Lagrange interpolation and approximation in Banach spaces

(joint work with Lisa Nilsson and Damián Pinasco)

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Universidad Torcuato Di Tella

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Outline

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- *Integral representation formula on Banach spaces*

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Lisa Nilsson and Seán Dineen later obtained the same formula in the context of fully nuclear spaces with a basis.

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The Newton form: $p_k(z) = \sum_{j=0}^{k} c_j(z - a_0)\cdots(z - a_{j-1})$, where the coefficients c_j are the "divided differences" defined inductively.

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so $c_j = \frac{1}{j!}$, and $p_m(z) = \sum_{j=0}^{n} \frac{1}{j!} z(z-1)\cdots(z-(j-1))$.

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$$\limsup_k \frac{|a_k|}{k} < \ln(2).$$

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$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \int_{E'} \frac{\gamma(x)^k}{k!} \tilde{f}(\gamma) dW(\gamma) \\ &= \sum_{k=0}^{\infty} P_k(x), \qquad \text{the Taylor series of } f. \end{aligned}$$

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Theorem

Let $f : E \longrightarrow \mathbb{C}$ be a representable function, and $x_0, x_1, \dots, x_n, \dots$ a sequence of points in E verifying

$$\limsup_k \frac{M_k}{k^\alpha} < \infty \quad \textit{for some } \alpha < 1/2.$$

Then the Lagrange polynomials L_k converge to f uniformly on bounded subsets of E.

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$$\frac{M_k^k \|\gamma\|^k e^{M_k\|\gamma\|}}{k!} \leq \frac{M_k^k \|\gamma\|^k e^{\frac{M_k^r}{r}} e^{\frac{\|\gamma\|^s}{s}}}{(k!)^{\frac{1}{r}}(k!)^{\frac{1}{s}}}$$

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&= \frac{e^{\frac{M_k^r}{r}} M_k^k}{(k!)^{\frac{1}{r}}} \cdot \frac{e^{\frac{\|\gamma\|^s}{s}} \|\gamma\|^k}{(k!)^{\frac{1}{s}}}.
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Now take $2 < r < \frac{1}{\alpha}$ and $s < 2$ such that $\frac{1}{r} + \frac{1}{s} = 1$. Then, using Young's inequality

$$\begin{aligned}
\frac{M_k^k \|\gamma\|^k e^{M_k\|\gamma\|}}{k!} &\leq \frac{M_k^k \|\gamma\|^k e^{\frac{M_k^r}{r}} e^{\frac{\|\gamma\|^s}{s}}}{(k!)^{\frac{1}{r}}(k!)^{\frac{1}{s}}} \\
&= \frac{e^{\frac{M_k^r}{r}} M_k^k}{(k!)^{\frac{1}{r}}} \cdot \frac{e^{\frac{\|\gamma\|^s}{s}} \|\gamma\|^k}{(k!)^{\frac{1}{s}}}.
\end{aligned}$$

The second factor is bounded by

$$e^{\frac{\|\gamma\|^s}{s}} \left(e^{\|\gamma\|^s}\right)^{\frac{1}{s}} = e^{\frac{2\|\gamma\|^s}{s}}.$$

In the first factor consider, for large k, $M_k \approx ck^{\alpha}$:

$$\frac{M_k^k e^{\frac{M_k^r}{r}}}{(k!)^{\frac{1}{r}}} \approx \frac{c^k k^{k\alpha} e^{\frac{c^r k^{r\alpha}}{r}}}{(k!)^{\frac{1}{r}}}$$

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because $\alpha - \frac{1}{r} < 0$, as is $r\alpha - 1$.

We therefore have

$$|f(x) - L_k(x)| \leq \int_{E'} |e^{\gamma(x)} - L_{k,\gamma}(\gamma(x))||\tilde{f}(\gamma)|dW(\gamma)$$

We therefore have

$$\begin{aligned}|f(x)-L_k(x)| &\le \int_{E'} |e^{\gamma(x)} - L_{k,\gamma}(\gamma(x))||\tilde{f}(\gamma)|dW(\gamma)\\ &\le \frac{M_k^k e^{\frac{M_k^r}{r}}}{(k!)^{\frac{1}{r}}} \int_{E'} e^{\frac{2\|\gamma\|^s}{s}} \left|\tilde{f}(\gamma)\right| dW(\gamma).\end{aligned}$$

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Now, if $\tilde{f} \in L^p(W)$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1,$ using Holder's inequality this is

$$\leq \frac{M_k^k e^{\frac{M_k^r}{r}}}{(k!)^{\frac{1}{r}}} \left(\int_{E'} \left[e^{\frac{2\|\gamma\|^s}{s}} \right]^q dW(\gamma) \right)^{1/q} \|\tilde{f}\|_p$$

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&= \frac{M_k^k e^{\frac{M_k^r}{r}}}{(k!)^{\frac{1}{r}}} \left\| e^{\frac{2q\|\gamma\|^s}{s}} \right\|_1^{1/q} \|\tilde{f}\|_p \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty.
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We therefore have

$$|f(x)-L_k(x)| \leq \int_{E'} |e^{\gamma(x)} - L_{k,\gamma}(\gamma(x))||\tilde{f}(\gamma)|dW(\gamma)$$
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Here we have used Fernique's theorem to assure the integrability of $e^{\frac{2q\|\gamma\|^s}{s}}$ for $s < 2$.

We therefore have

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\begin{aligned}
|f(x) - L_k(x)| &\leq \int_{E'} |e^{\gamma(x)} - L_{k,\gamma}(\gamma(x))||\tilde{f}(\gamma)|dW(\gamma) \\
&\leq \frac{M_k^k e^{\frac{M_k^r}{r}}}{(k!)^{\frac{1}{r}}} \int_{E'} e^{\frac{2\|\gamma\|^s}{s}} \left|\tilde{f}(\gamma)\right| dW(\gamma).
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Now, if $\tilde{f} \in L^p(W)$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1,$ using Holder's inequality this is

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&= \frac{M_k^k e^{\frac{M_k^r}{r}}}{(k!)^{\frac{1}{r}}} \left\| e^{\frac{2q\|\gamma\|^s}{s}} \right\|_1^{1/q} \|\tilde{f}\|_p \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty.
\end{aligned}
$$

Here we have used Fernique's theorem to assure the integrability of $e^{\frac{2q\|\gamma\|^s}{s}}$ for $s < 2$. Thus L_k converges to f uniformly on bounded subsets of E.

Thank you!