Lower bounds for norms of products of polynomials on $L_p$ spaces

Jorge Tomás Rodríguez
Joint work with Daniel Carando and Damián Pinasco

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Introduction

Objective: find lower bounds for the norm of the product of scalar polynomials on Banach space $E$.

Where the norm of a polynomial $P: E \rightarrow K$ is the usual $\|P\| = \sup_{\|x\| = 1} |P(x)|$.

We study this problem for:

A) Continuous homogeneous polynomials.

B) Continuous polynomials.
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A) Continuous homogeneous polynomials.

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Problems

Problem A
Given $k_1, \cdots, k_n \in \mathbb{N}$, find the optimal constant $C(\mathcal{E}, k_1, \cdots, k_n)$, such that for every set of continuous homogeneous polynomials $P_1, \cdots, P_n : \mathcal{E} \to \mathbb{K}$, of degrees $k_1, \cdots, k_n$, the next inequality holds

$$C(\mathcal{E}, k_1, \cdots, k_n) \prod_{j=1}^n \|P_j\| \leq \|\prod_{j=1}^n P_j\|$$

Problem B
Given $k_1, \cdots, k_n \in \mathbb{N}$, find the optimal constant $D(\mathcal{E}, k_1, \cdots, k_n)$, such that for every set of (not necessarily homogeneous) continuous polynomials $P_1, \cdots, P_n : \mathcal{E} \to \mathbb{K}$, of degrees $k_1, \cdots, k_n$, the next inequality holds

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Previous results

C. Benítez, Y. Sarantopoulos and A. Tonge found a lower bound for these constants.

Theorem (Benítez, Sarantopoulos, Tonge - 1998)

For any complex Banach space $E$

$$D(E, k_1, \ldots, k_n) \geq k_1 \cdots k_n (k_1 + \cdots + k_n)^{k_1 \cdots k_n}.$$  

As an immediate consequence, for any complex Banach space $E$

$$C(E, k_1, \ldots, k_n) \geq k_1 \cdots k_n (k_1 + \cdots + k_n)^{k_1 \cdots k_n}.$$  

These bounds are optimal.
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These bounds are optimal.
In the complex Banach space $\ell_1$, define the polynomials $P_1, \ldots, P_n$ by
$$P_j \left( (a_i^{i}) \in \mathbb{N} \right) = a_{k^j}.$$ 
Then $\|P_j\| = 1$ and
$$\left\| \prod_{j=1}^{n} P_j \right\| = k_{1}^{k_1} + \cdots + k_{n}^{k_n} \left( k_{1} + \cdots + k_{n} \right) k_{1}^{k_1} + \cdots + k_{n}^{k_n}.$$ 
Therefore
$$\left( k_{1} + \cdots + k_{n} \right) k_{1}^{k_1} + \cdots + k_{n}^{k_n} \leq \left\| \prod_{j=1}^{n} P_j \right\| = 1,$$
Hence
$$D(\ell_1, k_1, \ldots, k_n) = C(\ell_1, k_1, \ldots, k_n) = k_{1}^{k_1} + \cdots + k_{n}^{k_n}.$$
Example (Benítez, Sarantopouloš, Tonge - 1998)

In the complex Banach space $\ell_1$, define the polynomials $P_1, \ldots, P_n$ by

$$P_j((a_i)_{i \in \mathbb{N}}) = a_j^k.$$
Example (Benítez, Sarantopoulos, Tonge - 1998)

In the complex Banach space $\ell_1$, define the polynomials $P_1, \cdots, P_n$ by $P_j((a_i)_{i\in\mathbb{N}}) = a_j^{k_j}$. Then

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Hence $C(\ell_1, k_1, \cdots, k_n) \leq \frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1+\cdots+k_n}}.$
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Conclusion

$$D(\ell_1, k_1, \ldots, k_n)$$
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In the complex Banach space $\ell_1$, define the polynomials $P_1, \cdots, P_n$ by $P_j((a_i)_{i \in \mathbb{N}}) = a_j^{k_j}$. Then

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Conclusion

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D(\ell_1, k_1, \cdots, k_n) = C(\ell_1, k_1, \cdots, k_n)
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For Hilbert spaces D. Pinasco determined the exact value of $C(H, k_1, \ldots, k_n)$.

