# Regular Holomorphic Functions on Complex Banach Lattices

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## 1. Holomorphy — some of the history

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  in a neighbourhood of $a$.
  $P_n$: (bounded) $n$-homogeneous polynomials.
- Grothendieck, Nachbin, Gupta (1950's and 60's): Duality in terms of **nuclear functions**/tensor products:

$$\mathcal{P}({}^nE') = \left(\mathcal{P}_N({}^nE)\right)' \quad \text{(subject to AP)}$$

- Boland, Dineen (1970's): Holomorphic functions on **nuclear locally convex spaces**. For suitable nuclear spaces with basis, the monomials are a basis for the space of holomorphic functions.

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  $\mathcal{H}_v(E)$: the space of holomorphic functions representable locally by unconditionally convergent monomial expansions.
- Defant, Díaz, García, Kalton, Maestre (2001 and 2005) proved Dineen's conjecture: if $E$ is a Banach space and $n \geqslant 2$, then $\mathcal{P}(^nE)$ has an unconditional basis if and only if $E$ is finite dimensional.

## 2. The Matos-Nachbin Holomorphy Type

$E$: a Banach space with unconditional Schauder basis $(e_j)$.

Every $P \in \mathcal{P}(^nE)$ has a monomial expansion:

$$P(z) = A(z, \dots, z), \quad z = \sum_j z_j e_j$$

$$P(z) = \sum_{|\alpha|=n} c_\alpha z^\alpha$$

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$\mathcal{P}_v(^n E)$: the subspace of polynomials for which the monomial expansion is unconditionally convergent at every point.

If $P \in \mathcal{P}_\nu(^n E)$, then

$$\tilde{P}(z) := \sum_{|\alpha|=n} |c_\alpha| z^\alpha$$

also belongs to $\mathcal{P}_\nu(^n E)$. A norm is defined on $\mathcal{P}_\nu(^n E)$ by

$$\nu(P) := \|\tilde{P}\| = \sup\Big\{ \Big| \sum |c_\alpha| z^\alpha \Big| : \|z\| \leqslant 1 \Big\}$$

$\mathcal{P}_\nu(^n E)$ is a Banach space with this norm.

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$$\mathcal{P}_N({}^n E) \subset \mathcal{P}_\nu({}^n E) \subset \mathcal{P}({}^n E)$$

**Extreme cases:**

1. $E = c_0$: $\mathcal{P}_\nu({}^n E) = \mathcal{P}_N({}^n E)$
2. $E = \ell_1$: $\mathcal{P}_\nu({}^n E) = \mathcal{P}({}^n E)$

(with equivalent norms in each case.)

**Holomorphic functions.**

In the complex case, we have a holomorphy type:

for $P \in \mathcal{P}_\nu({}^n E)$ and $z \in E$,

$$\nu\Big(\frac{1}{k!}\hat{d}^k P(z)\Big) \leqslant 4^n \|z\|^{n-k}\, \nu(P)$$

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**Theorem (Matos-Nachbin):** A holomorphic function $f$ on a domain $U \subset E$ belongs to $\mathcal{H}_\nu(U)$ if and only if $f$ is representable locally in $U$ by unconditionally pointwise convergent monomial expansions.

**Matos-Nachbin (1992):**

$E$ a complex Banach space with an unconditional basis. Let $U$ be a Reinhardt domain in $E$ containing $0$.
the following are equivalent:

1. $U$ is the domain of convergence of a multiple power series around $0$.
2. $U$ is modularly decreasing and logarithmically convex.
3. $U$ is the domain of existence of some $f \in \mathcal{H}_\nu(()U)$.
4. $U$ is a domain of $\nu$-holomorphy.
5. $U$ is a domain of holomorphy.
6. $U$ is pseudo-convex.

