alpha,\frac{\alpha}{2}}(K)}$$

are uniformly bounded for every compact set $K\subset\mathbb{R}^{N-1}\times[-\tau,\infty)$. *Then, it holds that* $\overline{u}\in C^{\alpha,\frac{\alpha}{2}}(\{0\le x_1\le R,\ -\tau\le t\})$, $\overline{u}^{\varepsilon_j}(x,-\tau_{\varepsilon_j})\to\overline{u}(x,-\tau)$ *uniformly on compact subsets of* $\{0<x_1<R\}$,

$$\nabla\overline{u}\in C\big(\{(x,t)\,,\,0\le x_1<\mu_0\,,\,|x'|<\mu_0\,,\,t\ge-\tau\}\big),$$

and

$$\nabla\overline{u}\in C\big(\{(x,t)\,,\,R-\mu_0<x_1\le R\,,\,|x'|<\mu_0\,,\,t\ge-\tau\}\big).$$

In any case (τ,ξ *infinite or finite) it holds that, when* $\bar f_{\varepsilon_j}(0,0)=0$,

$$|\nabla\overline{u}^{\varepsilon_j}(0,0,0)|\to|\nabla\overline{u}(0,0,0)|,$$

and, when $\bar g_{\varepsilon_j}(0,0)=R$, $|\nabla\overline{u}^{\varepsilon_j}(R,0,0)|\to|\nabla\overline{u}(R,0,0)|$.

Proof. The proof follows the lines of Lemma 4.4 in [17]. Here we use a change of variables similar to that in Proposition 3.1 in [19] in order to straighten up both lateral boundaries at the same time. This is, we take

$$y_1=\frac{R}{\bar g_{\varepsilon_j}(x',t)-\bar f_{\varepsilon_j}(x',t)}\big(x_1-\bar f_{\varepsilon_j}(x',t)\big),\quad y'=x'.\qquad\square$$

We end this section by stating the corresponding result of approximation of a supersolution.

Theorem 5.2. *Let w be a classical supersolution to $\mathcal{P}$ in Q, with $\frac{\partial w}{\partial \eta} = 0$ on $\partial_N Q$, satisfying* **(H1)**. *Assume, in addition, that there exists $\delta_0 > 0$, with $|\nabla w^-|^2 > \delta_0$ on the free boundary $Q \cap \partial\{w > 0\}$, such that*

$$|\nabla w^+|^2 - |\nabla w^-|^2 = 2M - \delta_0 \qquad \text{on } Q \cap \partial\{w > 0\}.$$

Let $A = A(L, M, \delta_0)$ be the constant in Lemma 4.2, where $L > 0$ is such that $|\nabla w| \leq L$ in a neighborhood of the free boundary. Then, there exists a family $v^\varepsilon \in C(\overline{Q})$, with $\nabla v^\varepsilon \in L^2_{loc}(\overline{Q})$, of weak supersolutions to $\mathcal{P}_\varepsilon$ in Q, with $\frac{\partial v^\varepsilon}{\partial \eta} = 0$ on $\partial_N Q$, such that, as $\varepsilon \to 0$, $v^\varepsilon \to w$ uniformly in $\overline{Q}$.

Moreover, $v^\varepsilon = w^\varepsilon$ in $\{|w| \geq A\varepsilon\}$ and $\nabla v^\varepsilon \in C(\{|w| \leq A\varepsilon\} \cap \{t > 0\})$.

Proof. We construct the family w^ε exactly as we did in Theorem 5.1, but this time we take as A the constant in Lemma 4.2 instead of Lemma 4.1, and $\varphi(R; s)$ is the solution to (4.5) corresponding to this choice of constant A. Then, we let as in Theorem 5.1

$$v^\varepsilon = \begin{cases} w & \text{in } \{|w| \geq A\varepsilon\}, \\ w^\varepsilon & \text{in } \mathcal{D}^\varepsilon. \end{cases}$$

As before, an application of the maximum principle gives the uniform convergence of v^ε to w in $\overline{Q}$.