Theorem (D. Pinasco - 2012)

For any complex Hilbert space $H$, with $\dim(H) \geq n$,

$C(H, k_1, \ldots, k_n) = \left( k_1 k_1 \cdots k_n n \right)^{\frac{1}{k_1 + \cdots + k_n}}.$

Summarizing...

$C(\ell_1, k_1, \ldots, k_n) = \left( k_1 k_1 \cdots k_n n \right)^{\frac{1}{k_1 + \cdots + k_n}},$

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$C(\ell_2, k_1, \ldots, k_n) = \left( k_1 k_1 \cdots k_n n \right)^{\frac{1}{k_1 + \cdots + k_n}}.$
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$$C(\ell_1, k_1, \cdots, k_n) = \left( k_1 \cdots k_n \left( k_1 + \cdots + k_n \right) \right)^{1/n}$$ and 

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$$C(\ell_1, k_1, \cdots, k_n) = \left( \frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1+\cdots+k_n}} \right)^{\frac{1}{4}}.$$
Previous results

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and

$$C(\ell_2, k_1, \cdots, k_n) = \left( \frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}} \right)^{\frac{1}{2}}.$$
Case $\ell_p$, $1 < p < 2$

Question: What is the value of $C(\ell_p, k_1, \cdots, k_n)$?

Theorem (D. Carando, D. Pinasco, J. T. Rodríguez - 2013)

For the complex Banach space $\ell_p$, with $1 < p < 2$, $C(\ell_p, k_1, \cdots, k_n) = \left(\frac{1}{k_1 + \cdots + k_n}\right)^{\frac{1}{p}}$.

What happens if we consider the example from above in $\ell_p$?

Example: In the complex Banach space $\ell_p$, define the polynomials $P_1, \cdots, P_n$ by $P_j((a_i)_{i \in N}) = a_{k_j}$. Then $\|P_j\| = 1$ and $\|n \prod_{j=1}^n P_j\| = \left(\frac{1}{k_1 + \cdots + k_n}\right)^{\frac{1}{p}}$. The Buenos Aires Lower bound for norms of products of polynomials on $L_p$ spaces Buenos Aires, July 2014
Case $\ell_p, 1 < p < 2$

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**What happens if we consider the example from above in \( \ell_p \)?**

**Example**

In the complex Banach space \( \ell_p \), define the polynomials \( P_1, \cdots, P_n \) by \( P_j((a_i)_{i \in \mathbb{N}}) = a_j^{k_j} \). Then

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\|P_j\| = 1
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**Example**

In the complex Banach space $\ell_p$, define the polynomials $P_1, \cdots, P_n$ by $P_j((a_i)_{i\in\mathbb{N}}) = a_j^{k_j}$. Then

$$\|P_j\| = 1 \text{ and } \left\| \prod_{j=1}^{n} P_j \right\| = \left( \frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1+\cdots+k_n}} \right)^{\frac{1}{p}}.$$
Hence $C(\ell_p, k_1, \cdots, k_n) \leq \left( \frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}} \right)^{\frac{1}{p}}$. 

To see the other inequality we use the following simplifications.

1. Can assume $n = 2$.
2. Instead on working on $\ell_p$, is enough to consider the finite dimensional space $\ell_d$ (with $d$ arbitrarily large).

The first simplification can be made by induction. The second one follows from the fact that for any continuous polynomial $P$: $\ell_p \to C$, $\|P\| = \lim_{d \to \infty} \|P_d\|$, where $P_d((a_1, a_2, \cdots, a_d, 0, 0, \cdots)) = P((a_1, a_2, \cdots, a_d, 0, 0, \cdots))$. 

