Hypercyclic operators on spaces of holomorphic functions

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Joint work with S. Muro and D. Pinasco

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Introduction

Definition

Let $T \in \mathcal{L}(X)$, X a Fréchet space.

- T is transitive if for each $U, V \subset X$ open sets, $T^n(U) \cap V \neq \emptyset$ for some n.
- T is hypercyclic if $\{x, Tx, T^2x, \dots\}$ is dense for some $x \in X$

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Theorem (G. Godefroy - J. H. Shapiro, 1991)

Every convolution operator (i.e. an operator that commutes with translations) on $H(\mathbb{C}^n)$ which is not a scalar multiple of identity is hypercyclic.

A non-convolution operator

For $\lambda, b \in \mathbb{C}$, let $T \in \mathcal{L}(H(\mathbb{C}))$ be defined as

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Theorem (Aron-Markose, 2004)

(a) *If* $|\lambda| \geq 1$ *then* T *is hypercyclic.*

(b) *If* $|\lambda| < 1$ *and* $b = 0$ *then* T *is not hypercyclic.*

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The case $b = 0$ is the easiest to prove.

$$T^n f(z) = \lambda^{\frac{n(n-1)}{2}} f^{(n)}(\lambda^n z).$$

(a) use the hypercyclicity criterion.

(b) by the Cauchy's estimates

$$|T^n f(0)| \leq C |\lambda|^{\frac{n(n-1)}{2}} n! \underset{n\to\infty}{\longrightarrow} 0.$$

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Definition (Bayart-Grivaux, 2006)

An operator is frequently hypercyclic if there is a vector $x \in X$ such that for each open set V, there is $C > 0$ such that $\{T^k(x) : k \leq cn\}$ intersects V at least n times.

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A measure-preserving mapping $T : (X, \mu) \to (X, \mu)$ is

- *ergodic* if for every pair of sets U, V with $\mu(U)\mu(V) > 0$,

$$T^n(U) \cap V \neq \emptyset \quad \text{for some } n \in \mathbb{N},$$

- *strongly mixing* if for every pair of measurable sets U, V,

$$\lim_n \mu(U \cap T^{-n}(V)) = \mu(U)\mu(V).$$

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strongly mixing $\Rightarrow$ ergodic $\underset{T \text{ linear}}{\Rightarrow}$ frequently hypercyclic

Other forms of hypercyclicity

Theorem (Bayart-Matheron)

Let $T \in \mathcal{L}(X)$, X a separable complex Fréchet space. Suppose that for every dense $D \subset \mathbb{T}$, the set $\{Ker(T - \lambda) : \lambda \in D\}$ spans a dense subspace in X.
Then there is a Gaussian T-invariant strongly mixing Borel probability measure on X with full support.

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Theorem (Murillo Arcila - Peris)

Let $T \in \mathcal{L}(X)$, X a separable complex Fréchet space. If there is a dense subset X_0 of X and a sequence of maps $S_n : X_0 \to X$ such that, for each $x \in X_0$,

1. *$\sum_{n=0}^{\infty} T^n x$ converges unconditionally,*
2. *$\sum_{n=0}^{\infty} S_n x$ converges unconditionally, and*
3. *$T^n S_n x = x$ and $T^m S_n x = S_{n-m} x$ if $n > m$,*

then there is a T-invariant strongly mixing Borel probability measure on X with full support.

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If $\lambda = 1$, then T is a convolution operator. If $\lambda \neq 1$, let $a = \frac{b}{1-\lambda}$ and $T_0 f(z) = f'(\lambda z)$. Then

$$\begin{array}{ccc} H(\mathbb{C}) & \xrightarrow{T} & H(\mathbb{C}) \\ {\scriptstyle \tau_a}\downarrow & & \uparrow{\scriptstyle \tau_{-a}} \\ H(\mathbb{C}) & \xrightarrow[T_0]{} & H(\mathbb{C}) \end{array}$$

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- If $|\lambda| < 1$ we know by Aron-Markose that T is not hypercyclic.
- For $|\lambda| \geq 1$ we can use the Murillo Arcila - Peris criterion.

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For $\lambda, b \in \mathbb{C}^N$, $\alpha \in \mathbb{N}_0^N$ let $T \in \mathcal{L}(H(\mathbb{C}^N))$ be defined as

$$Tf(z) = D^\alpha f(\lambda z + b).$$

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- Conjugation by τ_a, $a = \sum_{k>j} \frac{b_k}{1-\lambda_k} e_k$, allows us to assume

$$Tf(z) = D^\alpha f(z_1 + b_1, \ldots, z_j + b_j, \lambda_{j+1} z_{j+1}, \ldots, \lambda_N z_N).$$

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- $T^n f(z) = \lambda^{\frac{n(n-1)}{2}\alpha} D^{n\alpha} f\left(\lambda^n z\right).$
- By Cauchy inequalities, $|T^n f(0)| \longrightarrow_{n\to\infty} 0.$

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c) If $|\lambda^\alpha| < 1$, $\lambda_i = 1$ and $b_i \neq 0$ for some i then T is frequently hypercyclic.

