Lower bounds for norms of products of polynomials on L_p spaces

Jorge Tomás Rodríguez
Joint work with Daniel Carando and Damián Pinasco

Buenos Aires , July 2014

Objective: find lower bounds for the norm of the product of scalar polynomials on Banach space E.

Objective: find lower bounds for the norm of the product of scalar polynomials on Banach space E. Where the norm of a polynomial $P : E \to \mathbb{K}$ is the usual

$$\|P\| = \sup_{\|x\|=1} |P(x)|$$

Objective: find lower bounds for the norm of the product of scalar polynomials on Banach space E. Where the norm of a polynomial $P : E \to \mathbb{K}$ is the usual

$$\|P\| = \sup_{\|x\|=1} |P(x)|$$

We study this problem for:

Objective: find lower bounds for the norm of the product of scalar polynomials on Banach space E. Where the norm of a polynomial $P : E \to \mathbb{K}$ is the usual

$$\|P\| = \sup_{\|x\|=1} |P(x)|$$

We study this problem for:

A) Continuous homogeneous polynomials.

Objective: find lower bounds for the norm of the product of scalar polynomials on Banach space E. Where the norm of a polynomial $P : E \to \mathbb{K}$ is the usual

$$\|P\| = \sup_{\|x\|=1} |P(x)|$$

We study this problem for:

A) Continuous homogeneous polynomials.

B) Continuous polynomials.

Problems

Problem **A**

Given $k_1, \cdots, k_n \in \mathbb{N}$, find the optimal constant $C(E, k_1, \cdots, k_n)$, such that for every set of continuous homogeneous polynomials $P_1, \cdots, P_n : E \to \mathbb{K}$, of degrees $k_1, \cdots, k_n$, the next inequality holds

$$C(E, k_1, \cdots, k_n) \prod_{j=1}^{n} \|P_j\| \leq \left\| \prod_{j=1}^{n} P_j \right\|$$

Problem A

Given $k_1, \cdots, k_n \in \mathbb{N}$, find the optimal constant $C(E, k_1, \cdots, k_n)$, such that for every set of continuous homogeneous polynomials $P_1, \cdots, P_n : E \to \mathbb{K}$, of degrees $k_1, \cdots, k_n$, the next inequality holds

$$C(E, k_1, \cdots, k_n) \prod_{j=1}^{n} \|P_j\| \leq \left\| \prod_{j=1}^{n} P_j \right\|$$

Problem B

Given $k_1, \cdots, k_n \in \mathbb{N}$, find the optimal constant $D(E, k_1, \cdots, k_n)$, such that for every set of (not necessarily homogeneous) continuous polynomials $P_1, \cdots, P_n : E \to \mathbb{K}$, of degrees $k_1, \cdots, k_n$, the next inequality holds

$$D(E, k_1, \cdots, k_n) \prod_{j=1}^{n} \|P_j\| \leq \left\| \prod_{j=1}^{n} P_j \right\|$$

Previous results

C. Benítez, Y. Sarantopoulos and A. Tonge found a lower bound for this constants

C. Benítez, Y. Sarantopoulos and A. Tonge found a lower bound for this constants

Theorem (Benítez, Sarantopoulos, Tonge - 1998)

For any complex Banach space E

$$D(E, k_1, \cdots, k_n) \geq \frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}}.$$

C. Benítez, Y. Sarantopoulos and A. Tonge found a lower bound for this constants

Theorem (Benítez, Sarantopoulos, Tonge - 1998)

For any complex Banach space E

$$D(E, k_1, \cdots, k_n) \geq \frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}}.$$

As an inmediate consecuence, for any complex Banach space E

$$C(E, k_1, \cdots, k_n) \geq \frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}}.$$

C. Benítez, Y. Sarantopoulos and A. Tonge found a lower bound for this constants

Theorem (Benítez, Sarantopoulos, Tonge - 1998)

For any complex Banach space E

$$D(E, k_1, \cdots, k_n) \geq \frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}}.$$

As an inmediate consecuence, for any complex Banach space E

$$C(E, k_1, \cdots, k_n) \geq \frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}}.$$

These bounds are optimal.

Example (Benítez, Sarantopoulos, Tonge - 1998)

Example (Benítez, Sarantopoulos, Tonge - 1998)

In the complex Banach space ℓ_1, define the polynomials $P_1, \cdots, P_n$ by $P_j((a_i)_{i\in\mathbb{N}}) = a_j^{k_j}$.

Example (Benítez, Sarantopoulos, Tonge - 1998)

In the complex Banach space ℓ_1, define the polynomials $P_1, \cdots, P_n$ by $P_j((a_i)_{i\in\mathbb{N}}) = a_j^{k_j}$. Then

$$\|P_j\| = 1$$

Example (Benítez, Sarantopoulos, Tonge - 1998)

In the complex Banach space ℓ_1, define the polynomials $P_1, \cdots, P_n$ by $P_j((a_i)_{i\in\mathbb{N}}) = a_j^{k_j}$. Then

$$\|P_j\| = 1 \text{ and } \left\|\prod_{j=1}^{n} P_j\right\| = \frac{k_1^{k_1}\cdots k_n^{k_n}}{(k_1+\cdots+k_n)^{k_1+\cdots+k_n}}.$$

Example (Benítez, Sarantopoulos, Tonge - 1998)

In the complex Banach space ℓ_1, define the polynomials $P_1, \cdots, P_n$ by $P_j((a_i)_{i\in\mathbb{N}}) = a_j^{k_j}$. Then

$$\|P_j\| = 1 \text{ and } \left\|\prod_{j=1}^{n} P_j\right\| = \frac{k_1^{k_1}\cdots k_n^{k_n}}{(k_1+\cdots+k_n)^{k_1+\cdots+k_n}}.$$

Therefore

$$\frac{k_1^{k_1}\cdots k_n^{k_n}}{(k_1+\cdots+k_n)^{k_1+\cdots+k_n}} \prod_{j=1}^{n} \|P_j\| = \left\|\prod_{j=1}^{n} P_j\right\|.$$

Example (Benítez, Sarantopoulos, Tonge - 1998)

In the complex Banach space ℓ_1, define the polynomials $P_1, \cdots, P_n$ by $P_j((a_i)_{i\in\mathbb{N}}) = a_j^{k_j}$. Then

$$\|P_j\| = 1 \text{ and } \left\| \prod_{j=1}^{n} P_j \right\| = \frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1+\cdots+k_n}}.$$

Therefore

$$\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1+\cdots+k_n}} \prod_{j=1}^{n} \|P_j\| = \left\| \prod_{j=1}^{n} P_j \right\|.$$

Hence $C(\ell_1, k_1, \cdots, k_n) \leq \frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1+\cdots+k_n)^{k_1+\cdots+k_n}}$.

