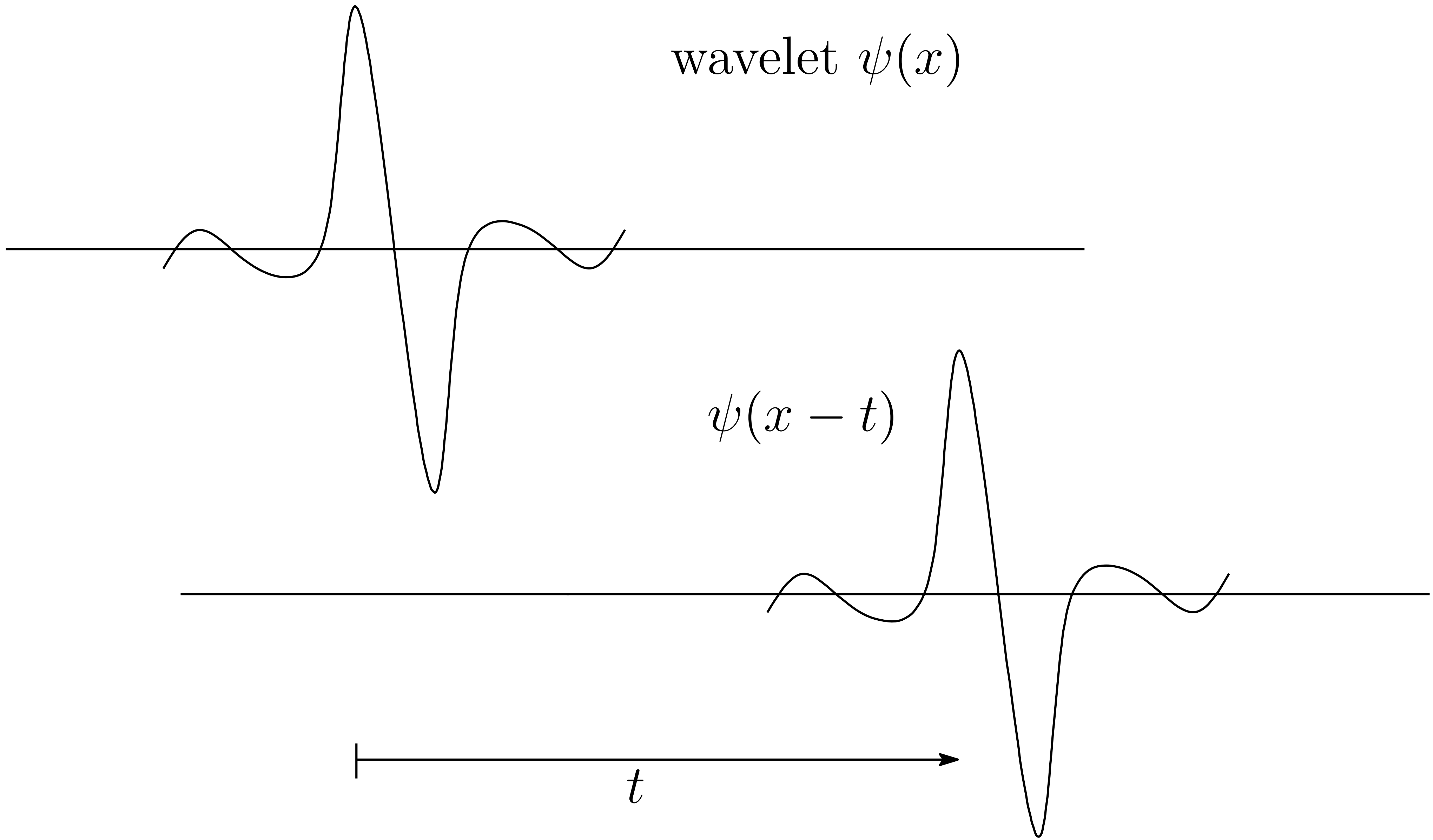
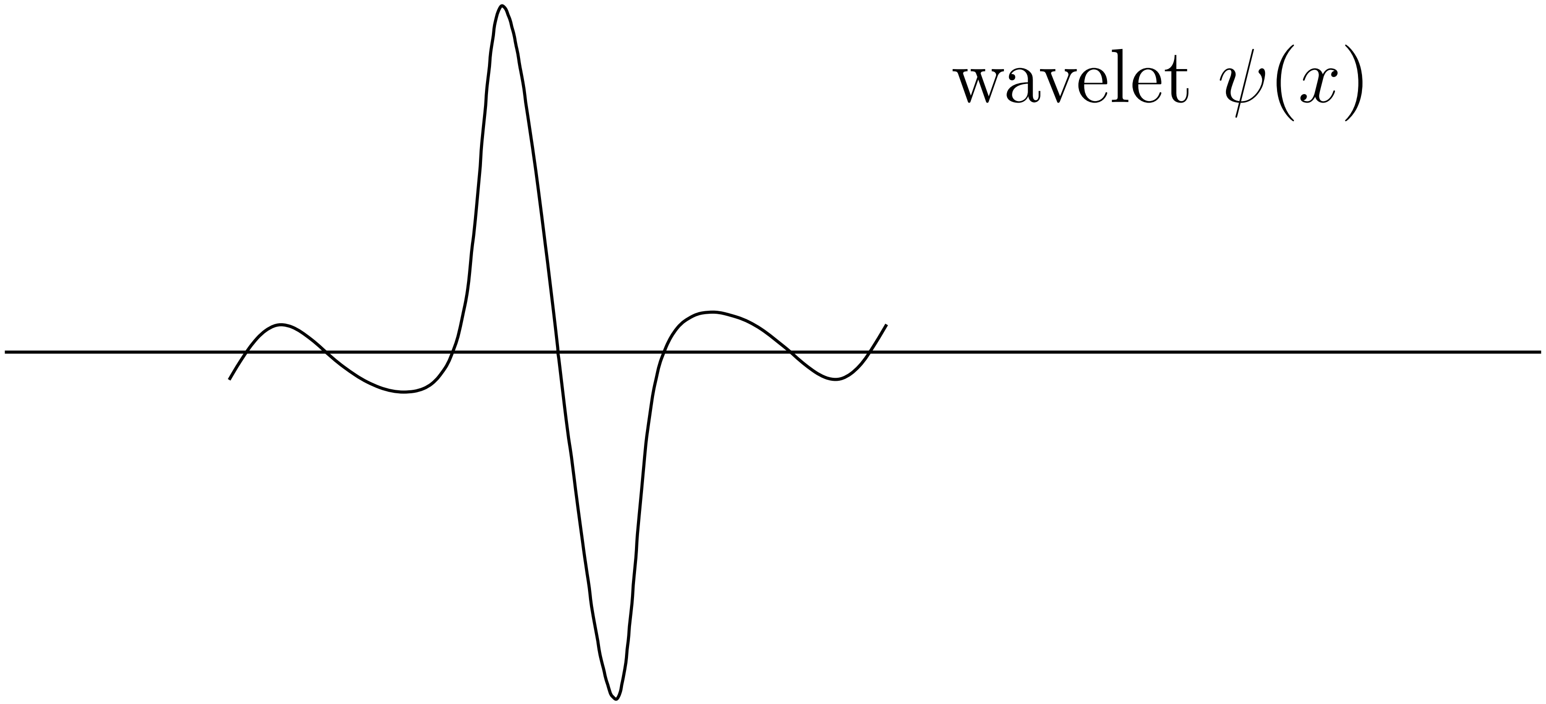