- In this case, T is Runge transitive: suppose C_ϕ is a composition operator. Take neighbourhoods $U_1 = \{f : \|f - g_1\|_K < \varepsilon\}$, $U_2 = \{f : \|f - g_2\|_K < \varepsilon\}$.

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- If $\phi^n(K) \cap K = \emptyset$, then by Runge's theorem there is a polynomial p such that

$$\|C_\phi^{-n}(g_1) - p\|_{\phi^n(K)} < \varepsilon \quad \text{and} \quad \|g_2 - p\|_K < \varepsilon.$$

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- Therefore $C_\phi^n(p) \in U_1$ and $p \in U_2$, which implies $C_\phi^n(U_2) \cap U_1 \neq \emptyset$.

Holomorphic functions on Banach spaces

$P : E \to \mathbb{C}$ is a k-homogeneous polynomial if $P(x) = A(x, \dots, x)$ for some (unique symmetric) k-linear form $A : E \times \cdots \times E \to \mathbb{C}$.

$\mathcal{P}(^k E)$ = space of k-homogeneous polynomials.

Example. Finite type polynomials: $P(x) = \sum_{j=1}^{k} \gamma_j(x)^n$, for $\gamma_j \in E'$.

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$f : E \to \mathbb{C}$ is holomorphic if it has Taylor expansion at each point.

- An entire function is of *bounded type* if it is bounded on each bounded set of E.
- $H_b(E)$ = space of entire bounded type functions.
- If E' separable and finite type polynomials are dense in $\mathcal{P}(^k E)$, then $H_b(E)$ is a separable Fréchet space.

Functions of bounded $\mathcal{A}$-type

Notation: for $P \in \mathcal{P}^k(E)$, $a \in E$, $l \leq k$ define $P_{a^l} \in \mathcal{P}^{k-l}(E)$ by

$$P_{a^l}(x) = \overset{\vee}{P}(a^l, x^{k-l}) = \overset{\vee}{P}(\underbrace{a, ..., a}_{l}, \underbrace{x, ..., x}_{k-l})$$

Note that $\frac{k!}{l!} P_{a^l} = d^{k-l} P(a)$

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Definition (Nachbin, 1969)

A sequence $\mathcal{A} = \{\mathcal{A}_k\}_{k=0}^{\infty}$, where $(\mathcal{A}_k, \| \cdot \|_{\mathcal{A}_k})$ is a Banach ideal of k-homogeneous polynomials is a **holomorphy type** if there exist constants $c_{k,l}$ such that for every Banach space E:

$$P \in \mathcal{A}_k(E),\ a \in E \Rightarrow P_{a^l} \in \mathcal{A}_{k-l}(E) \text{ and} \tag{1}$$

$$\|P_{a^l}\|_{\mathcal{A}_{k-l}(E)} \leq c_{k,l} \|P\|_{\mathcal{A}_k(E)} \|a\|^l \tag{2}$$

Functions of bounded $\mathcal{A}$-type

Definition

f is an entire function of $\mathcal{A}$-bounded type ($f \in H_{b\mathcal{A}}(E)$) if $d^k f(0) \in \mathcal{A}_k(E)$ and

$$p_s(f) := \sum_{k \geq 0} \left\| \frac{d^k f(x)}{k!} \right\|_{\mathcal{A}_k(E)} s^k < \infty \text{ for every } s > 0.$$

$H_{b\mathcal{A}}(E)$ is the space of entire functions with infinite $\mathcal{A}$-radius of convergence at zero.

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- For $\mathcal{A} = \mathcal{P} \rightsquigarrow H_{b\mathcal{A}}(E) = H_b(E)$.
- For $\mathcal{A}$ =nuclear polynomials $\rightsquigarrow H_{Nb}(E)$ (Gupta-Nachbin 1970).
- $\mathcal{A}$ =Hilbert-Schmidt polynomials $\rightsquigarrow H_{hs}(E)$ (Dwyer 1971).
- $\mathcal{A}$ = approximable polynomials$\rightsquigarrow H_{bc}(E)$ (Aron 1979).
- $\mathcal{A}$ = integral polynomials $\rightsquigarrow H_{bI}(E)$ (Dimant-Galindo-Maestre-Zalduendo 2004).
- w-continuous on bounded sets, extendible, Schatten, ...

A non-convolution operator on $H_{b\mathcal{A}}(E)$

- Let E be a Banach with 1-unconditional shrinking canonical basis $(e_j)_j$ ($E = c_0$ or $E = \ell_p$ with $1 \lneq p < \infty$).
- For $b \in E$, $\lambda \in \ell_\infty$, $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ let $T \in \mathcal{L}(H_{b\mathcal{A}}(E))$ be defined as

$$Tf(z) = D^\alpha f(\lambda z + b),$$

where $\lambda z = (\lambda_j z_j)_j$, and

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Theorem

If $|\lambda^\alpha| \geq 1$ then T is strongly mixing.