The proof of the fact that v^ε is a supersolution to $\mathcal{P}_\varepsilon$ if $\varepsilon \leq \varepsilon_0$ follows in a way similar to the corresponding proof for subsolutions in Theorem 5.1 by using Corollary 4.2 and Lemma 4.4 instead of Corollary 4.1 and Lemma 4.3. Thus, we omit it here.

6. Existence and uniqueness of the limit solution

In this section we prove that, under certain assumptions, a classical solution to the initial and boundary value problem associated to $\mathcal{P}$ is the uniform limit of any family of solutions to $\mathcal{P}_\varepsilon$ with corresponding boundary data. This in particular implies that such a limit exists and is unique.

In particular, under the assumptions of this section our classical solution is the unique classical solution and also the unique viscosity solution (by the results of Section 3).

Our first result is the approximation in a bounded cylinder. For the sake of simplicity we will assume that $\Omega = (0, d) \times \Sigma$.

Theorem 6.1. *Let $\Omega = (0, d) \times \Sigma$, $Q = \Omega \times (0, T)$, $\partial_N Q = (0, d) \times \partial\Sigma \times (0, T)$ and $\partial_D Q = \partial_p Q \setminus \partial_N Q$. Let u be a bounded classical solution to $\mathcal{P}$ in Q, with $\frac{\partial u}{\partial \eta} = 0$ on $\partial_N Q$, such that $u_{x_1} < 0$ on $\partial_D Q$. Assume that $u(0, x', t) > 0$ for $(x', t) \in \overline{\Sigma} \times [0, T]$ with $u(0, x', t) \in C^{2,1}(\overline{\Sigma} \times [0, T])$ and $u(d, x', t) < 0$*

for $(x',t) \in \overline{\Sigma} \times [0,T]$ with $u(d,x',t) \in C^{2,1}(\overline{\Sigma} \times [0,T])$. Let $u^\varepsilon \in C(\overline{Q})$ with $\nabla u^\varepsilon \in C(Q \cup \partial_N Q) \cap L^2_{loc}(\overline{Q})$ be a family of bounded weak solutions to $\mathcal{P}_\varepsilon$ in Q, with $\frac{\partial u^\varepsilon}{\partial \eta} = 0$ on $\partial_N Q$, such that $u^\varepsilon \to u$ uniformly on $\partial_D Q$.

Then $u^\varepsilon \to u$ uniformly in $\overline{Q}$.

Proof. Let $a > 0$; we will extend u to $Q_a = (-a, d+a) \times \Sigma \times (0,T)$ in such a way that $u \in C(\overline{Q}_a)$, $u > 0$ in $-a < x_1 < 0$, $u < 0$ in $d < x_1 < d+a$, $\mathcal{L}u \leq 0$ in $Q_a \cap \{u > 0\}$ and $\mathcal{L}u \geq 0$ in $Q_a \cap \{u < 0\}$.

Let $c > 0$ be such that $u_{x_1} \leq -c$ on $\partial_D Q$. Let $F_0(x',t) \in C^{2,1}(\overline{\Sigma} \times [0,T])$ be such that $u(0,x',t) = F_0(x',t)$, and let $F_d(x',t) \in C^{2,1}(\overline{\Sigma} \times [0,T])$ be such that $u(d,x',t) = F_d(x',t)$. Finally, let us define $u(x_1,x',t)$ in $Q_a \setminus Q$ in the following way:

$$u(x_1,x',t) = F_0(x',t) - cx_1 - kx_1^2 \quad \text{for } x_1 \in (-a,0),$$

$$u(x_1,x',t) = F_d(x',t) - c(x_1-d) + k(x_1-d)^2 \quad \text{for } x_1 \in (d,d+a).$$

Thus, clearly if k is large enough u satisfies all the requirements.

