Approximation by Toeplitz operators on Bergman spaces

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1 Introduction

Let $\mathbb{B}_n$ denote the unit ball in $\mathbb{C}^n$. For $\alpha > -1$, we let

$$dv_\alpha(z) := c_\alpha\,(1-|z|^2)^\alpha\,dv(z), \quad \text{with} \quad v_\alpha\,(\mathbb{B}_n) = 1.$$

For $1 \le p < \infty$ let $L^p_\alpha := L^p_\alpha(\mathbb{B}_n, dv_\alpha)$ and $A^p_\alpha \subset L^p_\alpha$ be the subspace of analytic functions. If $1 < p < \infty$,

$$P_\alpha(f)(z) := \int_{\mathbb{B}_n} \frac{f(w)}{(1-z\overline{w})^{n+1+\alpha}}\,dv_\alpha(w).$$

is a bounded projection from L^p_α to A^p_α.

If $a \in L^\infty(\mathbb{B}_n)$, the Toeplitz operator $T_a : A^p_\alpha \to A^p_\alpha$ is

$$T_a f := P_\alpha(af).$$

It is immediate to see that $\|T_a\|_{\mathcal{L}(L^p_\alpha, A^p_\alpha)} \lesssim \|a\|_{L^\infty}$.

For $1 < p < \infty$, $\alpha > -1$ and for $z \in \mathbb{B}_n$ let

$$k_z^{(p,\alpha)}(w) = \frac{(1 - |z|^2)^{\frac{n+1+\alpha}{q}}}{(1 - \overline{z}w)^{n+1+\alpha}},$$

where $q = \frac{p}{(p-1)}$. We have $\left\| k_z^{(p,\alpha)} \right\|_{A_\alpha^p} \approx 1$, with implied constants depending on p, α, n.

The Berezin transform of an operator $S \in \mathcal{L}(A_\alpha^p)$ is

$$B_\alpha(S)(z) := \left\langle S k_z^{(p,\alpha)}, k_z^{(q,\alpha)} \right\rangle_{dv_\alpha}.$$

Thus, $|B_\alpha(S)| \leq \|S k_z^{(p,\alpha)}\|_{A_\alpha^p} \|k_z^{(q,\alpha)}\|_{A_\alpha^q} \lesssim \|S\|$.

Hence, $B_\alpha : \mathcal{L}(A_\alpha^p) \to L^\infty$, and it is one-to-one.

2 Carleson measures

A measure $\mu \geq 0$ on $\mathbb{B}_n$ is called a Carleson measure for A^p_α if there is $C > 0$ independent of f, such that

$$\int_{\mathbb{B}_n} |f(z)|^p \, d\mu(z) \leq C \int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z)$$

The best constant C is $\|\imath_p\|^p$. For a finite measure $\mu \geq 0$,

$$T_\mu f(z) := \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \overline{w}z)^{n+1+\alpha}} d\mu(w)$$

is an analytic function for any polynomial f. Thus T_μ is densely defined on A^p_α.

For any $z \in \mathbb{B}_n$, there is a unique automorpishm $\varphi_z : \mathbb{B}_n \to \mathbb{B}_n$ such that $\varphi_z \circ \varphi_z = id$ and $\varphi_z(0) = z$.

$T_\mu \in \mathcal{L}(A^p_\alpha)$ if and only if μ is a Carleson for A^p_α. In this case, $B_\alpha(T_\mu)$ is bounded:

$$B_\alpha(\mu)(z) := \int_{\mathbb{B}_n} \frac{(1-|\varphi_z(w)|^2)^{n+1+\alpha}}{(1-|w|^2)^{n+1+\alpha}}\, d\mu(w).$$

Lemma 2.1. *If μ is a positive measure on $\mathbb{B}_n$,*

$$\|B_\alpha(\mu)\|_\infty \approx \|\imath_p\|^p \approx \|T_\mu\|_{\mathcal{L}(A^p_\alpha)},$$

where the constants depend on n, p and α.

If μ is a complex measure,

$$\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4,$$

where $\mu_j \geq 0$ and $|\mu| \approx \sum_{j=1}^4 \mu_j$. Then $|\mu|$ is Carleson iff all the μ_j are Carleson, and T_μ is bounded.

3 Approximation

Theorem 3.1. *Let μ be a complex measure such that $|\mu|$ is Carleson for A^p_α. Then there are functions $B_k(\mu) \in L^\infty$ such that $T_{B_k(\mu)} \to T_\mu$ in $\mathcal{L}(A^p_\alpha)$-norm.*

For $k \geq \alpha$ define the function

$$B_k(\mu)(z) := \frac{c_k}{c_\alpha} \int_{\mathbb{B}_n} \frac{(1 - |\varphi_z(w)|^2)^{n+1+k}}{(1 - |w|^2)^{n+1+\alpha}} \, d\mu(w).$$

Since $k \geq \alpha$, $\quad |B_k(\mu)(z)| \leq \dfrac{c_k}{c_\alpha} B_\alpha(|\mu|)(z).$

By the duality $(A^p_\alpha)^* \simeq A^q_\alpha$, where $q = p/(p-1)$, the theorem means that for $f \in A^p_\alpha$ and $g \in A^q_\alpha$,

$$\langle (T_{B_k(\mu)} - T_\mu) f, g\rangle = \int_{\mathbb{B}_n} B_k(\mu) f \overline{g}\, dv_\alpha - \int_{\mathbb{B}_n} f \overline{g}\, d\mu$$

tends to 0 when $k \to \infty$ uniformly on $\|f\|_p$ and $\|g\|_q$.