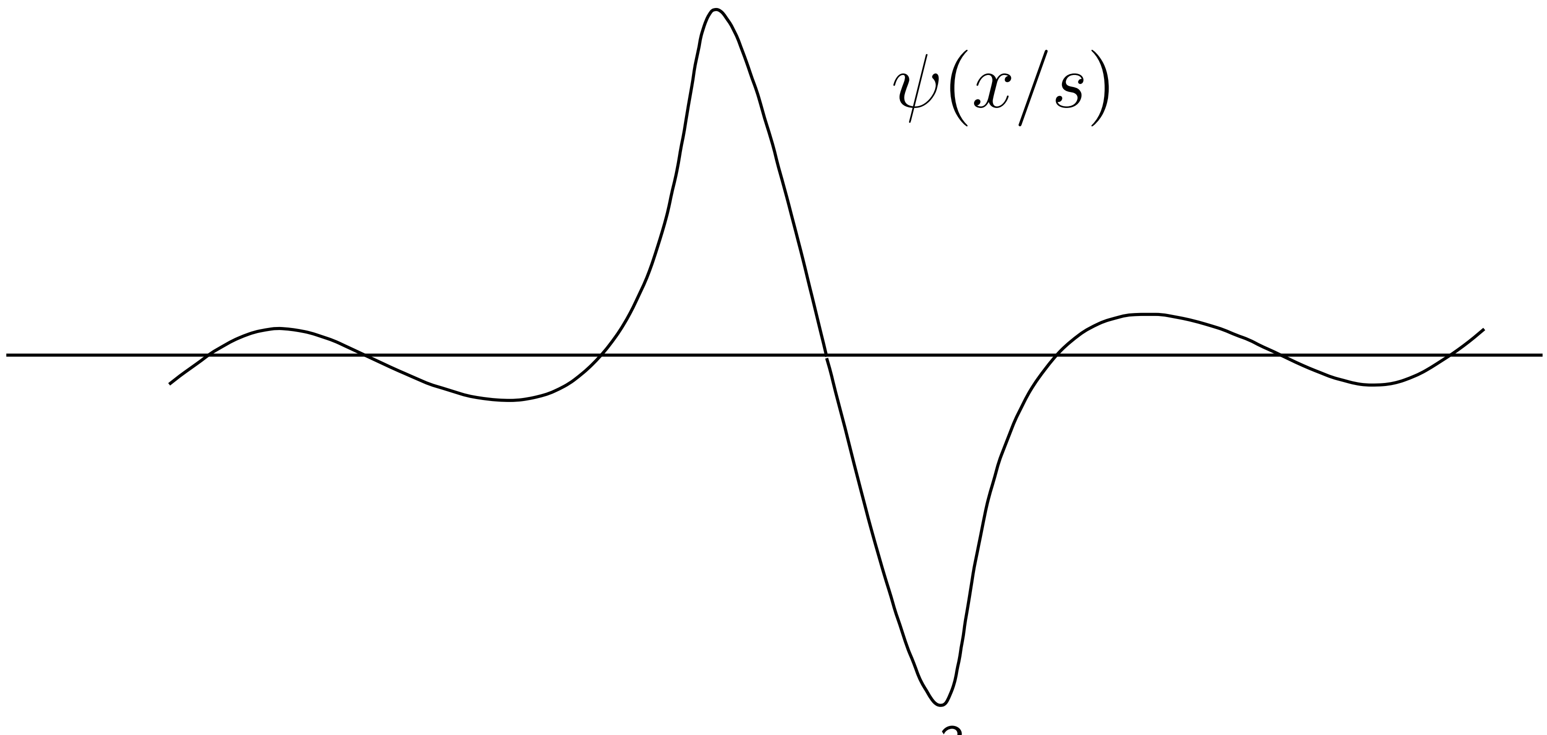
Some roles of function spaces in wavelet theory – detection of singularities –