Example (Benítez, Sarantopoulos, Tonge - 1998)

In the complex Banach space ℓ_1, define the polynomials $P_1, \cdots, P_n$ by $P_j((a_i)_{i\in\mathbb{N}}) = a_j^{k_j}$. Then

$$\|P_j\| = 1 \text{ and } \left\|\prod_{j=1}^{n} P_j\right\| = \frac{k_1^{k_1}\cdots k_n^{k_n}}{(k_1+\cdots+k_n)^{k_1+\cdots+k_n}}.$$

Therefore

$$\frac{k_1^{k_1}\cdots k_n^{k_n}}{(k_1+\cdots+k_n)^{k_1+\cdots+k_n}}\prod_{j=1}^{n}\|P_j\| = \left\|\prod_{j=1}^{n} P_j\right\|.$$

Hence $C(\ell_1, k_1, \cdots, k_n) \leq \frac{k_1^{k_1}\cdots k_n^{k_n}}{(k_1+\cdots+k_n)^{k_1+\cdots+k_n}}$.

Conclusion

$$D(\ell_1, k_1, \cdots, k_n)$$

Example (Benítez, Sarantopoulos, Tonge - 1998)

In the complex Banach space ℓ_1, define the polynomials $P_1, \cdots, P_n$ by $P_j((a_i)_{i\in\mathbb{N}}) = a_j^{k_j}$. Then

$$\|P_j\| = 1 \text{ and } \left\| \prod_{j=1}^{n} P_j \right\| = \frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1+\cdots+k_n}}.$$

Therefore

$$\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1+\cdots+k_n}} \prod_{j=1}^{n} \|P_j\| = \left\| \prod_{j=1}^{n} P_j \right\|.$$

Hence $C(\ell_1, k_1, \cdots, k_n) \leq \frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1+\cdots+k_n)^{k_1+\cdots+k_n}}$.

Conclusion

$$D(\ell_1, k_1, \cdots, k_n) = C(\ell_1, k_1, \cdots, k_n)$$

Example (Benítez, Sarantopoulos, Tonge - 1998)

In the complex Banach space ℓ_1, define the polynomials $P_1, \cdots, P_n$ by $P_j((a_i)_{i\in\mathbb{N}}) = a_j^{k_j}$. Then

$$\|P_j\| = 1 \text{ and } \left\|\prod_{j=1}^{n} P_j\right\| = \frac{k_1^{k_1}\cdots k_n^{k_n}}{(k_1+\cdots+k_n)^{k_1+\cdots+k_n}}.$$

Therefore

$$\frac{k_1^{k_1}\cdots k_n^{k_n}}{(k_1+\cdots+k_n)^{k_1+\cdots+k_n}}\prod_{j=1}^{n}\|P_j\| = \left\|\prod_{j=1}^{n} P_j\right\|.$$

Hence $C(\ell_1, k_1, \cdots, k_n) \leq \frac{k_1^{k_1}\cdots k_n^{k_n}}{(k_1+\cdots+k_n)^{k_1+\cdots+k_n}}$.

Conclusion

$$D(\ell_1, k_1, \cdots, k_n) = C(\ell_1, k_1, \cdots, k_n) = \frac{k_1^{k_1}\cdots k_n^{k_n}}{(k_1+\cdots+k_n)^{k_1+\cdots+k_n}}$$

Previous results

For Hilbert spaces D. Pinasco determined the exact value of $C(H, k_1, \cdots, k_n)$.

For Hilbert spaces D. Pinasco determined the exact value of $C(H, k_1, \cdots, k_n)$.

Theorem (D. Pinasco - 2012)

For any complex Hilbert space H, with $dim(H) \geq n$,

$$C(H, k_1, \cdots, k_n) = \left(\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}} \right)^{\frac{1}{2}}.$$

For Hilbert spaces D. Pinasco determined the exact value of $C(H, k_1, \cdots, k_n)$.

Theorem (D. Pinasco - 2012)

For any complex Hilbert space H, with $dim(H) \geq n$,

$$C(H, k_1, \cdots, k_n) = \left(\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}} \right)^{\frac{1}{2}}.$$

Summarizing...

For Hilbert spaces D. Pinasco determined the exact value of $C(H, k_1, \cdots, k_n)$.

Theorem (D. Pinasco - 2012)

For any complex Hilbert space H, with $dim(H) \geq n$,

$$C(H, k_1, \cdots, k_n) = \left(\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}} \right)^{\frac{1}{2}}.$$

Summarizing...

$$\boxed{C(\ell_1, k_1, \cdots, k_n) = \left(\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}} \right)^{\frac{1}{\mathbf{1}}}}$$

For Hilbert spaces D. Pinasco determined the exact value of $C(H, k_1, \cdots, k_n)$.

Theorem (D. Pinasco - 2012)

For any complex Hilbert space H, with $dim(H) \geq n$,

$$C(H, k_1, \cdots, k_n) = \left(\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}} \right)^{\frac{1}{2}}.$$

Summarizing...

$$\boxed{C(\ell_1, k_1, \cdots, k_n) = \left(\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}} \right)^{\frac{1}{\mathbf{1}}}}$$

and

$$\boxed{C(\ell_2, k_1, \cdots, k_n) = \left(\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}} \right)^{\frac{1}{\mathbf{2}}}}$$

Case ℓ_p, $1 < p < 2$

Question

What is the value of $C(\ell_p, k_1, \cdots, k_n)$?

Question

What is the value of $C(\ell_p, k_1, \cdots, k_n)$?

Theorem (D. Carando, D. Pinasco, J. T. Rodríguez - 2013)

For the complex Banach space ℓ_p, with $1 < p < 2$,

$$C(\ell_p, k_1, \cdots, k_n) = \left(\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}} \right)^{\frac{1}{p}} .$$

Question

What is the value of $C(\ell_p, k_1, \cdots, k_n)$?