Given $\rho > 0$ and $\sigma > 0$ small, we define in $\overline{Q}$, $u_{\rho,\sigma}(x,t) = (1+\sigma)\,u(x_1 + \rho, x', t)$. Then, $u_{\rho,\sigma}$ is a classical subsolution to $\mathcal{P}$ in $Q \cap \{x_1 \leq d - \mu\}$ with vanishing Neumann data on $\partial_N Q \cap \{x_1 \leq d - \mu\}$ where $\mu > 0$ is chosen small so that $u_{\rho,\sigma}(0,x',t) > 0$ if $\rho < \mu$.

Given $\delta > 0$ we choose ρ and σ small so that $u_{\rho,\sigma} \geq u - \delta$. On the other hand, using Corollary 3.3 and the results of [15], which imply that $|\nabla u^-| > 0$ on $\overline{Q \cap \partial\{u > 0\}}$, we see that $u_{\rho,\sigma}$ is under the hypotheses of Theorem 5.1 in $Q \cap \{x_1 \leq d - \mu\}$. Therefore, there exists a family v^ε (depending on ρ and σ) of subsolutions to $\mathcal{P}_\varepsilon$ in $Q \cap \{x_1 \leq d - \mu\}$, such that

$$\frac{\partial v^\varepsilon}{\partial \eta} = 0 \quad \text{on } \partial_N Q \cap \{x_1 \leq d - \mu\},$$

$$v^\varepsilon \to u_{\rho,\sigma} \quad \text{uniformly in } \overline{Q} \cap \{x_1 \leq d - \mu\}, \text{ as } \varepsilon \to 0. \tag{6.1}$$

Let us observe that by the construction, $v^\varepsilon = u_{\rho,\sigma}$ in a neighborhood of $\{x_1 = d - \mu\}$, so that we can extend v^ε to Q letting $v^\varepsilon = u_{\rho,\sigma}$ in $d - \mu \leq x_1 \leq d$, and it follows that the v^ε are subsolutions to $\mathcal{P}_\varepsilon$ in Q satisfying (6.1) up to $x_1 = d$.

In addition, it follows from the uniform convergence of v^ε to $u_{\rho,\sigma}$ that σ can be chosen small enough (and ρ small depending on σ) so that we have, for $\varepsilon \leq \varepsilon_0(\delta)$,

$$v^\varepsilon \leq u^\varepsilon \quad \text{on } \partial_D Q,$$

$$\frac{\partial v^\varepsilon}{\partial \eta} = \frac{\partial u^\varepsilon}{\partial \eta} = 0 \quad \text{on } \partial_N Q.$$

Consequently, $v^\varepsilon \le u^\varepsilon$ in $\overline{Q}$. Therefore, $u_{\rho,\sigma} - \delta \le u^\varepsilon$ in $\overline{Q}$, and finally we obtain

$$u - 2\delta \le u^\varepsilon \quad \text{in } \overline{Q}.$$

In order to show that $u^\varepsilon \le u + 2\delta$, we proceed in a similar way.

For $\rho > 0$ and $\sigma > 0$ small, we define in $\overline{Q}$

$$u^{\rho,\sigma}(x,t) = (1-\sigma)\, u(x_1 - \rho, x', t).$$

We choose ρ and σ small enough so that $u^{\rho,\sigma} \le u + \delta$. Then, if σ is small enough, $u^{\rho,\sigma}$ is under the hypotheses of Theorem 5.2 in the domain $Q \cap \{x_1 \ge \mu\}$ where $\mu > 0$ is chosen in such a way that $u^{\rho,\sigma}(d, x', t) < 0$ if $\rho < \mu$. Therefore, there exists a family v^ε (depending on ρ and σ) of supersolutions to $\mathcal{P}_\varepsilon$ in $Q \cap \{x_1 > \mu\}$, such that

$$\begin{aligned} &\frac{\partial v^\varepsilon}{\partial \eta} = 0 \quad \text{on } \partial_N Q \cap \{x_1 > \mu\}, \\ &v^\varepsilon \to u^{\rho,\sigma} \quad \text{uniformly in } \overline{Q} \cap \{x_1 \ge \mu\}, \text{ as } \varepsilon \to 0. \end{aligned} \tag{6.2}$$

Since by the construction in Theorem 5.2, $v^\varepsilon = u^{\rho,\sigma}$ in a neighborhood of $x_1 = \mu$, if we extend v^ε to the whole region $\overline{Q}$ letting $v^\varepsilon = u^{\rho,\sigma}$ in $0 \le x_1 \le \mu$, it follows that the v^ε are supersolutions to $\mathcal{P}_\varepsilon$ in Q satisfying (6.2) up to $x_1 = 0$.