Shinya MORITOH
Nara Women's University

July 22, 2014


wavelet $\psi(x)$
$\psi(x-t)$
t


wavelet $\psi(x)$


$\psi(x/s)$

Every function can be described as a superposition of wavelets.

Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

wavelet series

$$f(x) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{jk}\, \psi(2^j x - k) = \sum_{j,k} c_{jk}\, \psi_{jk}$$

Contents

1. **Introduction**

   function spaces

   wavelets

   singularities at a point

2. **Two-microlocal spaces and ridgelets: detection of line singularities**

   What is "two-microlocal"?

   Moritoh-Yamada 2004, Moritoh-Tanaka 2013

   Radon transformation, ridgelets

1. Introduction

Function spaces: L_2, L_p, W_p^s, B_{pq}^s, F_{pq}^s

Wavelet transformation: For a wavelet function $\psi(x)$, let

$$\psi_{s,t}(x) := s^{-1/2}\,\psi\big((x-t)/s\big), \quad s > 0,\ t \in \mathbb{R},$$

and

$$(W_\psi f)(s,t) := \int_{\mathbb{R}} f(x)\,\overline{\psi_{s,t}(x)}\,dx.$$

Reproducing formula (Calderón):

Let ψ satisfy the condition that

$$\int_0^\infty \frac{|\hat{\psi}(\xi)|^2}{\xi}\,d\xi = \int_0^\infty \frac{|\hat{\psi}(-\xi)|^2}{\xi}\,d\xi \ \ (=: C_\psi) < \infty.$$