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Case $\ell_p$, $1 < p < 2$

Hence $C(\ell_p, k_1, \cdots, k_n) \leq \left( \frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}} \right)^{\frac{1}{p}}$.

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2. Instead of working on \( \ell_p, \) is enough to consider the finite-dimensional space \( \ell_d \) (with \( d \) arbitrarily large).

The first simplification can be made by induction. The second one follows from the fact that for any continuous polynomial \( P: \ell_p \to C, \) \( \|P\| = \lim_{d \to \infty} \|P_d\|, \) where \( P_d((a_1, a_2, \cdots, a_d, 0, 0, \cdots)) = P((a_1, a_2, \cdots, a_d)). \)
Hence \( C(\ell_p, k_1, \ldots, k_n) \leq \left( \frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}} \right)^{\frac{1}{p}} \).

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Case $\ell_p$, $1 < p < 2$

Hence $C(\ell_p, k_1, \cdots, k_n) \leq \left( \frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}} \right)^{\frac{1}{p}}$.

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Hence $C(\ell_p, k_1, \cdots, k_n) \leq \left(\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}}\right)^{\frac{1}{p}}$.

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**Simplifications:**

1. Can assume $n = 2$.
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The first simplification can be made by induction. The second one follows from the fact that for any continuous polynomial $P : \ell_p \to \mathbb{C}$

$$
\|P\| = \lim_{d \to \infty} \|P_d\|,
$$

where $P_d((a_1, a_2, \cdots, a_d)) = P((a_1, a_2, \cdots, a_d, 0, 0, \cdots))$. 

Lower bounds for norms of products of polynomials on $L_p$ spaces  

Buenos Aires, July 2014
Case $\ell_p, 1 < p < 2$

Proof of the Theorem: we divide the proof into three cases.

First case: the polynomials $P_1$ and $P_2$ depend on different variables. Just like in the examples above, when the polynomials depend on different variables, we have

$$\|P_1 P_2\| = \left(\frac{k_1}{k_1 + k_2}\right)^{1/p} \|P_1\| \|P_2\|$$

Lower bounds for norms of products of polynomials on $L_p$ spaces

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First case: the polynomials $P_1$ and $P_2$ depend on different variables

Just like in the examples above, when the polynomials depend on different variable, we have

$$
\|P_1 P_2\| = \left( \frac{k_1^{k_1} k_2^{k_2}}{(k_1 + k_2)^{k_1+k_2}} \right)^{\frac{1}{p}} \|P_1\| \|P_2\|
$$
Case $\ell_p$, $1 < p < 2$

Second case: $k_1 = k_2$

For the second case we use the following result due to D. Lewis.

Theorem (D. Lewis - 1978)

For any $d$-dimensional subspace $E \subset L^p$, with $1 \leq p \leq \infty$,

$$d(E,\ell^2) \leq d^{\frac{1}{p} - \frac{1}{2}}.$$

Combining Lewis' result and the constant for Hilbert spaces obtained by Pinasco, we get

$$k_1^{k_1} k_2^{k_2} (k_1 + k_2) \leq \|P_1 P_2\|.$$

We can do the same for $k_1 \neq k_2$, but we would not obtain an optimal constant.
Case $\ell_p$, $1 < p < 2$

Second case: $k_1 = k_2 = k$

Theorem (D. Lewis - 1978)

For any $d$-dimensional subspace $E \subset L^p$, with $1 \leq p \leq \infty$, 

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Combining Lewis’ result and the constant for Hilbert spaces obtained by Pinasco, we get 

$$\left(k_1 + k_2\right) \frac{k_1 + k_2}{k_1 + k_2} \leq \|P_1 P_2\|.$$

We can do the same for $k_1 \neq k_2$, but we would not obtain an optimal constant.
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Theorem (D. Lewis - 1978)

For any $d$-dimensional subspace $E \subset \ell_p$, with $1 \leq p \leq \infty$,

$$d \left( E, \ell_d \right) \leq d \left| 1/p - 1/2 \right|.$$

Combining Lewis' result and the constant for Hilbert spaces obtained by Pinasco, we get

$$\left( k_1 k_1 + k_2 k_2 \right)^k_1 + k_2 \leq \| P_1 P_2 \|.$$

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For the second case we use the following result due to D. Lewis.

