Lower bounds for norms of products of polynomials on L_p spaces

Jorge Tomás Rodríguez Joint work with Daniel Carando and Damián Pinasco

Buenos Aires, July 2014

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- **B)** Continuous polynomials.

Problems Lower bounds for norms of products of polynomials on L_D spaces Buenos Aires , July 2014

Problems

Problem A

Given $k_1, \dots, k_n \in \mathbb{N}$, find the optimal constant $C(E, k_1, \dots, k_n)$, such that for every set of continuous homogeneous polynomials $P_1, \dots, P_n : E \to \mathbb{K}$, of degrees k_1, \dots, k_n , the next inequality holds

$$C(E, k_1, \dots, k_n) \prod_{j=1}^n ||P_j|| \le \left\| \prod_{j=1}^n P_j \right\|$$

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Problem **B**

Given $k_1, \cdots, k_n \in \mathbb{N}$, find the optimal constant $D(E, k_1, \cdots, k_n)$, such that for every set of (not necessarily homogeneous) continuous polynomials $P_1, \cdots, P_n : E \to \mathbb{K}$, of degrees k_1, \cdots, k_n , the next inequality holds

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For any complex Banach space E

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These bounds are optimal.



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The first simplification can be made by induction.

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The first simplification can be made by induction. The second one follows from the fact that for any continuous polynomial $P:\ell_p\to\mathbb{C}$

$$||P|| = \lim_{d \to \infty} ||P_d||,$$

where
$$P_d((a_1, a_2, \dots, a_d)) = P((a_1, a_2, \dots, a_d, 0, 0, \dots)).$$

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Just like in the examples above, when the polynomials depends on different variable, we have

$$\|P_1P_2\| = \left(\frac{k_1^{k_1}k_2^{k_2}}{(k_1+k_2)^{k_1+k_2}}\right)^{\frac{1}{p}}\|P_1\|\|P_2\|$$

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Theorem (D. Lewis - 1978)

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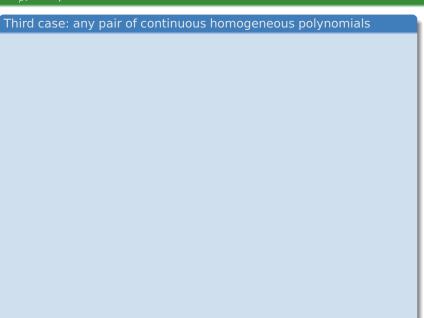
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We can do the same for $k_1 \neq k_2$, but we would not obtain an optimal constant.



Third case: any pair of continuous homogeneous polynomials

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$$p > 2$$
, $\left| \frac{1}{p} - \frac{1}{2} \right| \neq \frac{1}{p} - \frac{1}{2}$.

What happens if we follow the same reasoning for p>2? Recall the result from Lewis

For any d-dimensional subspace $E \subset L_p$, with $1 \le p \le \infty$,

$$d(E,\ell_2^d) \leq d^{\left|\frac{1}{p}-\frac{1}{2}\right|}.$$

For p > 2, $\left| \frac{1}{p} - \frac{1}{2} \right| \neq \frac{1}{p} - \frac{1}{2}$. Then, we get

$$C(\ell_p, k_1, \cdots, k_n) \geq \frac{C(\ell_2, k_1, \cdots, k_n)}{(n^{k_1 + \cdots + k_n})^{\frac{1}{2} - \frac{1}{p}}}$$



Finite dimensional case

Question

What can be said about $C(X, k_1, \dots, k_n)$ for arbitrary n and d—dimensional spaces

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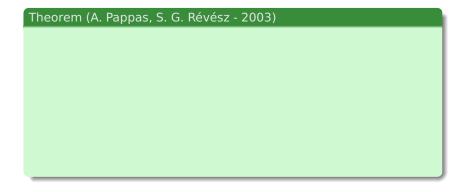
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Theorem (A. Pappas, S. G. Révész - 2003)

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Corollary (J. Brudnyi and I. Ganzburg - 1993)

Let $P:\mathbb{R}^d \to \mathbb{R}$ be a continuous polynomial of degree k and norm 1, then

$$\lambda(\{x \in K : |P(x)| \le t\}) \le 4d\left(\frac{t}{2}\right)^{\frac{1}{k}}$$

We then obtain:

$$\int_{B_X} \ln(|P(x)|) d\lambda(x) \geq \ln\left(\frac{2}{(4d)^k}\right) - k$$

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Question

What about an upper bound for $C(X, k_1, \cdots, k_n)$ or estimates for

$$\overline{\lim}_{n\to\infty} C(X,k_1,\cdots,k_n)^{\frac{1}{\sum k_j}}$$

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In the linear case we can estimate the limit for the space ℓ_p^d .

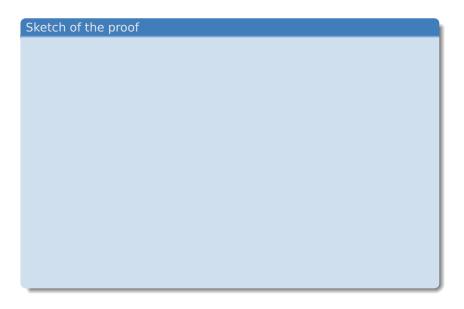
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$$\overline{\lim_{n\to\infty}}C(\ell_p^d,1,\cdots,1)^{\frac{1}{n}}\leq \exp\left\{-L(\mathbb{K},d)\right\}\|x_0\|_2^2\left(\frac{\operatorname{vol}(B_{\ell_{p'}^d})}{\operatorname{vol}(B_{\ell_2^d})}\right)^{\frac{1}{d}}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and x_0 is some point of S_X .





Take in S_{X^*} any probability measure μ .

Sketch of the proof

Take in S_{X^*} any probability measure μ . Using a probabilistic tools like the Law of Large Numbers construct a sequence $\{\varphi_j\}\subseteq S_{X^*}$ such that

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\sum_{j=1}^n\ln(|\varphi_j(x_n)|)\leq \int_{S_{X^*}}\ln(|\varphi(x_0)|)d\mu(\varphi)$$

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Thanks!