Then we have

$$f(x) = {C_\psi}^{-1} \int_0^\infty \int_{\mathbb{R}} (W_\psi f)(s,t)\,\psi_{s,t}(x)\,dt\frac{ds}{s^2}.$$

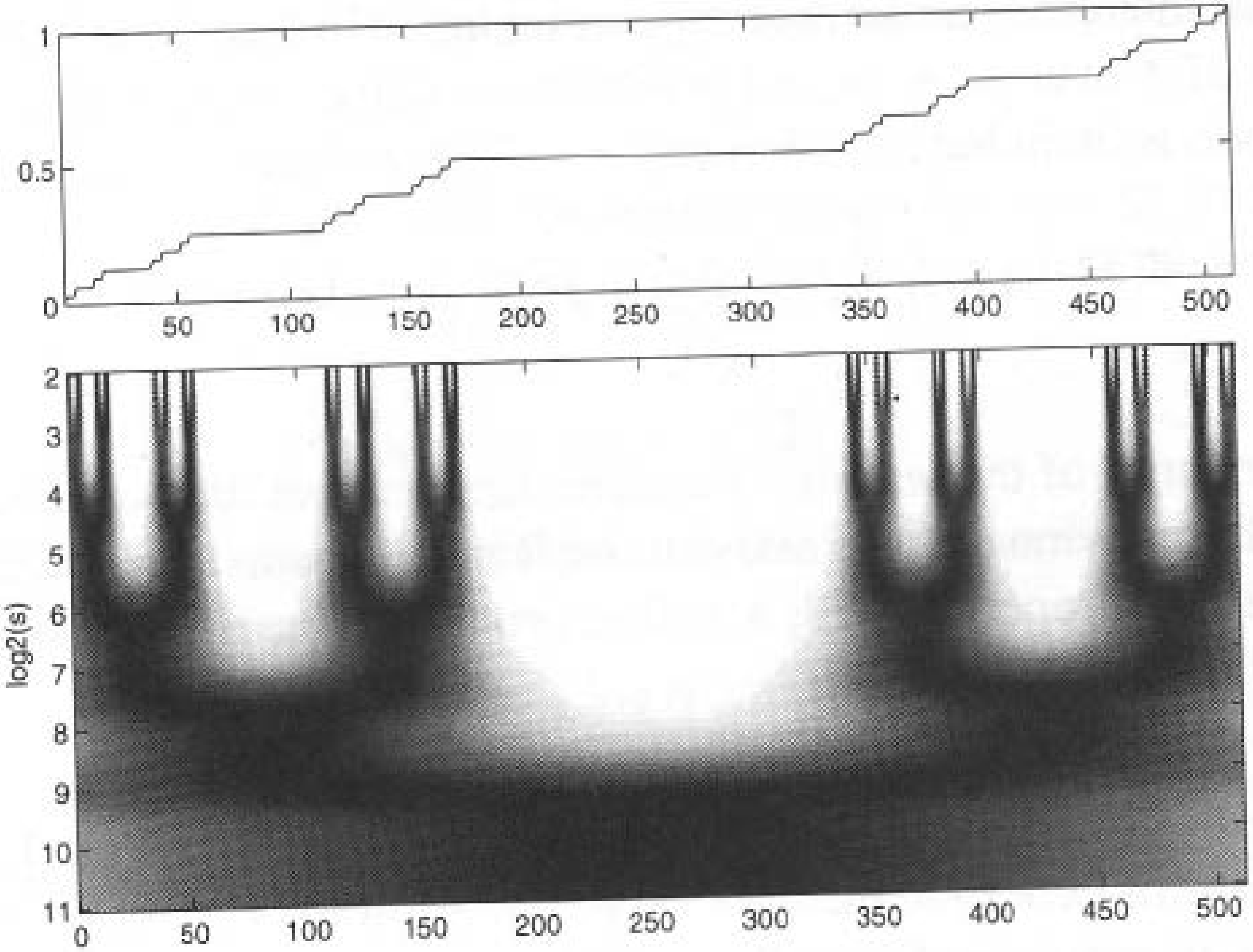
Discretization:

Multiresolution analysis (**MRA**) with a scaling function $\phi(x)$ gives us a **wavelet** $\psi(x)$.
Then, a complete orthonormal system of $L_2(\mathbb{R})$ is given by $\{2^{j/2}\,\psi(2^j x - k)\}_{j,k\in\mathbb{Z}}$.