**Theorem (D. Lewis - 1978)**

For any \( d \)-dimensional subspace \( E \subseteq L_p \), with \( 1 \leq p \leq \infty \),

\[
d(E, \ell_2^d) \leq d\left|\frac{1}{p} - \frac{1}{2}\right|.
\]

Combining Lewis’ result and the constant for Hilbert spaces obtained by Pinasco, we get

\[
\left( \frac{k_1^{k_1} k_2^{k_2}}{(k_1 + k_2)^{k_1+k_2}} \right)^{\frac{1}{p}} \|P_1\|\|P_2\| \leq \|P_1P_2\|.
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**Theorem (D. Lewis - 1978)**

For any $d$–dimensional subspace $E \subset L_p$, with $1 \leq p \leq \infty$,\[ d(E, \ell_2^d) \leq d^{1/p - 1/2}. \]

Combining Lewis’ result and the constant for Hilbert spaces obtained by Pinasco, we get

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\bigg( \frac{k_1^{k_1} k_2^{k_2}}{(k_1 + k_2)^{k_1+k_2}} \bigg)^{\frac{1}{p}} \|P_1\|\|P_2\| \leq \|P_1P_2\|.
\]

We can do the same for $k_1 \neq k_2$, but we would not obtain an optimal constant.
Third case: any pair of continuous homogeneous polynomials

Suppose that $k_1 > k_2$ and let $S$ be a norm 1 homogeneous polynomial of degree $m$, with $m = k_1 - k_2$, depending on different variables than $P_1$ and $P_2$ (if necessary, we increase the dimension to add a new variable).

$$\|P_1 P_2\| = \|P_1 P_2 S\| \left( (k_1 + k_2) (k_1 + k_2) + m (k_1 + k_2) \right)^{1/p} \geq \|P_1\| \|P_2\| \|S\| \left( (k_1 + k_2) (k_1 + k_2) + m k_1 k_2 + m k_1 k_2 \right)^{1/p} \left( (k_1 + k_2) (k_1 + k_2) + m k_1 k_2 + m k_1 k_2 \right)^{1/p}$$
Third case: any pair of continuous homogeneous polynomials

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Lower bounds for norms of products of polynomials on $L_p$ spaces
Case $\ell_p$, $1 < p < 2$

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$$\|P_1P_2\|$$
Third case: any pair of continuous homogeneous polynomials

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\|P_1 P_2\| = \|P_1 P_2\| \|S\|
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$$\|P_1 P_2\| = \|P_1 P_2\| \|S\|$$

$$= \|P_1 P_2 S\| \left( \frac{(k_1 + k_2 + m)(k_1+k_2)+m}{(k_1 + k_2)(k_1+k_2)m^m} \right)^{\frac{1}{p}}$$
Case $\ell_p$, $1 < p < 2$

Third case: any pair of continuous homogeneous polynomials

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\[
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\]

\[
= \|P_1P_2S\| \left( \frac{((k_1 + k_2) + m)(k_1 + k_2) + m}{(k_1 + k_2)(k_1 + k_2)m} \right)^{\frac{1}{p}},
\]

\[
\geq \|P_1\| \|P_2S\| \left( \frac{((k_1 + k_2) + m)(k_1 + k_2) + m}{(k_1 + k_2)(k_1 + k_2)m^m((k_1 + k_2) + m)^{k_1 + k_2 + m}} \right)^{\frac{1}{p}}.
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Suppose that $k_1 > k_2$ and let $S$ be a norm 1 homogeneous polynomial of degree $m$, with $m = k_1 - k_2$, depending on different variables than $P_1$ and $P_2$ (if necessary, we increase the dimension to add a new variable).

