Bounded sets with respect to an operator ideal

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A. Pietsch, Ideals of multilinear functionals, Proceedings of the Second International Conference on Operator Algebras, Ideals and Their Applications in Theoretical Physics, Teubner-Texte, Leipzig, (1983) 185-199.

- Botelho, G. Ideals of polynomials generated by weakly compact operators. Note Mat. 25 (2005/06)
- Botelho, G.; Pellegrino, D. M. A note on polynomial characterizations of Asplund spaces. Proyecciones 24 (2005)
- Botelho, G.; Pellegrino, D. M. Two new properties of ideals of polynomials and applications. Indag. Math. (N.S.) 16 (2005)
- Botelho, G.; Pellegrino, D.; Rueda, P. On composition ideals of multilinear mappings and homogeneous polynomials. Publ. Res. Inst. Math. Sci. 43 (2007)
- Carando, D.; Dimant, V.; Sevilla-Peris, P. Ideals of multilinear formsa limit order approach. Positivity 11 (2007)
- Carando, D.; Dimant, V.; Muro, S. Coherent sequences of polynomial ideals on Banach spaces. Math. Nachr. 282 (2009)

- Aron, R.; Botelho, G.; Pellegrino, D.; Rueda, P. Holomorphic mappings associated to composition ideals of polynomials. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 21 (2010)
- Carando, D.; Galicer, D. Extending polynomials in maximal and minimal ideals. Publ. Res. Inst. Math. Sci. 46 (2010)
- Carando, D.; Galicer, D. Unconditionality in tensor products and ideals of polynomials, multilinear forms and operators. Q. J. Math. 62 (2011)
- Berrios, S.; Botelho, G. Approximation properties determined by operator ideals and approximability of homogeneous polynomials and holomorphic functions. Studia Math. 208 (2012)

- Carando, D.; Dimant, V.; Muro, S. Holomorphic functions and polynomial ideals on Banach spaces. Collect. Math. 63 (2012)
- Carando, D.; Dimant, V.; Muro, S. Every Banach ideal of polynomials is compatible with an operator ideal. Monatsh. Math. 165 (2012)
- Pellegrino, D.; Ribeiro, J. On multi-ideals and polynomial ideals of Banach spaces: a new approach to coherence and compatibility. Monatsh. Math. 173 (2014)
- Botelho, G.; Caliskan, E.; Moraes, G. The polynomial dual of an operator ideal. Monatsh. Math. 173 (2014)

Compact and weakly compact operator ideals

(Weakly) compact m-homogeneous polynomials $=$ map bounded sets to relatively (weakly) compact sets.

Polynomial ideal

$P : E \to F$ is a (continuous) n-homogeneous polynomial if $P(x) = A(x, \ldots, x)$ for some (continous) n-linear mapping A.

$\mathcal{P}(^nE, F) =$ all continuous n-homogeneous polynomials from E to F.

$\|P\| := \sup_{x \in B_E} \|P(x)\|$.

A polynomial ideal:
$\mathcal{Q}$ is a subclass of the class of all continuous homogeneous polynomials between Banach spaces such that all the components $\mathcal{Q}(^nE, F) = \mathcal{P}(^nE, F) \cap \mathcal{Q}$ satisfy:

1. $\mathcal{Q}(^nE, F) \hookrightarrow \mathcal{P}(^nE, F)$ which contains the n-homogeneous polynomials of finite type.
2. The ideal property: if $u \in \mathcal{L}(G, E)$, $P \in \mathcal{Q}(^nE, F)$ and $t \in \mathcal{L}(F, H)$, then the composition $t \circ P \circ u$ is in $\mathcal{Q}(^nG, H)$.

A normed (Banach) polynomial ideal:
$[\mathcal{Q}, \|\cdot\|_{\mathcal{Q}}]$ if $\|\cdot\|_{\mathcal{Q}} : \mathcal{Q} \to \mathbb{R}^+$ satisfies

1. $(\mathcal{Q}(^nE; F), \|\cdot\|_{\mathcal{Q}})$ is a normed (Banach) space for all E, F and n,
2. $\|P^n : \mathbb{K} \to \mathbb{K} : P^n(x) = x^n\|_{\mathcal{Q}} = 1$ for all n, and
3. If $u \in \mathcal{L}(G, E)$, $P \in \mathcal{Q}(^nE, F)$ and $t \in \mathcal{L}(F, H)$, then $\|t \circ P \circ u\|_{\mathcal{Q}} \leq \|t\|\|P\|_{\mathcal{Q}}\|u\|^n$,