Expansion: $$f(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{jk}\,\psi_{jk}(x) \ \text{ in } L_2(\mathbb{R}).$$

Analogue of Parseval: $$\int_{\mathbb{R}} |f(x)|^2\,dx = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |c_{jk}|^2.$$

Examples: fractals (Cantor, ...)



↑ high frequency

phase space

Application: pointwise regularity of a function

Ref. Bony (1984), Jaffard-Meyer (1996), M (2004, 2013)

Def. A function $f(x)$ is said to have the **Hölder** continuity of order α at $x = x_0$, if for every $x \in \mathbb{R}$,

$$|f(x) - f(x_0)| \le C|x - x_0|^{\alpha}.$$

Then, we write $f \in B^{\alpha}_{\infty,\infty}(x_0)$.

Theorem. For $f \in B^{\alpha}_{\infty,\infty}(x_0)$, we have

$$\big|(W_\psi f)(s,t)\big| \le C s^{\alpha+1/2}\big(1 + |(t-x_0)/s|^{\alpha}\big).$$

If we have, for $\beta < \alpha$,

$$\big|(W_\psi f)(s,t)\big| \le C s^{\alpha+1/2}\big(1 + |(t-x_0)/s|^{\beta}\big),$$

then we have $f \in B^{\alpha}_{\infty,\infty}(x_0)$.

Remark. The factor $(t-x_0)/s$ represents the **uncertainty principle**.

Remark. The function space $B^{\alpha}_{\infty,\infty}$ is a special case of the **Besov spaces** $B^{\alpha}_{p,q}$, where $p = q = \infty$.

Outline of the proof.

1.
$$\hat{\psi}(0) = 0 \quad \Longleftrightarrow \quad \int_{\mathbb{R}} \psi(x)\,dx = 0.$$

2. **Littlewood-Paley decomposition**

$$f(x) = \sum_{j=-\infty}^{\infty} f_j(x),$$

where

$$f_j(x) = C_\psi^{-1} \int_{2^j}^{2^{j+1}} \int_{\mathbb{R}} (Wf)(s,t)\, s^{-1/2}\, \psi\big((x-t)/s\big)\, dt\, \frac{ds}{s^2}.$$

Key: $|f_j(x)| \leq C\, 2^{j\alpha} \left(1 + (2^{-j}|x - x_0|)^\beta\right). \quad \square$

Basic references

[D] I. Daubechies, Ten lectures on wavelets, CBMS-NSF Regional Conference Series in Applied Mathematics **61**, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

[H] B. B. Hubbard, The world according to wavelets, The story of a mathematical technique in the making, A K Peters, Ltd., Wellesley, MA, 1996.

[M] Y. Meyer, Wavelets and operators, Translated from the 1990 French original by D. H. Salinger, Cambridge Studies in Advanced Mathematics **37**, Cambridge University Press, Cambridge, 1992.

Remark.

1. 1970, **Sato**, **Hörmander**, basics of linear PDE
   (= microlocal analysis based on Fourier)

2. 1997 $\sim$, M, **wavelet version:** Motivated by

$$P(x, D)f(x) = \int_{\mathbb{R}} p(x, \xi)\, \hat{f}(\xi)\, e^{ix\xi}\, d\xi,$$

we define an operator P_ψ:

$$P_\psi(x, D)f(x) = \int_0^\infty \int_{\mathbb{R}} p(t, s)\, W_\psi f(s, t)\, \psi_{s,t}(x)\, dt\, \frac{ds}{s^2}.$$

Let their difference be $Q_\psi(x, D)f(x)$. Then, in what sense can Q_ψ be considered as an "error"?