\[
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\[
\|P_1P_2\| \geq \|P_1P_2S\| \left( \frac{(k_1 + k_2) + m}{(k_1 + k_2)(k_1 + k_2)m^m((k_1 + k_2) + m)^{(k_1 + k_2)+m}} \right)^\frac{1}{p} \]

\[
\|P_1P_2\| = \|P_1\| \|P_2\| \|S\| \left( \frac{k_1^k (k_2 + m)^{k_2+m}}{(k_1 + k_2)(k_1 + k_2)m^m} \right)^\frac{1}{p} \left( \frac{k_2^m m^m}{(k_2 + m)^{k_2+m}} \right)^\frac{1}{p} \]
Third case: any pair of continuous homogeneous polynomials

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\geq \|P_1\| \|P_2S\| \left( \frac{(k_1 + k_2)^{(k_1+k_2)+m}k_1^{k_1}(k_2 + m)^{k_2+m}}{(k_1 + k_2)^{(k_1+k_2)}m^m((k_1 + k_2) + m)^{(k_1+k_2)+m}} \right)^{\frac{1}{p}} \\
= \|P_1\| \|P_2\| \|S\| \left( \frac{k_1^{k_1}(k_2 + m)^{k_2+m}}{(k_1 + k_2)^{(k_1+k_2)}m^m} \right)^{\frac{1}{p}} \left( \frac{k_2^m m^m}{(k_2 + m)^{k_2+m}} \right)^{\frac{1}{p}} \\
= \|P_1\| \|P_2\| \left( \frac{k_1^{k_1}k_2^{k_2}}{(k_1 + k_2)^{(k_1+k_2)}} \right)^{\frac{1}{p}}
Case $\ell_p$, $p > 2$

Recall the result from Lewis For any $d$-dimensional subspace $E \subset L^p$, with $1 \leq p \leq \infty$, 

$$d(\mathcal{E}, \ell^d_2) \leq d^{1/p - 1/2}.$$ 

For $p > 2$, $\left|\left|\frac{1}{p} \right| - \frac{1}{2}\right| \neq \frac{1}{p} - \frac{1}{2}$.

Then, we get

$$C(\ell^p_k, \cdots, k_n) \geq C(\ell^2_k, \cdots, k_n) \left(\frac{n}{k_1} + \cdots + k_n\right)^{1/2 - 1/p}$$

Lower bounds for norms of products of polynomials on $L_p$ spaces

Buenos Aires, July 2014
What happens if we follow the same reasoning for $p > 2$?
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What happens if we follow the same reasoning for \( p > 2 \)? Recall the result from Lewis

For any \( d \)-dimensional subspace \( E \subset L_p \), with \( 1 \leq p \leq \infty \),

\[
d(E, \ell^d_2) \leq d^{\frac{1}{p} - \frac{1}{2}}.
\]

For \( p > 2 \), \( \left| \frac{1}{p} - \frac{1}{2} \right| \neq \frac{1}{p} - \frac{1}{2} \). Then, we get

\[
C(\ell_p, k_1, \ldots, k_n) \geq \frac{C(\ell_2, k_1, \ldots, k_n)}{(n^{k_1 + \cdots + k_n})^{\frac{1}{2} - \frac{1}{p}}}
\]
Finite dimensional case

What can be said about $C(X, k_1, \cdots, k_n)$ for arbitrary $n$ and $d$-dimensional spaces (with $d$ fixed)?