Given an operator ideal $[\mathcal{I}, \iota]$,

- the composition ideal of polynomials $\mathcal{I} \circ \mathcal{P}$: all P s.t. $P = T \circ Q$, where Q is a homogeneous polynomial and T is a linear operator belonging to $\mathcal{I}$.
- The composition norm of $P \in \mathcal{I} \circ \mathcal{P}(^m E; F)$

  $$\|P\|_{\mathcal{I} \circ \mathcal{P}} := \inf\{\iota(T)\|Q\| : P = T \circ Q, Q \in \mathcal{P}(^m E; G), T \in \mathcal{I}(G; F)\},$$

  With this norm $\mathcal{I} \circ \mathcal{P}$ becomes a Banach polynomial ideal whenever $[\mathcal{I}, \iota]$ is a Banach operator ideal.

A method to construct polynomial ideals

Let $[\mathcal{I}, \iota]$ be a normed operator ideal and let F be a Banach space,

Stephani: $C_{\mathcal{I}}(F) :=$ $A \subset F$ s.t. $A \subset T(B_Z)$ for some Banach space Z and some $T \in \mathcal{I}(Z; F)$.

Any $A \in C_{\mathcal{I}}(F)$ is called a $\mathcal{I}-$bounded set.

- $P \in \mathcal{P}(^mE; F)$ is locally $\mathcal{I}-$bounded if for every $x \in E$ there exists a neighborhood V_x of x such that $P(V_x) \in C_{\mathcal{I}}(F)$.
- $\mathcal{P}_{\mathcal{I}}(^mE; F)$= all locally $\mathcal{I}-$bounded m-homogeneous polynomials from E to F.

$$P \in \mathcal{P}(^mE; F) \text{ is locally } \mathcal{I}\text{-bounded } \Leftrightarrow P(B_E) \in C_{\mathcal{I}}(F).$$

Given $P \in \mathcal{P}_{\mathcal{I}}({}^m E; F)$, we have $P(B_E) \subset T(B_Z)$ for some Banach space Z and some $T \in \mathcal{I}(Z; F)$. Define

$$\|P\|_{\mathcal{I}} := \inf \iota(T)$$

where T varies among those operators in $\mathcal{I}$ fulfilling the above condition.

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When $m = 1$ we write $\mathcal{L}_{\mathcal{I}}(E; F) = \mathcal{P}_{\mathcal{I}}(^1 E; F)$.

$$\mathcal{I}(E; F) \subset \mathcal{L}_{\mathcal{I}}(E; F) \subset \mathcal{L}(E; F)$$

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$$\mathcal{I}(E; F) \subset \mathcal{L}_{\mathcal{I}}(E; F) \subset \mathcal{L}(E; F)$$

$\mathcal{I}(E; F) = \mathcal{L}_{\mathcal{I}}(E; F)$ if and only if $\mathcal{I}$ is surjective.

$[\mathcal{P}_{\mathcal{I}}, \|\cdot\|_{\mathcal{I}}]$ is a (Banach) ideal of polynomials whenever $[\mathcal{I}, \iota]$ is a (Banach) ideal of operators.

Examples

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  $\mathcal{P}_{\mathcal{K}}(^m E; F)$ = compact polynomials.

Examples

- Sinha, Karn (2002): Let $1 \leq p, p' \leq \infty$ be conjugate indices. $K \subset E$ is relatively $p-$compact if for some sequence $(x_n) \in \ell_p(E)$, $K \subset \{\sum_n a_n x_n \mid (a_n) \in B_{\ell'_p}\}$,

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  An operator $T \in \mathcal{L}(E; F)$ is $p-$compact if $T(B_E)$ is relatively $p-$compact.

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  $k_p(T) = \inf\{\|(x_n)_n\|_p\}$
  $[\mathcal{K}_p, k_p]$ = the Banach ideal formed by all $p-$compact operators.
- $A \in C_{\mathcal{K}_p}(E)$ if and only if A is relatively $p-$compact.
- $\mathcal{P}_{\mathcal{K}_p}$ is the ideal of all p-compact homogeneous polynomials.