We finally choose σ small enough (and ρ small depending on σ) so that we have

$$\begin{gathered} v^\varepsilon \ge u^\varepsilon \quad \text{on } \partial_D Q, \\ \frac{\partial v^\varepsilon}{\partial \eta} = \frac{\partial u^\varepsilon}{\partial \eta} = 0 \quad \text{on } \partial_N Q \end{gathered}$$

for $\varepsilon \le \varepsilon_1(\delta)$. It follows that $v^\varepsilon \ge u^\varepsilon$ in $\overline{Q}$. Therefore,

$$u^{\rho,\sigma} + \delta \ge u^\varepsilon \quad \text{on } \overline{Q},$$

so that $u + 2\delta \ge u^\varepsilon$ on $\overline{Q}$. Thus, u^ε converges uniformly to u in $\overline{Q}$. □

A similar result holds for a full cylinder as spatial domain, under suitable monotonicity assumptions at $x_1 = \pm\infty$.

Theorem 6.2. *Let $\Omega = \mathbb{R} \times \Sigma$, $Q = \Omega \times (0,T)$, $\partial_N Q = \mathbb{R} \times \partial\Sigma \times (0,T)$ and $\partial_D Q = \partial_p Q \setminus \partial_N Q$. Let u be a bounded classical solution to $\mathcal{P}$ in Q, with $\frac{\partial u}{\partial \eta} = 0$ on $\partial_N Q$ and $||u||_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q})} < \infty$, such that $u\big|_{\partial_D Q}$ has a bounded, nonempty free boundary. Assume that $u_{x_1} < 0$ on $\partial_D Q$ and $u_{x_1}(x,0) \le -c_1 e^{-c_2|x_1|}$ for $|x_1| \ge a$ for some constants c_1, c_2, $a > 0$. Let $u^\varepsilon \in C(\overline{Q})$ with $\nabla u^\varepsilon \in C(Q \cup \partial_N Q) \cap L^2_{loc}(\overline{Q})$ be a family of bounded weak solutions to $\mathcal{P}_\varepsilon$ in Q, with $\frac{\partial u^\varepsilon}{\partial \eta} = 0$ on $\partial_N Q$, such that $u^\varepsilon \to u$ uniformly on $\partial_D Q$, with $|u^\varepsilon(x,0) - u(x,0)| \le k_1\, e^{-k_2 x_1^2}$ for $|x_1| \ge a$ for some constants k_1, $k_2 > 0$.*

Then $u^\varepsilon \to u$ uniformly in $\overline{Q}$.

Proof. Step I. Behavior of u^ε and u for $x_1 \to \pm\infty$. Let us first see that we can take a large enough so that, for some $\varepsilon_1 > 0$,

$$u^\varepsilon \geq \varepsilon \qquad \text{for } x_1 \leq -a, \tag{6.3}$$

$$u^\varepsilon \leq 0 \qquad \text{for } x_1 \geq a. \tag{6.4}$$

In fact, let $c, K_0 > 0$ be such that, for $\varepsilon \leq \varepsilon_0$, it holds that $u^\varepsilon(x,0) \geq 2c$ for $x_1 \leq -K_0$. For $\delta_0 > 0$, $K \geq K_0$ and $\mathcal{A} \geq ||a_1||_\infty$, let us consider the function