Theorem (D. Carando, D. Pinasco, J. T. Rodríguez - 2013)

For the complex Banach space ℓ_p, with $1 < p < 2$,

$$C(\ell_p, k_1, \cdots, k_n) = \left(\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}} \right)^{\frac{1}{p}}.$$

What happens if we consider the example from above in ℓ_p?

Question

What is the value of $C(\ell_p, k_1, \cdots, k_n)$?

Theorem (D. Carando, D. Pinasco, J. T. Rodríguez - 2013)

For the complex Banach space ℓ_p, with $1 < p < 2$,

$$C(\ell_p, k_1, \cdots, k_n) = \left(\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}} \right)^{\frac{1}{p}}.$$

What happens if we consider the example from above in ℓ_p?

Example

In the complex Banach space ℓ_p, define the polynomials $P_1, \cdots, P_n$ by $P_j((a_i)_{i \in \mathbb{N}}) = a_j^{k_j}$. Then

$$\|P_j\| = 1$$

Question

What is the value of $C(\ell_p, k_1, \cdots, k_n)$?

Theorem (D. Carando, D. Pinasco, J. T. Rodríguez - 2013)

For the complex Banach space ℓ_p, with $1 < p < 2$,

$$C(\ell_p, k_1, \cdots, k_n) = \left(\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}} \right)^{\frac{1}{p}} .$$

What happens if we consider the example from above in ℓ_p?

Example

In the complex Banach space ℓ_p, define the polynomials $P_1, \cdots, P_n$ by $P_j((a_i)_{i \in \mathbb{N}}) = a_j^{k_j}$. Then

$$\|P_j\| = 1 \text{ and } \left\| \prod_{j=1}^{n} P_j \right\| = \left(\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}} \right)^{\frac{1}{p}} .$$

Hence $C(\ell_p, k_1, \cdots, k_n) \leq \left(\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}} \right)^{\frac{1}{p}}$.

Hence $C(\ell_p, k_1, \cdots, k_n) \leq \left(\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}} \right)^{\frac{1}{p}}$.

To see the other inequality we use the following simplifications.

Hence $C(\ell_p, k_1, \cdots, k_n) \leq \left(\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}} \right)^{\frac{1}{p}}$.

To see the other inequality we use the following simplifications.

Simplifications:

Hence $C(\ell_p, k_1, \cdots, k_n) \leq \left(\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}} \right)^{\frac{1}{p}}$.

To see the other inequality we use the following simplifications.

Simplifications:

1. Can assume $n = 2$.

Hence $C(\ell_p, k_1, \cdots, k_n) \leq \left(\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1+\cdots+k_n)^{k_1+\cdots+k_n}} \right)^{\frac{1}{p}}$.

To see the other inequality we use the following simplifications.

Simplifications:

1. Can assume $n = 2$.
2. Instead on working on ℓ_p, is enough to consider the finite dimensional space ℓ_p^d

Hence $C(\ell_p, k_1, \cdots, k_n) \leq \left(\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}} \right)^{\frac{1}{p}}$.

To see the other inequality we use the following simplifications.

Simplifications:

1. Can assume $n = 2$.
2. Instead on working on ℓ_p, is enough to consider the finite dimensional space ℓ_p^d (with d arbitrarily large).

Hence $C(\ell_p, k_1, \cdots, k_n) \leq \left(\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}} \right)^{\frac{1}{p}}$.

To see the other inequality we use the following simplifications.

Simplifications:

1. Can assume $n = 2$.
2. Instead on working on ℓ_p, is enough to consider the finite dimensional space ℓ_p^d (with d arbitrarily large).

The first simplification can be made by induction.

Hence $C(\ell_p, k_1, \cdots, k_n) \leq \left(\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}} \right)^{\frac{1}{p}}$.

To see the other inequality we use the following simplifications.

Simplifications:

1. Can assume $n = 2$.
2. Instead on working on ℓ_p, is enough to consider the finite dimensional space ℓ_p^d (with d arbitrarily large).

The first simplification can be made by induction. The second one follows from the fact that for any continuous polynomial $P : \ell_p \to \mathbb{C}$

$$\|P\| = \lim_{d \to \infty} \|P_d\|,$$

where $P_d((a_1, a_2, \cdots, a_d)) = P((a_1, a_2, \cdots, a_d, 0, 0, \cdots))$.

Case ℓ_p, $1 < p < 2$

Proof of the Theorem: we divide the proof into three cases.

Proof of the Theorem: we divide the proof into three cases.

First case: the polynomials P_1 and P_2 depend on different variables

Proof of the Theorem: we divide the proof into three cases.

First case: the polynomials P_1 and P_2 depend on different variables

Just like in the examples above, when the polynomials depends on different variable,

Proof of the Theorem: we divide the proof into three cases.

First case: the polynomials P_1 and P_2 depend on different variables

Just like in the examples above, when the polynomials depends on different variable, we have

$$\|P_1 P_2\| = \left(\frac{k_1^{k_1} k_2^{k_2}}{(k_1 + k_2)^{k_1+k_2}} \right)^{\frac{1}{p}} \|P_1\| \|P_2\|$$

Case ℓ_p, $1 < p < 2$

Second case: $k_1 = k_2 = k$

Second case: $k_1 = k_2 = k$

For the second case we use the following resuelt due to D. Lewis.

Second case: $k_1 = k_2 = k$

For the second case we use the following resuelt due to D. Lewis.

Theorem (D. Lewis - 1978)

For any $d-$dimensional subspace $E \subset L_p$, with $1 \leq p \leq \infty$,

$$d(E, \ell_2^d) \leq d^{\left|\frac{1}{p} - \frac{1}{2}\right|}.$$

Second case: $k_1 = k_2 = k$

For the second case we use the following resuelt due to D. Lewis.

Theorem (D. Lewis - 1978)

For any $d-$dimensional subspace $E \subset L_p$, with $1 \leq p \leq \infty$,

$$d(E, \ell_2^d) \leq d^{\left|\frac{1}{p} - \frac{1}{2}\right|}.$$

Combining Lewis' result and the constant for Hilbert spaces obtained by Pinasco, we get

$$\left(\frac{k_1^{k_1} k_2^{k_2}}{(k_1 + k_2)^{k_1 + k_2}} \right)^{\frac{1}{p}} \|P_1\| \|P_2\| \leq \|P_1 P_2\|.$$

Second case: $k_1 = k_2 = k$

For the second case we use the following resuelt due to D. Lewis.