## 3. Regularity

**Riesz spaces (vector lattices):**

A real vector space $E$ with a compatible lattice structure:

$$x, y \in E \to x \vee y, \quad x \wedge y$$

For every $x \in E$,

$$x = x^+ - x^- \quad \text{where} \quad x^+ = x \vee 0, \quad x^- = (-x) \vee 0$$

Absolute values:

$$|x| := x \vee (-x) = x^+ + x^-$$

Normed Lattice: Riesz space with a norm satisfying

$$\||x|\| = \|x\|$$

**Dedekind complete:** every order bounded set has a supremum.

## Regular operators

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If $F$ is Dedekind complete, then regularity of an operator $T$ is equivalent to order boundedness and in this case, the space $\mathcal{L}_r(E; F)$ of regular operators is a Dedekind complete Riesz space.

$$|T|(x) = \sup\{|T(y)| : |y| \leqslant x\} \quad \text{for } x \in E^+$$

$$|T(x)| \leqslant |T|(|x|) \quad \forall x$$

**Order dual:**

$$\tilde{E} = \mathcal{L}_r(E; \mathbb{R})$$

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2. The space of regular operators is a Banach lattice with the **regular norm:**
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3. $\tilde{E} = E'$, the Banach dual, with equality of norms.
4. **Principal ideals:** for $u > 0$, the principal ideal
$$E_u := \{x \in E : |x| \leqslant ku \text{ for some } k \in \mathbb{N}\}$$
with norm given by the Minkowski functional of the order interval $[-u, u]$ is an AM-space with unit. It is lattice isometric to a $C(K)$ space.
$$E_u \approx C(K)$$

# Regular polynomials on Banach lattices

**Multilinear forms:**

$A \in \mathcal{L}(^n E_1, E_2, \ldots, E_n)$ is **positive** if

$$A(x_1, \ldots, x_n) \geqslant 0 \quad \text{for all } x_1, \ldots, x_n \geqslant 0$$

and $A$ is **regular** if it is the difference of two positive forms.

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$\mathcal{L}_r(^n E_1, \ldots, E_n)$: the Banach lattice of regular $n$-linear forms with the regular norm.

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**Homogeneous polynomials:**

The $n$-homogeneous polynomial $P = \hat{A}$ is **positive** if $A$ is positive.
$\mathcal{P}_r(^n E)$: the Banach lattice of regular $n$-homogeneous polynomials with the regular norm

$$\|P\|_r := \|\,|P|\,\|$$

If $P \in \mathcal{P}(^n E)$ is positive, then

1. $P(x) \geqslant 0$ for every $x \geqslant 0$.
2. $P$ is monotone on the positive cone:

$$\text{if} \quad 0 \leqslant x \leqslant y \quad \text{then} \quad P(x) \leqslant P(y)$$

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Absolute value:

$$|P(x)| \leqslant |P|(|x|) \quad \forall x \in E$$

$|P|$ is the smallest positive $n$-homogeneous polynomial satisfying this.

Every Banach space with a 1-unconditional basis is a Banach lattice, where the lattice operations are defined coordinatewise: if $x = \sum x_j e_j$, then

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**Grecu–Ryan (2004):** If $E$ is a Banach space with a

1-unconditional basis, then

$$\mathcal{P}_v({}^n E) = \mathcal{P}_r({}^n E)$$

with equality of norms.

# 4. Tensor Products

## The Fremlin Tensor Product (1972)

For (archimedean) Riesz spaces $E$ and $F$, the Fremlin tensor product $E \mathbin{\overline{\otimes}} F$ linearizes regular bilinear forms on $E \times F$.

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### The Fremlin-Wittstock Banach Lattice Tensor Products

$E \,\tilde{\otimes}_{|\pi|}\, F$ and $E \,\tilde{\otimes}_{|\varepsilon|}\, F$: Banach lattice versions of the projective and injective Banach space tensor products.

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None of Grothendieck's 14 natural tensor norms are Banach lattice norms.

**Labuschagne (2004):** If $E$ and $F$ are Banach lattices and $\alpha$ is a reasonable crossnorm on $E \otimes F$, then there is a reasonable crossnorm $|\alpha|$ on $E \otimes F$ such that $E \,\tilde{\otimes}_{|\alpha|}\, F$ is a Banach lattice with respect to the ordering induced by the $|\alpha|$-closure of the Fremlin cone of $E \otimes F$.

