Tangent structures to measures: an invitation to the scenery flow

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Magnifying a measure around a point

Definition

Given a Radon (locally finite and Borel regular) measure μ on $\mathbb{R}^d$ and a point $x \in \text{supp}\mu$, let $\mu_{x,t}$ be the measure obtained by restricting μ to the ball $B(x, e^{-t})$, normalizing so that the new measure has unit mass, and rescaling back to the unit ball.

More explicitly,

$$\mu_{x,t} = T_{x,t}\left(\frac{1}{\mu B(x, e^{-t})}\mu|_{B(x,e^{-t})}\right)$$

where $T_{x,t}(y) = e^t(y - x)$ is the homothety that maps $B(x, e^{-t})$ to $B = B(0, 1)$.

Note that $\mu_{x,t} \in \mathcal{P}_d$, the family of Borel probability measures on B.

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Examples

- If $\mu \ll \mathcal{L}_d$ = Lebesgue measure on$\mathbb{R}^d$, then $\mathrm{Tan}(\mu, x) = \{\mathcal{L}_d|_B\}$ for μ a.e. x.
- If M is a k-dimensional immersed submanifold of $\mathbb{R}^d$ and $\mathcal{H}^k$ is k-Hausdorff measure, then $\mathrm{Tan}(\mathcal{H}^k|_M, x) = \mathcal{H}^k|_{T_xM \cap B}$, where T_xM is the tangent plane to M at x.
- If μ is the natural measure on the middle-thirds Cantor set, then $\mathrm{Tan}(\mu, x)$ is uncountable at almost all x; however, by the self-similarity of μ, every $\nu \in \mathrm{Tan}(\mu, x)$ is obtained by restricting a homothetic copy of μ to B.

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Remarks on tangent measures

- Tan(μ, x) is always nonempty (by Alaoglu); in general, it is very large. It is known that for a residual set of measures μ, all measures in $\mathcal{P}_d$ are in Tan(μ, x) for μ almost all x.
- There is a close (and deep) link between the uniqueness of tangent measures and rectifiability properties of μ. David Preiss in 1987 introduced and applied tangent measures to settle long standing open problems in the theory of rectifiability.
- Tangent measures capture important local information (for example related to densities and to rectifiability) and are more regular than the original measures. But for certain problems, notably those involving dimension, tangent measures capture very little or no information.

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Local dimensions of measures

Definition

If μ is a Radon measure and $x \in \operatorname{supp}(\mu)$, the (upper and lower) local dimensions are

$$\overline{\dim}(\mu, x) = \limsup_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r},$$

$$\underline{\dim}(\mu, x) = \liminf_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r}$$

If the limit exists, we call it the local dimension and denote it $\dim(\mu, x)$.

If μ is such that $\dim(\mu, x)$ exists and is constant at μ almost all points x, we say that μ is exact dimensional. Note that this means that $\mu(B(x,r)) \sim r^s$ for some s (the value of the dimension), μ typical x and all small radii r.

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Remarks on local dimensions

- Local dimensions are closely connected to Hausdorff and packing dimensions (of sets).
- If $\mu \ll \mathcal{L}_d$, then μ has exact dimension d, that is, $\dim(\mu, x) = d$ for a.e. x. The reciprocal is far from true.
- If μ is k-surface area on a k-dim. immersed submanifold M, then $\dim(\mu, x) = k$ for all $x \in M$.
- The natural measure μ on the middle-thirds Cantor set is exact dimensional; the dimension is $\log 2/\log 3$.
- In general, however, the local dimension may fail to exist at all points, the the value of the upper and lower local dimension may also be different at all points.
- Tangent measures give nearly no information on the (local) dimension of a measure. For example, there are two exact dimensional measures on $\mathbb{R}^d$, of dimensions 0 and d (minimal and maximal) which have the same set of tangent measures at all points.

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The scenery flow

- Let $\mathcal{M}_d$ be the family of Radon measures μ on $\mathbb{R}^d$ such that $0 \in \text{supp}\mu$. This is a Borel (but not closed) set in the weak* topology.
- If $\mu \in \mathcal{M}_d$, then set $S_t\mu = \mu_{0,t}$. Recall that this is the measure obtained by restricting μ to $B(0, e^{-t})$, normalizing, and rescaling back to the unit ball B.
- Note that $S_{t+s} = S_t \circ S_s$ by the choice of exponential radius. This says that $\{S_t\}$ is a $\mathcal{P}_d$-valued (semi)flow on $\mathcal{M}_d$. We call this flow the scenery flow.
- A Borel probability measure P on $\mathcal{M}_d$ is called invariant under the scenery flow if

$$P(A \Delta S_t^{-1} A) = 0 \quad \text{for all } t > 0, \text{ Borel set } A \subset \mathcal{P}_d.$$

  Note that P is a measure on measures (or the distribution of a random measure).