Theorem (D. Lewis - 1978)

For any $d-$dimensional subspace $E \subset L_p$, with $1 \leq p \leq \infty$,

$$d(E, \ell_2^d) \leq d^{\left|\frac{1}{p} - \frac{1}{2}\right|}.$$

Combining Lewis' result and the constant for Hilbert spaces obtained by Pinasco, we get

$$\left(\frac{k_1^{k_1} k_2^{k_2}}{(k_1 + k_2)^{k_1 + k_2}} \right)^{\frac{1}{p}} \|P_1\| \|P_2\| \leq \|P_1 P_2\|.$$

We can do the same for $k_1 \neq k_2$, but we would not obtain an optimal constant.

Third case: any pair of continuous homogeneous polynomials

Third case: any pair of continuous homogeneous polynomials

Suppose that $k_1 > k_2$

Third case: any pair of continuous homogeneous polynomials

Suppose that $k_1 > k_2$ and let S be a norm 1 homogeneous polynomial of degree m, with $m = k_1 - k_2$, depending on different variables than P_1 and P_2

Third case: any pair of continuous homogeneous polynomials

Suppose that $k_1 > k_2$ and let S be a norm 1 homogeneous polynomial of degree m, with $m = k_1 - k_2$, depending on different variables than P_1 and P_2 (if necessary, we increase the dimension to add a new variable).

Third case: any pair of continuous homogeneous polynomials

Suppose that $k_1 > k_2$ and let S be a norm 1 homogeneous polynomial of degree m, with $m = k_1 - k_2$, depending on different variables than P_1 and P_2 (if necessary, we increase the dimension to add a new variable).

$\|P_1 P_2\|$

Third case: any pair of continuous homogeneous polynomials

Suppose that $k_1 > k_2$ and let S be a norm 1 homogeneous polynomial of degree m, with $m = k_1 - k_2$, depending on different variables than P_1 and P_2 (if necessary, we increase the dimension to add a new variable).

$$\|P_1 P_2\| = \|P_1 P_2\| \, \|S\|$$

Third case: any pair of continuous homogeneous polynomials

Suppose that $k_1 > k_2$ and let S be a norm 1 homogeneous polynomial of degree m, with $m = k_1 - k_2$, depending on different variables than P_1 and P_2 (if necessary, we increase the dimension to add a new variable).

$$\begin{aligned} \|P_1 P_2\| &= \|P_1 P_2\| \, \|S\| \\ &= \|P_1 P_2 S\| \left(\frac{((k_1+k_2)+m)^{(k_1+k_2)+m}}{(k_1+k_2)^{(k_1+k_2)} m^m} \right)^{\frac{1}{p}} \end{aligned}$$

Third case: any pair of continuous homogeneous polynomials

Suppose that $k_1 > k_2$ and let S be a norm 1 homogeneous polynomial of degree m, with $m = k_1 - k_2$, depending on different variables than P_1 and P_2 (if necessary, we increase the dimension to add a new variable).

$$
\begin{aligned}
\|P_1 P_2\| &= \|P_1 P_2\| \, \|S\| \\
&= \|P_1 P_2 S\| \left(\frac{((k_1+k_2)+m)^{(k_1+k_2)+m}}{(k_1+k_2)^{(k_1+k_2)} m^m} \right)^{\frac{1}{p}} \\
&\geq \|P_1\| \, \|P_2 S\| \left(\frac{((k_1+k_2)+m)^{(k_1+k_2)+m} k_1^{k_1} (k_2+m)^{k_2+m}}{(k_1+k_2)^{(k_1+k_2)} m^m ((k_1+k_2)+m)^{(k_1+k_2)+m}} \right)^{\frac{1}{p}}
\end{aligned}
$$

Third case: any pair of continuous homogeneous polynomials

Suppose that $k_1 > k_2$ and let S be a norm 1 homogeneous polynomial of degree m, with $m = k_1 - k_2$, depending on different variables than P_1 and P_2 (if necessary, we increase the dimension to add a new variable).

$$\begin{aligned}
\|P_1P_2\| &= \|P_1P_2\|\,\|S\| \\
&= \|P_1P_2S\| \left(\frac{((k_1+k_2)+m)^{(k_1+k_2)+m}}{(k_1+k_2)^{(k_1+k_2)} m^m} \right)^{\frac{1}{p}} \\
&\geq \|P_1\|\,\|P_2S\| \left(\frac{((k_1+k_2)+m)^{(k_1+k_2)+m} k_1^{k_1} (k_2+m)^{k_2+m}}{(k_1+k_2)^{(k_1+k_2)} m^m ((k_1+k_2)+m)^{(k_1+k_2)+m}} \right)^{\frac{1}{p}} \\
&= \|P_1\|\,\|P_2\|\|S\| \left(\frac{k_1^{k_1} (k_2+m)^{k_2+m}}{(k_1+k_2)^{(k_1+k_2)} m^m} \right)^{\frac{1}{p}} \left(\frac{k_2^{k_2} m^m}{(k_2+m)^{k_2+m}} \right)^{\frac{1}{p}}
\end{aligned}$$

Third case: any pair of continuous homogeneous polynomials

Suppose that $k_1 > k_2$ and let S be a norm 1 homogeneous polynomial of degree m, with $m = k_1 - k_2$, depending on different variables than P_1 and P_2 (if necessary, we increase the dimension to add a new variable).

$$\begin{aligned}
\|P_1P_2\| &= \|P_1P_2\|\,\|S\| \\
&= \|P_1P_2S\| \left(\frac{((k_1+k_2)+m)^{(k_1+k_2)+m}}{(k_1+k_2)^{(k_1+k_2)} m^m} \right)^{\frac{1}{p}} \\
&\geq \|P_1\|\,\|P_2S\| \left(\frac{((k_1+k_2)+m)^{(k_1+k_2)+m} k_1^{k_1} (k_2+m)^{k_2+m}}{(k_1+k_2)^{(k_1+k_2)} m^m ((k_1+k_2)+m)^{(k_1+k_2)+m}} \right)^{\frac{1}{p}} \\
&= \|P_1\|\,\|P_2\|\|S\| \left(\frac{k_1^{k_1} (k_2+m)^{k_2+m}}{(k_1+k_2)^{(k_1+k_2)} m^m} \right)^{\frac{1}{p}} \left(\frac{k_2^{k_2} m^m}{(k_2+m)^{k_2+m}} \right)^{\frac{1}{p}} \\
&= \|P_1\|\,\|P_2\| \left(\frac{k_1^{k_1} k_2^{k_2}}{(k_1+k_2)^{k_1+k_2}} \right)^{\frac{1}{p}}
\end{aligned}$$

Case ℓ_p, $p > 2$

What happens if we follow the same reasoning for $p > 2$?