$$\begin{aligned} v_-(x,t) = c\Big(1 - \exp\Big\{\frac{\alpha}{c}(x_1 + \mathcal{A}t) + \frac{\alpha^2}{c^2}t + K\frac{\alpha}{c}\Big\}\Big)^+ \\ - \frac{\gamma}{\alpha}c\Big(1 - \exp\{\frac{\alpha}{c}(x_1 + \mathcal{A}t) + \frac{\alpha^2}{c^2}t + K\frac{\alpha}{c}\}\Big)^- \end{aligned}$$

where $\gamma > 0$ and $\alpha = \sqrt{2M + \delta_0 + \gamma^2}$. Then, for every $R > 0$, v_- is a bounded classical subsolution to $\mathcal{P}$ in $Q \cap \{x_1 \leq R\}$, with $\frac{\partial v_-}{\partial \eta} = 0$ on $\partial_N Q \cap \{x_1 \leq R\}$, such that $Q \cap \partial\{v_- > 0\}$ is bounded and $|\nabla v_-^+|^2 - |\nabla v_-^-|^2 = 2M + \delta_0$ on $Q \cap \partial\{v_- > 0\}$. Moreover, it holds that $v_-(x,0) \leq c$ for $x \in \Omega$.

Let us choose K large enough so that $v_-(x,t) \leq -2\mathcal{C}$ for $x_1 \geq -K_0$, $t \in [0,T]$, where $\mathcal{C} \geq ||u^\varepsilon||_{L^\infty(Q)}$. Since v_- satisfies the hypotheses of Theorem 5.1 with $\Omega = (-\infty, R) \times \Sigma$ for every $R \geq -K_0$, we can construct a family v_-^ε of weak subsolutions to $\mathcal{P}_\varepsilon$ in $Q \cap \{x_1 \leq R\}$ with $\frac{\partial v_-^\varepsilon}{\partial \eta} = 0$ on $\partial_N Q \cap \{x_1 < R\}$, such that $v_-^\varepsilon \to v_-$ uniformly in $\overline{Q} \cap \{x_1 \leq R\}$.

Then, there exists $0 < \varepsilon_1 \leq \varepsilon_0$ such that $v_-^\varepsilon(x,0) \leq 2c$ in $\Omega \cap \{x_1 \leq R\}$, $v_-^\varepsilon(x,0) \leq -\mathcal{C}$ for $-K_0 \leq x_1 \leq R$ and $v_-^\varepsilon(x,t) \leq -\mathcal{C}$ for $x_1 = R$, $t \in (0,T)$ if $\varepsilon \leq \varepsilon_1$. Thus, $v_-^\varepsilon(x,0) \leq u^\varepsilon(x,0)$ in $\Omega \cap \{x_1 \leq R\}$ for $\varepsilon \leq \varepsilon_1$ and

$$v_-^\varepsilon(x,t) \leq u^\varepsilon(x,t) \quad \text{on } x_1 = R, \text{ for } t \in (0,T).$$

Therefore, $v_-^\varepsilon \leq u^\varepsilon$ in $\overline{Q} \cap \{x_1 \leq R\}$ so that, if a is taken large enough, (6.3) holds.

Let us now see that we can take a so large that (6.4) also holds. In fact, let $K_0 > 0$ such that $u^\varepsilon(x,0) \leq 0$ for $x_1 \geq K_0$. Let

$$v_+(x,t) = 2\mathcal{C}\Big(1 - \exp\Big\{\frac{\sqrt{2M - \delta_0}}{2\mathcal{C}}(x_1 + \mathcal{A}t) + \frac{2M - \delta_0}{4\mathcal{C}^2}t - \frac{\sqrt{2M - \delta_0}}{2\mathcal{C}}K\Big\}\Big)^+.$$