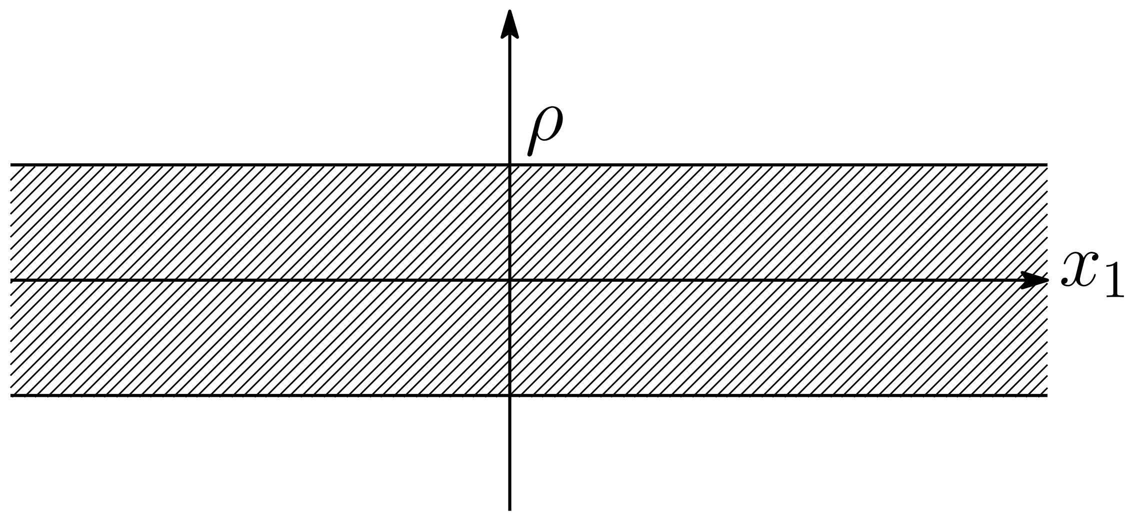
Cf. **Córdoba-Fefferman** (1978), wave packet transform.

2. Two-microlocal spaces and ridgelets: detection of line singularities

We have the following two ideas:

1. Two-microlocal analysis: **Uncertainty Principle**.

horizontal strip A_ρ :



For a function f, we consider the following norm

$$\left[\int_0^R \left(\rho^s \, \|f \mid F(A_\rho)\|\right)^p \, d\rho/\rho\right]^{1/p},$$

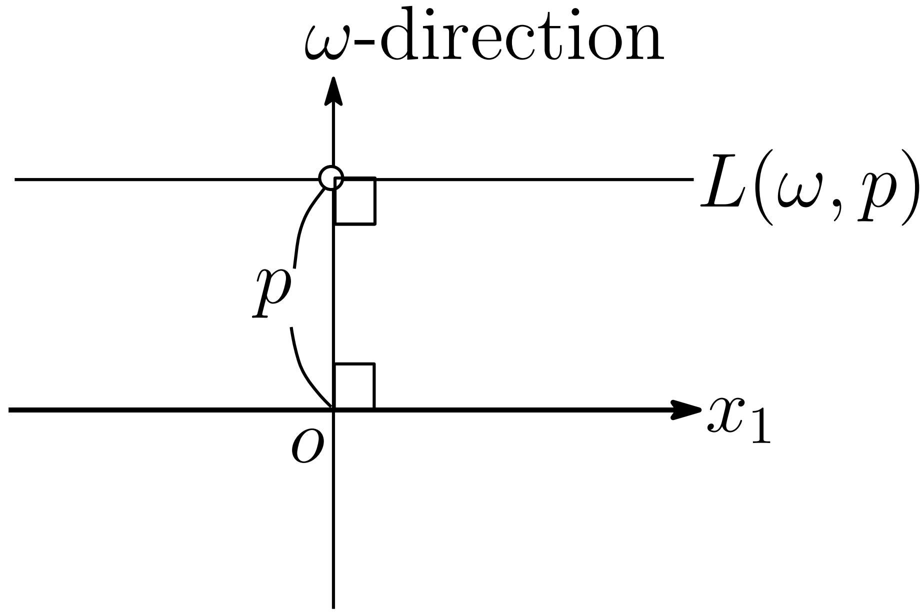
where $F(A_\rho)$ stands for some function space on A_ρ.

Remark. Singularities of the function f **along the** x_1**-axis** can be captured.

Remark. This is an analogue of Hörmander's norm, which appears in his discussion of the **hypoellipticity** for operators.

2. Ridgelet analysis: **Radon transformation**.

hyperplane $L(\omega, p)$:



We capture the singularities of a function f **along the x_1-axis** by considering the following norm of f:

$$\|(\mathrm{R}f)(\omega,\cdot)\,|\,F(-\rho,\rho)\|\,,$$

where the **Radon transform** $(\mathrm{R}f)(\omega,p)$ is the integral of f on the hyperplane $L(\omega,p)$, and $F(-\rho,\rho)$ stands for some **function space** on the interval $(-\rho,\rho)$.