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## Linearizing regular polynomials

**Loane (2007) :** Construction of a symmetric $n$-fold Fremlin tensor product satisfying

$$\Big(\bigotimes_{n,|\pi|,s} E\Big)' = \mathcal{P}_r({}^n E)$$

for Banach lattices $E$.

## 5. Complex Banach Lattices

**Mittelmeyer-Wolff (1974) Axiomatization:**

$E$ a complex vector space, with a function $m\colon E \to R^+$, satisfying

1. $m(\lambda x) = |\lambda| m(x)$;
2. $m(m(m(x)+m(y)) - m(x+y)) = m(x)+m(y)+m(x+y)$;
3. $m(m(y) - km(x)) = m(y) - km(x) \quad \forall k > 0$ implies $x = 0$;
4. $E$ is the $\mathbb{R}$-linear span of $m(E)$.

This structure is called a **Complex Riesz Space**. $E$ is the algebraic complexification of the real vector space $E_{\mathbb{R}} = m(E)$ and this space has a vector lattice structure with $m$ as absolute value.

## The Krivine Functional Calculus for Banach Lattices

Fix $x_1, \ldots, x_n \in E$.

Let $\mathcal{C}_n$ be the vector lattice of all continuous, positively homogeneous real functions on $\mathbb{R}^n$, with

$$\|f\| = \sup\{|f(t_1, \ldots, t_n)| : |t_1| \vee \cdots \vee |t_n| = 1\}$$

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There exists a unique map

$$\tau : \mathcal{C}_n \to E$$

satisfying

- $\tau(t_j) = x_j$ for each $j = 1, \dots, n$.
- $\tau$ is linear and preserves the lattice operations.
- $\|\tau(f)\| \leqslant \|f\| \, |x_1| \vee \cdots \vee |x_n|$.

## Complex Banach Lattices:

For $z = x + iy \in E_{\mathbb{C}}$, the **modulus** is defined by

$$|z| = \sqrt{|x|^2 + |y|^2} = \sup_{0 \leqslant \theta \leqslant 2\pi} \left| x \cos\theta + y \sin\theta \right|$$

$E_{\mathbb{C}}$ is a Banach space with the norm

$$\|z\| = \| \, |z| \, \|$$

## 6. Regular Holomorphy

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**Definition:** A power series $\sum_n P_n$ of $n$-homogeneous polynomials on $E$ is **regularly convergent at** $z \in E$ if each $P_n$ is regular and

$$\sum_n |P_n|(|z|) < \infty$$

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$$\sum_n |P_n|(|z|) < \infty$$

**Theorem:** $f \in \mathcal{H}(U)$ is a regular holomorphic function if and only if $f$ is representable locally in $U$ by regularly pointwise convergent power series.

## Properties of the domain of regular convergence

Let $\sum_n P_n$ be a power series whose terms are all regular. Let $D$ be the set of points at which the series is regularly convergent.

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**The finite dimensional case:**

When $E = \mathbb{C}^k$, the domain of convergence is **logarithmically convex**:

$$\text{if } z,\ w \in D, \text{ then} \quad |z|^{\theta}|w|^{1-\theta} \in D \quad \text{for every } \theta \in (0,1)$$

## A Hölder Inequality for homogeneous polynomials:

Let $P$ be a regular homogeneous polynomial on the (real or complex) Banach lattice $E$ and let $a, b$ be positive elements of $E$. Then

$$\left|P(a^\theta b^{1-\theta})\right| \leqslant \left(|P|(a)\right)^\theta \left(|P|(b)\right)^{1-\theta}$$

for every $\theta \in (0, 1)$.

**Proof:**

$P$ restricts to a regular polynomial on the principal ideal generated by $u = a \vee b$.

By Fremlin's theorem, $P$ is an integral polynomial on $E_n \approx C(K)$.

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**Corollary:** The domain of regular convergence of a power series of regular homogeneous polynomials is logarithmically convex.