What happens if we follow the same reasoning for $p > 2$? Recall the result from Lewis

For any $d-$dimensional subspace $E \subset L_p$, with $1 \leq p \leq \infty$,

$$d(E, \ell_2^d) \leq d^{\left|\frac{1}{p} - \frac{1}{2}\right|}.$$

What happens if we follow the same reasoning for $p > 2$? Recall the result from Lewis

For any $d-$dimensional subspace $E \subset L_p$, with $1 \leq p \leq \infty$,

$$d(E, \ell_2^d) \leq d^{\left|\frac{1}{p} - \frac{1}{2}\right|}.$$

For $p > 2$, $\left|\frac{1}{p} - \frac{1}{2}\right| \neq \frac{1}{p} - \frac{1}{2}$.

What happens if we follow the same reasoning for $p > 2$? Recall the result from Lewis

For any $d-$dimensional subspace $E \subset L_p$, with $1 \leq p \leq \infty$,

$$d(E, \ell_2^d) \leq d^{\left|\frac{1}{p} - \frac{1}{2}\right|}.$$

For $p > 2$, $\left|\frac{1}{p} - \frac{1}{2}\right| \neq \frac{1}{p} - \frac{1}{2}$. Then, we get

$$\boxed{C(\ell_p, k_1, \cdots, k_n) \geq \frac{C(\ell_2, k_1, \cdots, k_n)}{(n^{k_1 + \cdots + k_n})^{\frac{1}{2} - \frac{1}{p}}}}$$

Finite dimensional case

Question

What can be said about $C(X, k_1, \cdots, k_n)$ for arbitrary n and $d-$dimensional spaces

Question

What can be said about $C(X, k_1, \cdots, k_n)$ for arbitrary n and $d-$dimensional spaces (with d fixed)?

Question

What can be said about $C(X, k_1, \cdots, k_n)$ for arbitrary n and $d-$dimensional spaces (with d fixed)?

To face this question we use the fact that

$$C(X, k_1, \cdots, k_n) = \inf \left\{ \left\| \prod_{j=1}^{n} P_j \right\| : P_j \in \mathscr{P}({}^{k_j}X), \|P_j\| = 1 \right\}$$

Question

What can be said about $C(X, k_1, \cdots, k_n)$ for arbitrary n and $d-$dimensional spaces (with d fixed)?

To face this question we use the fact that

$$C(X, k_1, \cdots, k_n) = \inf \left\{ \left\| \prod_{j=1}^{n} P_j \right\| : P_j \in \mathscr{P}(^{k_j}X), \|P_j\| = 1 \right\}$$

Let $P_1, \cdots, P_n$ be polynomials as above and let μ be any probability measure in $K = S_X$ or B_X,

Question

What can be said about $C(X, k_1, \cdots, k_n)$ for arbitrary n and $d-$dimensional spaces (with d fixed)?

To face this question we use the fact that

$$C(X, k_1, \cdots, k_n) = \inf \left\{ \left\| \prod_{j=1}^{n} P_j \right\| : P_j \in \mathscr{P}(^{k_j}X), \|P_j\| = 1 \right\}$$

Let $P_1, \cdots, P_n$ be polynomials as above and let μ be any probability measure in $K = S_X$ or B_X, then

$$\left\| \prod_{j=1}^{n} P_j \right\|$$

Question

What can be said about $C(X, k_1, \cdots, k_n)$ for arbitrary n and $d-$dimensional spaces (with d fixed)?

To face this question we use the fact that

$$C(X, k_1, \cdots, k_n) = \inf \left\{ \left\| \prod_{j=1}^{n} P_j \right\| : P_j \in \mathscr{P}({}^{k_j}X), \|P_j\| = 1 \right\}$$

Let $P_1, \cdots, P_n$ be polynomials as above and let μ be any probability measure in $K = S_X$ or B_X, then

$$\left\| \prod_{j=1}^{n} P_j \right\| = \exp \left\{ \ln \left(\max_{x \in K} \prod_{j=1}^{n} |P_j(x)| \right) \right\}$$

Question

What can be said about $C(X, k_1, \cdots, k_n)$ for arbitrary n and $d-$dimensional spaces (with d fixed)?

To face this question we use the fact that

$$C(X, k_1, \cdots, k_n) = \inf \left\{ \left\| \prod_{j=1}^{n} P_j \right\| : P_j \in \mathscr{P}({}^{k_j}X), \|P_j\| = 1 \right\}$$

Let $P_1, \cdots, P_n$ be polynomials as above and let μ be any probability measure in $K = S_X$ or B_X, then

$$\left\| \prod_{j=1}^{n} P_j \right\| = \exp \left\{ \ln \left(\max_{x \in K} \prod_{j=1}^{n} |P_j(x)| \right) \right\} = \exp \left\{ \max_{x \in K} \ln \left(\prod_{j=1}^{n} |P_j(x)| \right) \right\}$$

Question

What can be said about $C(X, k_1, \cdots, k_n)$ for arbitrary n and $d-$dimensional spaces (with d fixed)?

To face this question we use the fact that

$$C(X, k_1, \cdots, k_n) = \inf \left\{ \left\| \prod_{j=1}^{n} P_j \right\| : P_j \in \mathscr{P}(^{k_j}X), \|P_j\| = 1 \right\}$$

Let $P_1, \cdots, P_n$ be polynomials as above and let μ be any probability measure in $K = S_X$ or B_X, then

$$\begin{aligned} \left\| \prod_{j=1}^{n} P_j \right\| &= \exp\left\{ \ln\left(\max_{x \in K} \prod_{j=1}^{n} |P_j(x)| \right) \right\} = \exp\left\{ \max_{x \in K} \ln\left(\prod_{j=1}^{n} |P_j(x)| \right) \right\} \\ &= \exp\left\{ \max_{x \in K} \sum_{j=1}^{n} \ln\left(|P_j(x)| \right) \right\} \end{aligned}$$

Question

What can be said about $C(X, k_1, \cdots, k_n)$ for arbitrary n and $d-$dimensional spaces (with d fixed)?