We truncate B_k and take its adjoint. If $0 < r < 1$, let

$$B_{k,r}(\mu)(z) := \frac{c_k}{c_\alpha} \int_{|\varphi_z(w)|<r} \frac{(1-|\varphi_z(w)|^2)^{n+1+k}}{(1-|w|^2)^{n+1+\alpha}} \, d\mu(w).$$

and for $h \in L^1_\alpha$,

$$B^*_{k,r}(h)(w) := \frac{c_k}{c_\alpha} \int_{|\varphi_z(w)|<r} \frac{(1-|\varphi_z(w)|^2)^{n+1+k}}{(1-|w|^2)^{n+1+\alpha}} \, h(z) dv_\alpha(z).$$

$B^*_{k,r}$ is a contraction on L^1_α and acts on L^∞ with norm $\le C(n, \alpha, r)$. If $|\mu|$ is Carleson and $h \in L^1_\alpha$, by Fubini

$$\int_{\mathbb{B}_n} B_{k,r}(\mu)(z) h(z) dv_\alpha(z) = \int_{\mathbb{B}_n} B^*_{k,r}(h)(w) \, d\mu(w). \tag{3.1}$$

Theorem 3.2. *Let $|\mu|$ be a Carleson measure and $h = f\overline{g}$, with $f \in A^p_\alpha$ and $g \in A^q_\alpha$. Then*

$$\left|\int_{\mathbb{B}_n} B_k(\mu) f\overline{g} dv_\alpha - \int_{\mathbb{B}_n} f\overline{g} d\mu\right| \leq C(k) \left\|B_\alpha(|\mu|)\right\|_\infty \left\|f\right\|_p \left\|g\right\|_q,$$

where $C(k) \to 0$ as $k \to \infty$.

Proof. Assume that $\|B_\alpha(|\mu|)\|_\infty \leq 1$ and let $h = f\overline{g}$.

$$B_k(\mu) = B_{k,r}(\mu) + E_{k,r}(\mu) = \frac{c_k}{c_\alpha}\int_{|\varphi_z(w)|<r} \cdots + \frac{c_k}{c_\alpha}\int_{|\varphi_z(w)|\geq r} \cdots$$

From (3.1) we see that

$$\begin{aligned} J_k &:= \left|\int_{\mathbb{B}_n} B_k(\mu) h dv_\alpha - \int_{\mathbb{B}_n} h d\mu\right| \\ &\le \left|\int_{\mathbb{B}_n} B_{k,r}(\mu) h dv_\alpha - \int_{\mathbb{B}_n} h d\mu\right| + \int_{\mathbb{B}_n} \left|E_{k,r}(\mu) h\right| dv_\alpha \\ &\le \int_{\mathbb{B}_n} \left|B^*_{k,r}(h) - h\right| d|\mu| + \int_{\mathbb{B}_n} E_{k,r}(|\mu|) \, |h| \, dv_\alpha, \end{aligned}$$

and since

$$\begin{aligned} |B^*_{k,r}(h) - h| &\le |B^*_{k,r}(h) - B^*_{k,r}(1) h| + |B^*_{k,r}(1) - 1| \, |h| \\ &\le B^*_{k,r}(|h - h(w)|)(w) + |B^*_{k,r}(1) - 1| \, |h|, \end{aligned}$$

we get

$$
\begin{aligned}
J_k \;\leq\; & \int_{\mathbb{B}_n} B^*_{k,r}(|h-h(w)|)d|\mu|(w) \\
& + \int_{\mathbb{B}_n} |B^*_{k,r}(1)-1|\,|h|d|\mu| + \int_{\mathbb{B}_n} E_{k,r}(|\mu|)\,|h|\,dv_\alpha.
\end{aligned}
$$

For the last two integrals, we show that

$$
E_{k,r}(|\mu|)(z) \lesssim k^n \,\left(1-r^2\right)^{k-\alpha}
$$

and

$$
\left|B^*_{k,r}(1)-1\right| \lesssim \int_{\mathbb{B}_n} |u|\,dv_k(u) + v_k\left(\mathbb{B}_n \setminus r\mathbb{B}_n\right),
$$

where the constants of $\lesssim$ depend only on n and α. Hence, we only need to estimate the first of the above integrals.