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Remarks on the scenery flow

- Variants of the scenery flow were studied, often independently, by many people. This definition is due to Mike Hochman.
- The scenery flow is not continuous - there are discontinuities at times t for which the boundary of $B(x, e^{-t})$ has positive μ-measure.
- In the definition of the scenery flow, the origin 0 plays the role of a typical point. Given a measure μ and a point x in its support, one can translate μ so that x becomes the origin and then study the orbit of that measure under the scenery flow. Note that this is nothing but $(\mu_{x,t})_{t>0}$.

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Tangent distributions

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Let μ be a Radon measure and x a point in its support. Given $T > 0$, let $\langle\mu\rangle_{x,T}$ be distribution of the random measure $\mu_{x,t}$, where t is sampled uniformly from $[0, T]$. More formally,

$$\langle\mu\rangle_{x,T} = \frac{1}{T}\int_0^T \delta_{\mu_{x,t}}\, dt.$$

Note that $\langle\mu\rangle_{x,T}$ is a Borel probability measure on $\mathcal{P}_d$ (the latter endowed with the weak* topology). The set of weak* accumulation points of $\langle\mu\rangle_{x,T}$ as $T \to \infty$ are called tangent distributions of μ at x, and the set of all of them is denoted $\mathcal{TD}(\mu, x)$.

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Remarks on tangent distributions

- $\mathcal{TD}(\mu, x)$ is always supported on $\mathrm{Tan}(\mu, x)$, but not all tangent measures are seen by tangent distributions. Heuristically, tangent distributions see limits $\nu = \lim_n \mu_{x,t_n}$ where the sequence of times t_n has positive density (it is not very sparse).
- Again, $\mathcal{TD}(\mu, x)$ is nonempty by Alaoglu; in general it is not unique, but many fractal measures have the property that $\mathcal{TD}(\mu, x)$ is a singleton at μ almost all points. This is a weak notion of self-similarity, it says that although tangent measures are not unique, the statistics of what one sees as we zoom in towards a point is well defined.
- Tangent distributions have a surprising degree of regularity. For example, for μ almost all x, all $P \in \mathcal{TD}(\mu, x)$ are supported on exact dimensional measures, even if μ is far from exact dimensional.

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Tangent distributions and dimension

One can recover a lot of information about μ from its tangent distributions, in particular regarding local dimension. For example, at μ a.e. x,

$$\overline{\dim}(\mu, x) = \sup\left\{ \int \dim \nu \, dP : P \in \mathcal{TD}(\mu, x) \right\},$$

$$\underline{\dim}(\mu, x) = \inf\left\{ \int \dim \nu \, dP : P \in \mathcal{TD}(\mu, x) \right\}.$$

Quasi-palm distributions

Definition

Let P be a Borel probability measure on $\mathcal{P}_d$. We say that P is quasi-Palm if a set $A \subset \mathcal{P}_d$ has full P-measure if and only if for P-almost all measures μ and μ almost all points x, the measure $\mu_{x,0}$ (the translation of μ so that x becomes the origin) is in A.

Heurisically, the quasi-Palm property is a strong translation-invariance property; it says that the origin behaves exactly in the same way as a typical point for P-typical measures.

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Let P be a Borel probability measure on $\mathcal{P}_d$. We say that P is quasi-Palm if a set $A \subset \mathcal{P}_d$ has full P-measure if and only if for P-almost all measures μ and μ almost all points x, the measure $\mu_{x,0}$ (the translation of μ so that x becomes the origin) is in A.

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Tangent distributions are fractal distributions

Theorem (Mike Hochman)

Let μ be a Radon measure on $\mathbb{R}^d$. Then at μ almost all x, all tangent distributions $P \in \mathcal{TD}(\mu, x)$ satisfy the following:

1. *P is supported on $\mathcal{M}_d$ (the measures with 0 in their support).*
2. *P is invariant under the scenery flow.*
3. *P is quasi-Palm.*

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Distributions P with the above three properties are called fractal distributions.

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The space of fractal distributions

Theorem (Käenmäki-Sahslten-P.S. (2014))

Let $\mathcal{FD}$ be the space of all fractal distributions.

1. *$\mathcal{FD}$ is weak* closed (far from obvious since none of the defining properties are closed).*
2. *$\mathcal{FD}$ is a simplex (easy).*
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Tangent distributions=fractal distributions

Remark

Recall that Hochman proved that tangent distributions at a typical point are fractal distributions.

Theorem (Käenmäki-Sahslten-P.S. (2014))

1. *For each $P \in \mathcal{FD}$, there exists a measure μ such that $\mathcal{TD}(\mu, x) = \{P\}$ for μ almost all x.*
2. *For a residual set of Radon measures μ (in the weak* topology), at μ almost all points x it holds that $\mathcal{TD}(\mu, x) = \mathcal{FD}$*

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Final remarks

- Because tangent distributions are invariant under the scenery flow, one can use the tools of ergodic theory to study geometric problems, in particular (but not only) those involving dimension.
- With A. Käenmäki and T. Sahlsten, we use the machinery of the scenery flow to improve, unify and generalize many results related to conical densities and porosities (classical subjects in geometric measure theory).
- Although the machinery is very technical, once it is in place, it yields simple, transparent, "trivial" proofs of many results.
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