Then, if $K > K_0$ is large enough, it holds that $v_+(x,0) \geq \frac{3}{2}\mathcal{C}$ if $x_1 \leq K_0$. On the other hand, $v_+ \geq 0$ in Q and v_+ is a bounded classical supersolution to $\mathcal{P}$ in Q with bounded free boundary such that $|\nabla v_+^+|^2 = 2M - \delta_0$ on the free boundary. By Theorem 5.2 in [17] there exists a family v_+^ε of weak supersolutions to $\mathcal{P}_\varepsilon$ in Q with $\frac{\partial v_+^\varepsilon}{\partial \eta} = 0$ on $\partial_N Q$ such that $v_+^\varepsilon \to v_+$ uniformly in Q. Thus, there exists $\varepsilon_2 > 0$ such that for $\varepsilon \leq \varepsilon_2$

$$v_+^\varepsilon(x,0) \geq u^\varepsilon(x,0) \quad \text{in } \Omega.$$

Therefore, $v_+^\varepsilon \geq u^\varepsilon$ in Q, so that (6.4) holds if a is large.

On the other hand, u satisfies the hypotheses of Proposition 2.3. Therefore, $Q \cap \partial\{u > 0\}$ is bounded, so that we may choose a large enough in order to have, in addition, that

$$\begin{aligned} &u > 0 \qquad \text{in } x_1 \leq -a, \\ &u < 0 \qquad \text{in } x_1 \geq a. \end{aligned}$$

Therefore, for $\varepsilon \leq \varepsilon_1$ and $w^\varepsilon = u^\varepsilon - u$, it holds that

$$\begin{aligned} &\mathcal{L}w^\varepsilon = 0 && \text{in } |x_1| > a, \quad 0 < t < T \\ &w^\varepsilon(x,0) \leq k_1 e^{-k_2 x_1^2} && \text{in } |x_1| > a \\ &w^\varepsilon \leq L && \text{on } |x_1| \geq a, \quad 0 < t < T \end{aligned}$$

for some constant L independent of ε. Therefore, there exist $\bar{k}_1$, $\bar{k}_2 > 0$ such that, for some constant $l_1 > a$ independent of ε,

$$w^\varepsilon(x,t) \leq \bar{k}_1 e^{-\bar{k}_2 x_1^2} \qquad \text{in } |x_1| \geq l_1, \text{ if } \varepsilon \leq \varepsilon_1.$$

We may replace the function w^ε above by $-w^\varepsilon$. Therefore,

$$|u^\varepsilon(x,t) - u(x,t)| \leq \bar{k}_1 e^{-\bar{k}_2 x_1^2} \qquad \text{in } |x_1| \geq l_1, \text{ if } \varepsilon \leq \varepsilon_1. \tag{6.5}$$

Let us now analyze the behavior of u_{x_1} for $x_1 \to \pm\infty$. It holds that

$$\begin{aligned} &\mathcal{L}u_{x_1} = 0 && \text{in } |x_1| > a, \quad 0 < t < T \\ &u_{x_1}(x,0) \leq -c_1 e^{-c_2|x_1|} && \text{in } |x_1| > a \\ &u_{x_1} \leq -r && \text{on } |x_1| = a, \quad 0 < t < T \end{aligned}$$

for some positive constant r. Therefore, there exist $\bar{c}_1, \bar{c}_2 > 0$ and $l_2 > a$ such that

$$u_{x_1}(x,t) \leq -\bar{c}_1 e^{-\bar{c}_2|x_1|} \quad \text{in } |x_1| \geq l_2. \tag{6.6}$$

Step II. Let $\delta > 0$. We will show that

$$|u^\varepsilon - u| < 2\delta \qquad \text{in } \overline{Q}$$

if ε is small enough.

In fact, the proof follows exactly as that of Step II of Theorem 6.2 in [18]. In the present situation the comparison of the functions $u_{\rho,\sigma}$ (respectively $u^{\rho,\sigma}$) and u is done in the bounded cylinder $Q \cap \{|x_1| \leq l\}$ where l is chosen large enough. □

Using the ideas of Theorems 6.1 and 6.2 we can prove the following theorem in a semicylinder. For the sake of simplicity we will state it for the semicylinders $(0,\infty)\times\Sigma$ and $(-\infty,d)\times\Sigma$.