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- $T^n f(a) = \lambda^{\frac{n(n-1)}{2}\alpha} D^{n\alpha} f(a)$,
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3. $\frac{b}{1-\lambda} \notin E''$.

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A non-convolution operator on $H_{b\mathcal{A}}(E)$

- Note that $a = \frac{b}{1-\lambda} = \left(\frac{b_j}{1-\lambda_j}\right)_j$ is not necessarily in E.
- If $a = \frac{b}{1-\lambda} \in E$, then it is a fixed point of $\phi(z) = \lambda z + b$.
- $T^n f(a) = \lambda^{\frac{n(n-1)}{2}\alpha} D^{n\alpha} f(a)$,
- Using Cauchy-type inequalities for $H_{b\mathcal{A}}(E)$:

If $|\lambda^\alpha| < 1$ and $\frac{b}{1-\lambda} \in E$ then T is not hypercyclic.

- What happens if $\frac{b}{1-\lambda} \notin E$?

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If $\mathcal{A}$ is AB-closed, $|\lambda^\alpha| < 1$ and $\frac{b}{1-\lambda} \in E''$ then T is not hypercyclic.

The case $b/(1-\lambda) \notin E''$

1. For some j, $\lambda_j = 1$ and $b_j \neq 0$.

Then $\phi(z) = \lambda z + b$ is runaway (for every bounded set, $\phi^n(B) \cap B = \emptyset$ for n big enough).

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We need a Runge type result:

Let $\mathcal{A}$ be a multiplicative holomorphy type and f a holomorphic function of $\mathcal{A}$-bounded type $B(0, r+\delta) \cup B(a, s+\delta)$ (disjoint balls). Then there are polynomials in $H_{b\mathcal{A}}(E)$ that approximate f in $H_{b\mathcal{A}}(B(0, \frac{r}{3}))$ and $H_{b\mathcal{A}}(B(a, \frac{s}{3}))$.

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Multiplicative sequence $\{\mathcal{A}_k\}$ of Banach polynomial ideals

$$P \in \mathcal{A}_k(E),\ Q \in \mathcal{A}_l(E) \Rightarrow\ PQ \in \mathcal{A}_{k+l}(E) \text{ and} \tag{3}$$

$$\|PQ\|_{\mathcal{A}_{k+l}(E)} \leq c_{k,l}\|P\|_{\mathcal{A}_k(E)}\|Q\|_{\mathcal{A}_l(E)}. \tag{4}$$

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- If $c_{k,l} \leq M^{k+l}$ then $H_{b\mathcal{A}}(E)$ is an algebra.
- If $c_{k,l} \leq \dfrac{(k+l)^{k+l}}{(k+l)!}\dfrac{k!}{k^k}\dfrac{l!}{l^l}$ then $H_{b\mathcal{A}}(E)$ is a locally m-convex Fréchet algebra.
- Every mentioned example is multiplicative.

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It suffices to show that $\{\phi^k(0) = \sum_{j\in\mathbb{N}} b_j \frac{\lambda_j^k - 1}{\lambda_j - 1} e_j\}_k$ is not bounded.
Let $a = a^1 + a^2$, $a^1 = \sum_{j\in N^1} \frac{b_j}{1-\lambda_j}$, $N^1 = \{j \text{ such that } |\lambda_j| = 1\}$.

$$\begin{aligned}
\frac{1}{N}\sum_{j=1}^{N} \|\phi^j(0)\| &\geq \frac{1}{N}\sum_{j=1}^{N} \left\| \sum_{l\leq M,\, l\in N_1} (\lambda_l^j - 1)\frac{b_l}{\lambda_l - 1} e_l \right\| \\
&\geq \left\| \sum_{l\leq M,\, l\in N_1} \frac{b_l}{\lambda_l - 1} e_l \left[\frac{1}{N}\sum_{j=1}^{N} (\lambda_l^j - 1) \right] \right\| \\
&\geq \frac{1}{2} \left\| \sum_{l\leq M,\, l\in N_1} \frac{b_l}{\lambda_l - 1} e_l \right\| \longrightarrow \infty, \text{ if } a^1 \notin E''
\end{aligned}$$

$Tf(z) = D^\alpha f(\lambda z + b)$ on $H_{b\mathcal{A}}(E)$

Theorem (Muro-Pinasco-S.)

Suppose E' separable and finite type polynomials dense in $\mathcal{A}_k(E)$ for every k.

- *If $|\lambda^\alpha| \geq 1$ then T is strongly mixing.*
- *If $\frac{b}{1-\lambda} \notin E''$ then T is frequently hypercyclic ($\mathcal{A}$ multiplicative).*
- *If $\frac{b}{1-\lambda} \in E''$ and $|\lambda^\alpha| < 1$ then T is not hypercyclic ($\mathcal{A}$ AB-closed).*

Thank you!