$\mathcal{P}_{\mathcal{K}_p}(^m E; F)$

- Aron, Maestre and R.(2010): p-compact holomorphic functions.
- Aron and R.(2011): the composition of a p-compact homogeneous polynomial with any n-homogeneous polynomial is p-compact
- Aron and R.(2011): Any continuous homogeneous polynomial maps relatively p-compact sets to relatively p-compact sets.
- Lassalle, Turco(2012): p-approximation property on E $\Leftrightarrow$ p-compact homogeneous polynomials with range on E can be uniformly approximated by finite rank polynomials.
- Aron, Caliskan, Garcia, Maestre: Any holomorphic function maps relatively p-compact sets to relatively p-compact sets.

Tensor stability

A Banach operator ideal $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ is tensor stable with respect to a tensor norm α if $T \otimes S$ belongs to $\mathcal{I}(E\hat{\otimes}_\alpha F; G\hat{\otimes}_\alpha H)$ whenever $T \in \mathcal{I}(E; G)$ and $S \in \mathcal{I}(F; H)$.

- Vala (1964): compact operators are ϵ-stable.
- Holub (1970): studied the stability of absolutely p-summing operators and nuclear operators.
- Berrios, Botelho, Carl, Defant, Ramanujan, König, Pietsch,...

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  Each tensor θ in $\otimes_{n,s}E$ can be written as $\theta = \sum_{i=1}^{m} \lambda_i x_i \otimes \cdots \otimes x_i$,
- Projective s-tensor norm:

$$\pi_s(\theta) := \inf\{\sum_{i=1}^{m} |\lambda_j| \|x_i\|^n : \theta = \sum_{i=1}^{m} \lambda_i x_i \otimes \cdots \otimes x_i\}.$$

$$\hat{\otimes}_{n,s}^{\pi_s} E := \text{the completion of } (\otimes_{n,s}E, \pi_s)$$

Symmetric tensor stability

Given $T \in \mathcal{L}(E; F)$, let $\otimes_m T : \hat{\otimes}_{\pi_s}^{m,s} E \to \hat{\otimes}_{\pi}^{m,s} F$ be defined by

$$\otimes_m T(\sum_{i=1}^{n} \alpha_i x_i \otimes \cdots \otimes x_i) = \sum_{i=1}^{n} \alpha_i T(x_i) \otimes \cdots \otimes T(x_i)$$

and extended by continuity to the completions.
An operator ideal $[\mathcal{I}, \iota]$ is stable under the formation of symmetric tensor products if $\otimes_m T : \hat{\otimes}_{\pi_s}^{m,s} E \to \hat{\otimes}_{\pi}^{m,s} F$ belongs to $\mathcal{I}(\hat{\otimes}_{\pi_s}^{m,s} E; \hat{\otimes}_{\pi}^{m,s} F)$ whenever $T \in \mathcal{I}(E; F)$ and, in this case, $\iota(\otimes_m T) \leq C\iota(T)^m$ for some positive constant C.

Aron and R.(2012)

If an operator ideal $[\mathcal{I}, \iota]$ is stable under the formation of symmetric tensor products then any continuous $m-$homogeneous polynomial maps $\mathcal{I}-$bounded sets to $\mathcal{I}-$bounded sets.

Aron and R.(2011):

The ideal of $p-$compact operators is stable under the formation of symmetric tensor products.

Any continuous $m-$homogeneous polynomial maps p-compact sets to p-compact sets.

Aron and R.(2012)

Let $\mathcal{I}$ be an operator ideal and let E, F, G and Z be Banach spaces.

(i) If $P \in \mathcal{P}_{\mathcal{I}}(^m E; F)$ and $Q \in \mathcal{P}(^l Z; E)$ then $P \circ Q \in \mathcal{P}_{\mathcal{I}}(^{ml} Z; F)$. In this case,

$$\|P \circ Q\|_{\mathcal{I}} \leq \|P\|_{\mathcal{I}} \|Q\|^m.$$

(ii) If $\mathcal{I}$ is stable under the formation of symmetric tensor products then $R \circ P \in \mathcal{P}_{\mathcal{I}}(^{mk} E; G)$ for any $P \in \mathcal{P}_{\mathcal{I}}(^m E; F)$ and any $R \in \mathcal{P}(^k F; G)$. In this case, there exists $K > 0$ such that

$$\|R \circ P\|_{\mathcal{I}} \leq K \|R\| \|P\|^m_{\mathcal{I}}.$$

Symmetric tensor stability for polynomials

The lack of associativity in the projective symmetric tensor product does not permit us to define the tensor product $\otimes_n P$ of an $m-$homogeneous polynomials P for $n \neq m$.