Let f be a C^1 function on $\mathbb{B}_n$. The gradient of f is

$$\nabla f = \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}, \frac{\partial f}{\partial \overline{z}_1}, \ldots, \frac{\partial f}{\partial \overline{z}_n} \right)$$

and the invariant gradient is

$$\widetilde{\nabla} f(z) = \nabla \left(f \circ \varphi_z \right)(0).$$

By [Zhu, 2005], if $f : \mathbb{B}_n \to \mathbb{C}$ is holomorphic,

1. $|\widetilde{\nabla}(f \circ \varphi)(z)| = |(\widetilde{\nabla} f) \circ \varphi(z)|$ for all $\varphi \in \mathrm{Aut}(\mathbb{B}_n)$
2. $f \in A^p_\alpha \iff |\widetilde{\nabla} f| \in L^p_\alpha$
3. $\| \, |\widetilde{\nabla} f| \, \|_{L^p_\alpha} \lesssim \|f\|_{A^p_\alpha}$

Lemma 3.3. *If $0 < r < 1$ is fixed, there are positive constants $C(k) \to 0$ when $k \to \infty$, such that*

$$B^*_{k,r}(|h - h(w)|)(w) \leq$$
$$C(k) \int_{\mathbb{B}_n} \big(|g||\widetilde{\nabla} f| + |f||\widetilde{\nabla} g|\big)(\zeta) \, \frac{(1-|\varphi_\zeta(w)|^2)^{n+1+\alpha}}{(1-|w|^2)^{n+1+\alpha}} dv_\alpha(\zeta).$$

Hence, integrating this inequality with respect to $d|\mu|(w)$,

$$\int_{\mathbb{B}_n} B^*_{k,r}\,(|h - h(w)|)\, d|\mu|(w)$$
$$\leq \; C(k) \int_{\mathbb{B}_n} \big(|g|\,|\widetilde{\nabla} f| + |f|\,|\widetilde{\nabla} g|\big)(\zeta)\, \underbrace{B_\alpha(|\mu|)(\zeta)}_{\leq 1}\, dv_\alpha(\zeta)$$
$$\leq \; C(k) \left(\|g\|_{A^q_\alpha} \|\,|\widetilde{\nabla} f|\,\|_{L^p_\alpha} + \|f\|_{A^p_\alpha} \|\,|\widetilde{\nabla} g|\,\|_{L^q_\alpha} \right).$$

$\square$

4 Applications

Theorem 4.1. *Let μ be a measure such that $|\mu|$ is Carleson. If $B_\alpha(\mu) \equiv 0$ on $\partial\mathbb{B}_n$ then T_μ is compact.*

Proof.

- If $a \in L^\infty$, $a(z) \to 0$ when $z \to \partial\mathbb{B}_n$, T_a is compact.
- If $B_\alpha(\mu) \equiv 0$ on $\partial\mathbb{B}_n$ then $B_k(\mu) \equiv 0$ on $\partial\mathbb{B}_n$
- $T_\mu = \lim T_{B_k(\mu)}$

$\square$

Definition. A function $a \in L^1(dv_\alpha)$ is in BMO if

$$\|a\|_{BMO} := \sup_{w \in \mathbb{B}_n} B_\alpha(|a - (B_\alpha a)(w)|)(w) < \infty.$$

This vanishes on constants, but it becomes a norm by just adding $|\int a \, dv_\alpha|$.

In 2002 Zorboska proved for $n = 1,\ \alpha = 0$ and $p = 2$, that if $a \in BMO$ and $B_\alpha(a) \underset{|z|\to 1}{\longrightarrow} 0$ then T_a is compact. We'll see that this holds in general. Since for any $w \in \mathbb{B}_n$,

$$|a| \leq |a - (B_\alpha a)(w)| + |(B_\alpha a)(w)|,$$

taking B_α of this inequality and evaluating at w we get

$$B_\alpha(|a|)(w) \leq \|a\|_{BMO} + |(B_\alpha a)(w)|.$$

Thus, for $a \in BMO$,

$$\begin{aligned} B_\alpha(a) \in L^\infty \Rightarrow B_\alpha(|a|) \in L^\infty &\Rightarrow |a| dv_\alpha \text{ is Carleson} \\ &\Rightarrow T_a \text{ is bounded} \Rightarrow B_\alpha(T_a) \in L^\infty. \end{aligned}$$

Corollary 4.2. *Let $a \in BMO$. If $B_\alpha(a)(z) \to 0$ when $|z| \to 1$ then T_a is compact on A^p_α.*

Proof. Since $B_\alpha(a)$ is continuous on $\mathbb{B}_n$, the hypothesis implies that it is bounded. Thus, $|a| dv_\alpha$ is a Carleson measure, and since $B_\alpha(a) = B_\alpha(a dv_\alpha) \to 0$ at the boundary, the previous theorem says that T_a is compact. $\square$

5 A characterization of compactness

Definition. The Toeplitz algebra $\mathfrak{T}$ is the closure in $\mathcal{L}(A^p_\alpha)$ of

$$\Big\{ \sum_{\text{finite}} \prod_{\text{finite}} T_{a_{ij}} : a_{ij} \in L^\infty \Big\}$$

Theorem 5.1. *Let $Q \in \mathcal{L}(A^p_\alpha)$. Then Q is compact if and only if $Q \in \mathfrak{T}$ and $B_\alpha(Q)(z) \to 0$ when $|z| \to 1$.*

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