To face this question we use the fact that

$$C(X, k_1, \cdots, k_n) = \inf \left\{ \left\| \prod_{j=1}^n P_j \right\| : P_j \in P(k_j X), \left\| P_j \right\| = 1 \right\}$$

Let $P_1, \cdots, P_n$ be polynomials as above and let $\mu$ be any probability measure in $K = S_X$ or $B_X$, then

$$\left\| \prod_{j=1}^n P_j \right\| = \exp \left\{ \max_{x \in K} \ln \left( \left\| P_j(x) \right\| \right) \right\} \geq \exp \left\{ \int K \sum_{j=1}^n \ln \left( \left| P_j(x) \right| \right) d\mu(x) \right\}$$

Lower bounds for norms of products of polynomials on $L_p$ spaces

Buenos Aires, July 2014
Finite dimensional case

Question
What can be said about $C(X, k_1, \cdots, k_n)$ for arbitrary $n$ and $d$–dimensional spaces.

Lower bounds for norms of products of polynomials on $L_p$ spaces

Buenos Aires, July 2014
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Lower bounds for norms of products of polynomials on $L_p$ spaces

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$$\left\| \prod_{j=1}^{n} P_j \right\| = \exp \left\{ \ln \left( \max_{x \in K} \prod_{j=1}^{n} |P_j(x)| \right) \right\}$$
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= \exp \left\{ \max_{x \in K} \sum_{j=1}^{n} \ln (|P_j(x)|) \right\} \geq \exp \left\{ \int_{K} \sum_{j=1}^{n} \ln (|P_j(x)|) \, d\mu(x) \right\}
\]
Finite dimensional case

Then

\[ C(X, k_1, \cdots, k_n) \geq \inf_{\|P_j\|=1, \deg(P)=k_j} \left\{ \exp \left( \sum_j \int_K \ln |P_j(x)| d\mu(x) \right) \right\} . \]
Finite dimensional case

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Thus, if we find lower bounds for \( \int_K \ln |P_j(x)| d\mu(x) \) (depending only on \( k_j \)) we obtain a lower bound for \( C(X, k_1, \cdots, k_n) \).
Finite dimensional case

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Theorem (García-Vázquez, Villa - 1999)

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Finite dimensional case

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Theorem (García-Vázquez, Villa - 1999)

\[ C(\mathbb{R}^d, 1, \cdots, 1) \geq \exp\{-nL(d, \mathbb{R})\} \text{ with } \]

\[ n-\text{times} \]

\[ L(d, \mathbb{R}) = \begin{cases} 
\ln(2) + \sum_{m=1}^{d-2} \frac{1}{2m} & \text{if } d \equiv 0(2) \\
\ln(2) + \sum_{m=1}^{d-3} \frac{1}{2m+1} & \text{if } d \equiv 1(2)
\end{cases} \]
Finite dimensional case

Then

\[ C(X, k_1, \cdots, k_n) \geq \inf_{\|P\|=1, \deg(P)=k_j} \left\{ \exp \left( \sum_j \int_K \ln |P_j(x)| d\mu(x) \right) \right\}. \]

Thus, if we find lower bounds for \( \int_K \ln |P_j(x)| d\mu(x) \) (depending only on \( k_j \)) we obtain a lower bound for \( C(X, k_1, \cdots, k_n) \).

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Moreover

\[ \lim_{n \to \infty} C(\mathbb{R}^d, 1, \cdots, 1)^{\frac{1}{n}} = \exp\{-L(d, \mathbb{R})\} \]
Finite dimensional case


\[
C(d, 1, \cdots, 1) \geq \exp\{\frac{1}{2} d - \sum_{m=1}^{\infty} \frac{1}{m}\}
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Finite dimensional case


\[ C(\mathbb{C}^d, 1, \ldots, 1) \geq \exp\{-nL(d, \mathbb{C})\} \text{ with} \]

\[ L(d, \mathbb{C}) = \frac{1}{2} \sum_{m=1}^{d-1} \frac{1}{m} \]

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Moreover

\[ \lim_{n \to \infty} \frac{C(\mathbb{C}^d, 1, \cdots, 1)}{n} = \exp\{-L(d, \mathbb{C})\} \]
Finite dimensional case

Let $X = (\mathbb{R}^d, \| - \|)$, take $K = B_X$ and $\mu = \lambda$ the normalized Lebesgue measure.
Finite dimensional case

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Now we use the following Corollary of a Remez type inequality for several variables.
Finite dimensional case

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Now we use the following Corollary of a Remez type inequality for several variables.