Theorem 6.3. *Let $\Omega=(0,\infty)\times\Sigma$ (respectively $(-\infty,d)\times\Sigma$), $Q=\Omega\times(0,T)$, $\partial_N Q=(0,\infty)\times\partial\Sigma\times(0,T)$ (respectively $\partial_N Q=(-\infty,d)\times\partial\Sigma\times(0,T)$) and $\partial_D Q=\partial_p Q\backslash\partial_N Q$. Let u be a bounded classical solution to $\mathcal{P}$ in Q, with $\frac{\partial u}{\partial\eta}=0$ on $\partial_N Q$ and $||u||_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q})}<\infty$, such that $u\big|_{\partial_D Q}$ has a bounded, nonempty free boundary. Assume that $u_{x_1}<0$ on $\partial_D Q$ and $u_{x_1}(x,0)\le -c_1e^{-c_2|x_1|}$ for $x_1\ge a$ (respectively for $x_1\le -a$) for some constants c_1, c_2, $a>0$. Also, $u(0,x',t)\in C^{2,1}(\overline{\Sigma}\times[0,T])$ with $u(0,x',t)>0$ for $x'\in\overline{\Sigma}$, $t\in[0,T]$ (respectively $u(d,x',t)\in C^{2,1}(\overline{\Sigma}\times[0,T])$ with $u(d,x',t)<0$ for $x'\in\overline{\Sigma}$, $t\in[0,T]$). Let $u^\varepsilon\in C(\overline{Q})$ with $\nabla u^\varepsilon\in C(Q\cup\partial_N Q)\cap L^2_{loc}(\overline{Q})$ be a family of bounded weak solutions to $\mathcal{P}_\varepsilon$ in Q, with $\frac{\partial u^\varepsilon}{\partial\eta}=0$ on $\partial_N Q$, such that $u^\varepsilon\to u$ uniformly on $\partial_D Q$, with $|u^\varepsilon(x,0)-u(x,0)|\le k_1\,e^{-k_2\,x_1^2}$ for $x_1\ge a$ (respectively for $x_1\le -a$) for some constants k_1, $k_2>0$.*

Then $u^\varepsilon\to u$ uniformly in $\overline{Q}$.

7. Comments and concluding remarks

The results about uniqueness and coincidence of the classical, viscosity and limit solution established in this paper for the two-phase problem are parallel to similar results obtained for the one-phase problem in [17]. Also, the idea of constructing super- and subsolutions of the problems $\mathcal{P}_\varepsilon$ by rounding the classical super- or subsolution of the free-boundary problem near the free boundary is similar to the one-phase case. However, in the two-phase problem the values of the slope of the super- or subsolution at each side of the free boundary are not individually controlled since the free-boundary condition only gives a relation between both slopes. Consequently, there is a difference in technique that justifies the detailed derivation done in this paper. In particular, the construction of the two-phase auxiliary ODE profiles done in Section 4 is completely different. Moreover, the modified sub- and supersolutions to the problems $\mathcal{P}_\varepsilon$ constructed in Section 5 must be pasted to the respective free-boundary sub- and supersolution at the levels $u=A\varepsilon$ and $u=-A\varepsilon$ for a suitable constant A. There is also a delicate construction of the initial data since we must use different profiles in different directions. This entails a number of new theoretical steps.

Besides, there are other differences to be noted: conditions on the supports that had to be imposed in the one-phase case to ensure uniqueness of the limit and viscosity solutions disappear from the statements of the two-phase case.

The uniqueness result for the one-phase problem has been recently improved by Cafferelli and Petrosyan in [12], where they show that uniqueness of the limit solution holds without the assumption that a classical solution exists. They use a different monotonicity condition corresponding to a different geometry, but this difference is not essential. It is not known how to use their technique in the two-phase problem.

As was mentioned in the Introduction, the results of Sections 2 and 3 apply to the two-phase free-boundary problem with general jump condition $G(|\nabla u^+|, |\nabla u^-|) = 0$ with suitable assumptions on $G(a,b)$: it must be monotone nondecreasing in a and nonincreasing in b, and one of both monotonicities has to be strict, as used in [9] for the elliptic case.

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