To face this question we use the fact that

$$C(X, k_1, \cdots, k_n) = \inf \left\{ \left\| \prod_{j=1}^{n} P_j \right\| : P_j \in \mathscr{P}({}^{k_j}X), \|P_j\| = 1 \right\}$$

Let $P_1, \cdots, P_n$ be polynomials as above and let μ be any probability measure in $K = S_X$ or B_X, then

$$\begin{aligned} \left\| \prod_{j=1}^{n} P_j \right\| &= \exp\left\{ \ln \left(\max_{x \in K} \prod_{j=1}^{n} |P_j(x)| \right) \right\} = \exp\left\{ \max_{x \in K} \ln \left(\prod_{j=1}^{n} |P_j(x)| \right) \right\} \\ &= \exp\left\{ \max_{x \in K} \sum_{j=1}^{n} \ln\left(|P_j(x)|\right) \right\} \geq \exp\left\{ \int_K \sum_{j=1}^{n} \ln\left(|P_j(x)|\right) d\mu(x) \right\} \end{aligned}$$

Then

$$C(X,k_1,\cdots,k_n) \geq \inf_{\|P_j\|=1,\deg(P)=k_j} \left\{ \exp\left(\sum_j \int_K \ln|P_j(x)|d\mu(x) \right) \right\}.$$

Then

$$C(X, k_1, \cdots, k_n) \geq \inf_{\|P_j\|=1, \deg(P)=k_j} \left\{ \exp\left(\sum_j \int_K \ln |P_j(x)| d\mu(x) \right) \right\}.$$

Thus, if we find lower bounds for $\int_K \ln |P_j(x)| d\mu(x)$ (depending only on k_j) we obtain a lower bound for $C(X, k_1, \cdots, k_n)$.

Then

$$C(X, k_1, \cdots, k_n) \geq \inf_{\|P_j\|=1, \deg(P)=k_j} \left\{ \exp\left(\sum_j \int_K \ln|P_j(x)| d\mu(x) \right) \right\}.$$

Thus, if we find lower bounds for $\int_K \ln|P_j(x)| d\mu(x)$ (depending only on k_j) we obtain a lower bound for $C(X, k_1, \cdots, k_n)$.

Theorem (García-Vázquez, Villa - 1999)

Then

$$C(X, k_1, \cdots, k_n) \geq \inf_{\|P_j\|=1, \deg(P)=k_j} \left\{ \exp\left(\sum_j \int_K \ln |P_j(x)| d\mu(x) \right) \right\}.$$

Thus, if we find lower bounds for $\int_K \ln |P_j(x)| d\mu(x)$ (depending only on k_j) we obtain a lower bound for $C(X, k_1, \cdots, k_n)$.

Theorem (García-Vázquez, Villa - 1999)

$C(\mathbb{R}^d, \underbrace{1, \cdots, 1}_{n-times}) \geq \exp\{-nL(d, \mathbb{R})\}$ with

$$L(d, \mathbb{R}) = \begin{cases} \ln(2) + \sum_{m=1}^{\frac{d-2}{2}} \frac{1}{2m} & \text{if} \quad d \equiv 0(2) \\ \ln(2) + \sum_{m=1}^{\frac{d-3}{2}} \frac{1}{2m+1} & \text{if} \quad d \equiv 1(2) \end{cases}$$

Then

$$C(X, k_1, \cdots, k_n) \geq \inf_{\|P_j\|=1, \deg(P)=k_j} \left\{ \exp\left(\sum_j \int_K \ln |P_j(x)| d\mu(x) \right) \right\}.$$

Thus, if we find lower bounds for $\int_K \ln |P_j(x)| d\mu(x)$ (depending only on k_j) we obtain a lower bound for $C(X, k_1, \cdots, k_n)$.

Theorem (García-Vázquez, Villa - 1999)

$C(\mathbb{R}^d, \underbrace{1, \cdots, 1}_{n-times}) \geq \exp\{-nL(d, \mathbb{R})\}$ with

$$L(d, \mathbb{R}) = \begin{cases} \ln(2) + \sum_{m=1}^{\frac{d-2}{2}} \frac{1}{2m} & \text{if} \quad d \equiv 0(2) \\ \ln(2) + \sum_{m=1}^{\frac{d-3}{2}} \frac{1}{2m+1} & \text{if} \quad d \equiv 1(2) \end{cases}$$

Moreover

$$\lim_{n \to \infty} C(\mathbb{R}^d, 1, \cdots, 1)^{\frac{1}{n}} = \exp\{-L(d, \mathbb{R})\}$$

Theorem (A. Pappas, S. G. Révész - 2003)

Theorem (A. Pappas, S. G. Révész - 2003)

$C(\mathbb{C}^d, \underbrace{1, \cdots, 1}_{n-times}) \geq \exp\{-nL(d, \mathbb{C})\}$ with

$$L(d, \mathbb{C}) = \frac{1}{2} \sum_{m=1}^{d-1} \frac{1}{m}$$

Theorem (A. Pappas, S. G. Révész - 2003)

$C(\mathbb{C}^d, \underbrace{1, \cdots, 1}_{n-times}) \geq \exp\{-nL(d, \mathbb{C})\}$ with

$$L(d, \mathbb{C}) = \frac{1}{2} \sum_{m=1}^{d-1} \frac{1}{m}$$

Moreover

$$\lim_{n \to \infty} C(\mathbb{C}^d, 1, \cdots, 1)^{\frac{1}{n}} = \exp\{-L(d, \mathbb{C})\}$$

Let $X = (\mathbb{R}^d, \| - \|)$, take $K = B_X$ and $\mu = \lambda$ the normalized Lebesgue measure.