**Corollary (J. Brudnyi and I. Ganzburg - 1993)**

Let $P : \mathbb{R}^d \to \mathbb{R}$ be a continuous polynomial of degree $k$ and norm 1, then

\[
\lambda(\{x \in K : |P(x)| \leq t\}) \leq 4d \left( \frac{t}{2} \right)^{\frac{1}{k}}
\]
Finite dimensional case

We then obtain:

\[ \int_{B_x} \ln(|P(x)|)d\lambda(x) \geq \ln \left(\frac{2}{(4d)^k}\right) - k \]
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**Lower bound for** $C(X, k_1, \cdots, k_n)$

\[ C(X, k_1, \cdots, k_n) \geq \exp \left\{ \sum_j \ln \left( \frac{2}{(4d)^{k_j}} \right) - k_j \right\} \]
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Lower bound for \( C(X, k_1, \ldots, k_n) \)

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C(X, k_1, \ldots, k_n) \geq \exp \left\{ \sum_j \ln \left( \frac{2}{(4d)^{k_j}} \right) - k_j \right\} \\
= \prod_j \frac{2}{(4d)^{k_j}} \frac{1}{e^{k_j}} = \frac{2^n}{(4de)^{\sum_{j=1}^n k_j}}
\]
Question
What about an upper bound for \( C(X, k_1, \ldots, k_n) \) or estimates for
\[
\lim_{n \to \infty} C(X, k_1, \ldots, k_n) \frac{1}{\sum k_j}
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Question

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In the linear case we can estimate the limit for the space $\ell_p^d$.
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Question

What about an upper bound for \( C(X, k_1, \ldots, k_n) \) or estimates for \( \lim_{n \to \infty} C(X, k_1, \ldots, k_n) \sum_{j}^{1} k_j \)?

In the linear case we can estimate the limit for the space \( \ell^d_p \).

\[
\lim_{n \to \infty} C(\ell^d_p, 1, \ldots, 1)^{\frac{1}{n}} \leq \exp \{ -L(\mathbb{K}, d) \} \| x_0 \|^2 \left( \frac{\text{vol}(B_{\ell^d_{p'}})}{\text{vol}(B_{\ell^d_2})} \right)^{\frac{1}{d}}
\]

where \( \frac{1}{p} + \frac{1}{p'} = 1 \) and \( x_0 \) is some point of \( S_X \).
Finite dimensional case

Sketch of the proof

Take in $\mathcal{S}X^*$ any probability measure $\mu$. Using a probabilistic tools like the Law of Large Numbers construct a sequence \{\phi_j\} \subseteq \mathcal{S}X^* such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \ln(|\phi_j(x_n)|) \leq \int \mathcal{S}X^* \ln(|\phi(x_0)|) \, d\mu(\phi)$$

where $x_n \in \mathcal{S}X$ is such that $\|\prod_{j=1}^{n} \phi_j\| = |\prod_{j=1}^{n} \phi_j(x_n)|$ and $x_0$ is some accumulation point of the sequence \{x_n\}.

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$$\left\| \prod_{j=1}^{n} \phi_j \right\| \leq \exp \left\{ \frac{1}{n} \sum_{j=1}^{n} \ln(|\phi_j(x_n)|) \right\}.$$
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Lower bounds for norms of products of polynomials on $L_p$ spaces

Buenos Aires, July 2014
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Taking upper limit we obtain

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\lim_{n \to \infty} C(X, 1, \cdots, 1)^{\frac{1}{n}} \leq \exp \left\{ -L(K, d) \right\} \|x_0\|_2^2 \left( \frac{\text{vol}(B_{\ell_p^d})}{\text{vol}(B_{\ell_2^d})} \right)^{\frac{1}{d}}
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Thanks!