However, the next definition shows how to handle the case $n = m$.

Let $P \in \mathcal{P}(^m E; F)$. The $m-$tensor product of P is the $m-$homogeneous polynomial $\otimes_m P \in \mathcal{P}(^m \hat{\otimes}_{\pi_s}^{m,s} E; \hat{\otimes}_{\pi}^{m,s} F)$ given by $\otimes_m P = (\otimes_m P^L) \circ \delta_m$.

The commutativity of the diagram

$$\begin{tikzcd} \hat{\otimes}^{m,s}_{\pi_s} E \arrow[rr, "\otimes_m P"] \arrow[dr, "\delta_m"'] & & \hat{\otimes}^{m,s}_{\pi} F \\ & \hat{\otimes}^{m,s}_{\pi_s}(\hat{\otimes}^{m,s}_{\pi_s} E) \arrow[ur, "\otimes_m P^L"'] & \end{tikzcd}$$

makes clear that $(\otimes_m P)^L = \otimes_m P^L$. Notice that $\hat{\otimes}^{m,s}_{\pi_s}(\hat{\otimes}^{m,s}_{\pi_s} E)$ and $\hat{\otimes}^{m^2,s}_{\pi_s} E$ may differ. Then, although $\mathcal{P}({}^m\hat{\otimes}^{m,s}_{\pi_s} E; \hat{\otimes}^{m,s}_{\pi} F)$ and $\mathcal{L}(\hat{\otimes}^{m,s}_{\pi_s}(\hat{\otimes}^{m,s}_{\pi_s} E); \hat{\otimes}^{m,s}_{\pi} F)$ are isometrically isomorphic via the canonical linearization, we cannot conclude that $(\otimes_m P)^L$ belongs to $\mathcal{L}(\hat{\otimes}^{m^2,s}_{\pi_s} E; \hat{\otimes}^{m,s}_{\pi} F)$.

The composition ideal $\mathcal{I} \circ \mathcal{P}$ is stable under the formation of symmetric tensor products if $\otimes_m P$ belongs to $\mathcal{I} \circ \mathcal{P}({}^m\hat{\otimes}_{\pi_s}^{m,s} E; \hat{\otimes}_{\pi}^{m,s} F)$ for all $P \in \mathcal{I} \circ \mathcal{P}({}^m E; F)$.

Aron and R. (2011)

If an operator ideal $\mathcal{I}$ is stable under the formation of symmetric tensor products then so is the composition ideal of polynomials $\mathcal{I} \circ \mathcal{P}$.

If $P \in \mathcal{P}_{\mathcal{K}_p}({}^m E; F)$ then $\otimes_m P \in \mathcal{P}_{\mathcal{K}_p}({}^m\hat{\otimes}_{\pi_s}^{m,s} E; \hat{\otimes}_{\pi}^{m,s} F)$.

Comparing with the composition ideal of polynomials

An operator ideal $\mathcal{I}$ is said to satisfy Condition Γ if the closed absolutely convex hull $\overline{\Gamma}(A)$ of any $\mathcal{I}-$bounded set$A$ is $\mathcal{I}-$bounded.

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1. González and Gutiérrez: Any operator ideal which is surjective and closed satisfies Condition Γ. The most usual examples are the ideals of compact operators and weakly compact operators.
2. For $1 < p < \infty$, the ideal $[K_p, k_p]$ of all $p-$compact operators is surjective and satisfies Condition Γ, although K_p is not closed.

If $\mathcal{I}$ is surjective and satisfies Condition Γ then $\mathcal{P}_{\mathcal{I}}(^mE;F) = \mathcal{I} \circ \mathcal{P}(^mE;F)$ for all Banach spaces E and F, and $\|\cdot\|_{\mathcal{I}} \leq \|\cdot\|_{\mathcal{I}\circ\mathcal{P}}$.

If $\mathcal{I}$ is closed and surjective then $\|P\|_{\mathcal{I}} = \|P\|$ for all $P \in \mathcal{P}_{\mathcal{I}}(^mE;F)$.

THANK YOU!