Let $X = (\mathbb{R}^d, \| - \|)$, take $K = B_X$ and $\mu = \lambda$ the normalized Lebesgue measure.

$$\int_{B_X} \ln(|P(x)|) d\lambda(x)$$

Finite dimensional case

Let $X = (\mathbb{R}^d, \| - \|)$, take $K = B_X$ and $\mu = \lambda$ the normalized Lebesgue measure.

$$\int_{B_X} \ln(|P(x)|) d\lambda(x) \quad = \quad - \int_{B_X} - \ln(|P(x)|) d\lambda(x)$$

Finite dimensional case

Let $X = (\mathbb{R}^d, \| - \|)$, take $K = B_X$ and $\mu = \lambda$ the normalized Lebesgue measure.

$$\begin{aligned}\int_{B_X} \ln(|P(x)|) d\lambda(x) &= -\int_{B_X} -\ln(|P(x)|) d\lambda(x) \\ &= -\int_0^{+\infty} \lambda(\{x : -\ln(|P(x)|) \geq t\}) dt\end{aligned}$$

Let $X = (\mathbb{R}^d, \| - \|)$, take $K = B_X$ and $\mu = \lambda$ the normalized Lebesgue measure.

$$\begin{aligned}
\int_{B_X} \ln(|P(x)|) d\lambda(x) &= -\int_{B_X} -\ln(|P(x)|) d\lambda(x) \\
&= -\int_0^{+\infty} \lambda(\{x : -\ln(|P(x)|) \geq t\}) dt \\
&= -\int_0^{+\infty} \lambda(\{x : |P(x)| \leq e^{-t}\}) dt
\end{aligned}$$

Let $X = (\mathbb{R}^d, \| - \|)$, take $K = B_X$ and $\mu = \lambda$ the normalized Lebesgue measure.

$$\begin{aligned}
\int_{B_X} \ln(|P(x)|) d\lambda(x) &= -\int_{B_X} -\ln(|P(x)|) d\lambda(x) \\
&= -\int_0^{+\infty} \lambda(\{x : -\ln(|P(x)|) \geq t\}) dt \\
&= -\int_0^{+\infty} \lambda(\{x : |P(x)| \leq e^{-t}\}) dt
\end{aligned}$$

Now we use the following Corollary of a Remez type inequality for several variables.

Let $X = (\mathbb{R}^d, \| - \|)$, take $K = B_X$ and $\mu = \lambda$ the normalized Lebesgue measure.

$$\begin{aligned}\int_{B_X} \ln(|P(x)|) d\lambda(x) &= -\int_{B_X} -\ln(|P(x)|) d\lambda(x) \\ &= -\int_0^{+\infty} \lambda(\{x : -\ln(|P(x)|) \geq t\}) dt \\ &= -\int_0^{+\infty} \lambda(\{x : |P(x)| \leq e^{-t}\}) dt\end{aligned}$$

Now we use the following Corollary of a Remez type inequality for several variables.

Corollary (J. Brudnyi and I. Ganzburg - 1993)

Let $P : \mathbb{R}^d \to \mathbb{R}$ be a continuous polynomial of degree k and norm 1, then

$$\lambda(\{x \in K : |P(x)| \leq t\}) \leq 4d \left(\frac{t}{2}\right)^{\frac{1}{k}}$$

Finite dimensional case

We then obtain:

$$\int_{B_X} \ln(|P(x)|)d\lambda(x) \quad \geq \quad \ln\left(\frac{2}{(4d)^k}\right) - k$$

We then obtain:

$$\int_{B_X} \ln(|P(x)|)d\lambda(x) \quad \geq \quad \ln\left(\frac{2}{(4d)^k}\right) - k$$

Then

We then obtain:

$$\int_{B_X} \ln(|P(x)|)d\lambda(x) \quad \geq \quad \ln\left(\frac{2}{(4d)^k}\right) - k$$

Then

Lower bound for $C(X, k_1, \cdots, k_n)$

$$C(X, k_1, \cdots, k_n) \quad \geq \quad \exp\left\{\sum_j \ln\left(\frac{2}{(4d)^{k_j}}\right) - k_j\right\}$$

We then obtain:

$$\int_{B_X} \ln(|P(x)|)d\lambda(x) \quad \geq \quad \ln\left(\frac{2}{(4d)^k}\right) - k$$

Then

Lower bound for $C(X, k_1, \cdots, k_n)$

$$\begin{aligned} C(X, k_1, \cdots, k_n) \quad &\geq \quad \exp\left\{\sum_j \ln\left(\frac{2}{(4d)^{k_j}}\right) - k_j\right\} \\ &= \quad \prod_j \frac{2}{(4d)^{k_j}} \frac{1}{e^{k_j}} = \frac{2^n}{(4de)^{\sum_{j=1}^n k_j}} \end{aligned}$$

Question

What about an upper bound for $C(X, k_1, \cdots, k_n)$ or estimates for $\overline{\lim}_{n\to\infty} C(X, k_1, \cdots, k_n)^{\frac{1}{\sum k_j}}$

Question

What about an upper bound for $C(X, k_1, \cdots, k_n)$ or estimates for $\overline{\lim}_{n\to\infty} C(X, k_1, \cdots, k_n)^{\frac{1}{\sum k_j}}$

In the linear case we can estimate the limit for the space ℓ_p^d.

Question

What about an upper bound for $C(X, k_1, \cdots, k_n)$ or estimates for $\overline{\lim}_{n\to\infty} C(X, k_1, \cdots, k_n)^{\frac{1}{\sum k_j}}$

In the linear case we can estimate the limit for the space ℓ_p^d.

$$\boxed{\overline{\lim_{n\to\infty}} C(\ell_p^d, 1, \cdots, 1)^{\frac{1}{n}} \leq \exp\{-L(\mathbb{K}, d)\} \|x_0\|_2^2 \left(\frac{vol(B_{\ell_{p'}^d})}{vol(B_{\ell_2^d})} \right)^{\frac{1}{d}}}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and x_0 is some point of S_X.

Sketch of the proof

Sketch of the proof

Take in S_{X^*} any probability measure μ.

Sketch of the proof

Take in S_{X^*} any probability measure μ. Using a probabilistic tools like the Law of Large Numbers construct a sequence $\{\varphi_j\} \subseteq S_{X^*}$ such that

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{j=1}^{n} \ln(|\varphi_j(x_n)|) \leq \int_{S_{X^*}} \ln(|\varphi(x_0)|) d\mu(\varphi)$$

where $x_n \in S_X$ is such that $\| \prod_{j=1}^{n} \varphi_j \| = | \prod_{j=1}^{n} \varphi_j(x_n)|$ and x_0 is some accumulation point of the sequence $\{x_n\}$.

Sketch of the proof

Take in S_{X^*} any probability measure μ. Using a probabilistic tools like the Law of Large Numbers construct a sequence $\{\varphi_j\} \subseteq S_{X^*}$ such that

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{j=1}^{n} \ln(|\varphi_j(x_n)|) \leq \int_{S_{X^*}} \ln(|\varphi(x_0)|) d\mu(\varphi)$$

where $x_n \in S_X$ is such that $\| \prod_{j=1}^{n} \varphi_j \| = | \prod_{j=1}^{n} \varphi_j(x_n)|$ and x_0 is some accumulation point of the sequence $\{x_n\}$. Then

$$C(X, 1, \cdots, 1)^{\frac{1}{n}}$$

Sketch of the proof

Take in S_{X^*} any probability measure μ. Using a probabilistic tools like the Law of Large Numbers construct a sequence $\{\varphi_j\} \subseteq S_{X^*}$ such that

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{j=1}^{n} \ln(|\varphi_j(x_n)|) \leq \int_{S_{X^*}} \ln(|\varphi(x_0)|) d\mu(\varphi)$$

where $x_n \in S_X$ is such that $\| \prod_{j=1}^{n} \varphi_j \| = | \prod_{j=1}^{n} \varphi_j(x_n)|$ and x_0 is some accumulation point of the sequence $\{x_n\}$. Then

$$C(X, 1, \cdots, 1)^{\frac{1}{n}} \quad \leq \quad \left\| \prod_{j=1}^{n} \varphi_j \right\|^{\frac{1}{n}} = \left| \prod_{j=1}^{n} \varphi_j(x_n) \right|^{\frac{1}{n}}$$

Sketch of the proof

Take in S_{X^*} any probability measure μ. Using a probabilistic tools like the Law of Large Numbers construct a sequence $\{\varphi_j\} \subseteq S_{X^*}$ such that

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{j=1}^{n} \ln(|\varphi_j(x_n)|) \leq \int_{S_{X^*}} \ln(|\varphi(x_0)|) d\mu(\varphi)$$

where $x_n \in S_X$ is such that $\| \prod_{j=1}^n \varphi_j \| = | \prod_{j=1}^n \varphi_j(x_n) |$ and x_0 is some accumulation point of the sequence $\{x_n\}$. Then

$$\begin{aligned} C(X,1,\cdots,1)^{\frac{1}{n}} &\leq \left\| \prod_{j=1}^{n} \varphi_j \right\|^{\frac{1}{n}} = \left| \prod_{j=1}^{n} \varphi_j(x_n) \right|^{\frac{1}{n}} \\ &= \exp\left\{ \frac{1}{n} \sum_{j=1}^{n} \ln(|\varphi_j(x_n)|) \right\} \end{aligned}$$

Taking upper limit we obtain

$$\overline{\lim_{n\to\infty}} C(X, 1, \cdots, 1)^{\frac{1}{n}} \quad \leq \quad \exp\left\{ \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{j=1}^{n} \ln(|\varphi_j(x_n)| \right\}$$

Finite dimensional case

Taking upper limit we obtain

$$\begin{aligned}\overline{\lim_{n\to\infty}} C(X,1,\cdots,1)^{\frac{1}{n}} &\le \exp\left\{\overline{\lim_{n\to\infty}}\frac{1}{n}\sum_{j=1}^{n}\ln(|\varphi_j(x_n)|\right\} \\ &\le \exp\left\{\int_{S_{X^*}}\ln(|\varphi(x_0)|)d\mu(\varphi)\right\}\end{aligned}$$

Taking upper limit we obtain

$$\begin{aligned}\overline{\lim_{n\to\infty}} C(X,1,\cdots,1)^{\frac{1}{n}} &\leq \exp\left\{\overline{\lim_{n\to\infty}} \frac{1}{n}\sum_{j=1}^{n} \ln(|\varphi_j(x_n)|\right\} \\ &\leq \exp\left\{\int_{S_{X^*}} \ln(|\varphi(x_0)|) d\mu(\varphi)\right\}\end{aligned}$$

Therefore, if we get an upper bound for $\int_{S_{X^*}} \ln(|\varphi(x_0)|) d\mu(\varphi)$ we obtain an upper bound for $\overline{\lim}_{n\to\infty} C(X,1,\cdots,1)^{\frac{1}{n}}$.

Taking upper limit we obtain

$$\begin{aligned}\overline{\lim_{n\to\infty}} C(X,1,\cdots,1)^{\frac{1}{n}} &\leq \exp\left\{\overline{\lim_{n\to\infty}}\frac{1}{n}\sum_{j=1}^{n}\ln(|\varphi_j(x_n)|\right\} \\ &\leq \exp\left\{\int_{S_{X^*}}\ln(|\varphi(x_0)|)d\mu(\varphi)\right\}\end{aligned}$$

Therefore, if we get an upper bound for $\int_{S_{X^*}}\ln(|\varphi(x_0)|)d\mu(\varphi)$ we obtain an upper bound for $\overline{\lim}_{n\to\infty} C(X,1,\cdots,1)^{\frac{1}{n}}$.
In the case $X=\ell_p^d$, taking a suitable measure μ we obtain

Taking upper limit we obtain

$$\overline{\lim_{n\to\infty}} C(X,1,\cdots,1)^{\frac{1}{n}} \leq \exp\left\{\overline{\lim_{n\to\infty}}\frac{1}{n}\sum_{j=1}^{n}\ln(|\varphi_j(x_n)|\right\}$$
$$\leq \exp\left\{\int_{S_{X^*}}\ln(|\varphi(x_0)|)d\mu(\varphi)\right\}$$

Therefore, if we get an upper bound for $\int_{S_{X^*}}\ln(|\varphi(x_0)|)d\mu(\varphi)$ we obtain an upper bound for $\overline{\lim}_{n\to\infty} C(X,1,\cdots,1)^{\frac{1}{n}}$.
In the case $X=\ell_p^d$, taking a suitable measure μ we obtain

$$\boxed{\overline{\lim_{n\to\infty}} C(X,1,\cdots,1)^{\frac{1}{n}} \leq \exp\{-L(\mathbb{K},d)\}\,\|x_0\|_2^2\left(\frac{vol(B_{\ell_{p'}^d})}{vol(B_{\ell_2^d})}\right)^{\frac{1}{d}}}$$

Thanks!