[JM] S. Jaffard and Y. Meyer, *Wavelet methods for pointwise regularity and local oscillations of functions*, Mem. Amer. Math. Soc. **587** (1996).

[C] E. Candès, *Ridgelets: theory and applications*, Ph. D. thesis, Department of Statistics, Stanford University, 1998.

[MY] S. Moritoh and T. Yamada, *Two-microlocal Besov spaces and wavelets*, Rev. Mat. Iberoamericana **20** (2004), no. 1, 277–283.

[MT] S. Moritoh and Y. Tanaka, *Microlocal Besov spaces and dominating mixed smoothness*, preprint (2013).

What is "two-microlocal"?

Two-microlocal analysis of a function f measures **pointwise regularities** of f. The objects, whose pointwise oscillations are very rapid, are well described by two-microlocal analysis. Cf. turbulence.

1. **Moritoh-Yamada (2004)** is an extension of **Bony (1984)** and **Jaffard-Meyer (1996)** to Besov spaces.

2. **Moritoh-Yamada (2004)** treats **pointwise** regularities; **Moritoh-Tanaka (2013)** describes regularities **along the** x_1**-axis** in $\mathbb{R}^2$. For that purpose, function spaces with **dominating mixed smoothness** are used.

Moritoh-Yamada (2004)

For $s > 0$, $1 \leqslant p,\ q \leqslant \infty$, **homogeneous Besov space** $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is defined to be the collection of all tempered distributions f satisfying

$$\left[\sum_{j=-\infty}^{\infty}\left(2^{js}\,\|f_j(x)|L_p(\mathbb{R}^n)\|\right)^q\right]^{1/q} < \infty,$$

where

$$f \equiv \sum_{j\in\mathbb{Z}} f_j \quad (\textbf{Littlewood-Paley decomposition}).$$

Recall that the **translated** $(k2^{-j})$ **and dilated** (2^j) wavelet is given by

$$\psi_{j,k}(x) := 2^{nj/2}\,\psi(2^j x - k), \quad j \in \mathbb{Z},\ k \in \mathbb{Z}^n.$$

Then, $f \in \mathcal{S}'(\mathbb{R}^n)$ has the following expansion:

$$f(x) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} C_{j,k}\, \psi_{j,k}(x), \quad C_{j,k} = \langle f,\, \psi_{j,k} \rangle.$$

Fact: $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ if and only if

$$\sum_{j \in \mathbb{Z}} \left[2^{j\tilde{s}} \left(\sum_{k \in \mathbb{Z}^n} |C_{j,k}|^p \right)^{1/p} \right]^q < \infty.$$

Here, $\tilde{s} = s + n(1/2 - 1/p)$.

Philosophy

Parseval:

$$f \in L_2(\mathbb{R}^n) \iff \text{wavelet coefficients } \{C_{j,k}\} \in l_2 \quad (\textbf{seq. sp.})$$

Extension:

$$f \in \dot{B}^s_{p,q}(\mathbb{R}^n) \stackrel{\textbf{Fact}}{\iff} \text{wavelet coeff. } \{C_{j,k}\} \in \dot{b}^s_{p,q} \quad (\textbf{seq. sp.})$$

$$\downarrow \text{ Further extension}$$

$$\textbf{New function spaces} \stackrel{\textbf{def}}{\iff} \text{more general conditions on } C_{j,k}$$

Desired theorem: Every f belonging to the new function space has **a good decomposition**. **The error term** in this decomposition describes the very **singular part** of the function f.

Two-microlocal estimates illustrate the philosophy well:

Let $s' \in \mathbb{R}$. Then, $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to belong to the **two-microlocal Besov space** $B^{s,s'}_{p,q}(x_0)$, if the estimate of the **fact** with

$$C_{j,k} \to (1 + 2^j|2^{-j}k - x_0|)^{s'} C_{j,k}$$

holds for the wavelet coefficients $C_{j,k}$ of f.

Remark. $2^j|2^{-j}k - x_0|$ stands for the **uncertainty principle** in the phase space.

Notation (local Besov type condition). A nonnegative function $g(\rho), \rho > 0$, satisfies the condition

$$g(\rho) = \mathcal{O}^{(p)}(\rho^{-s})$$

if for every $R > 0$,

$$\int_0^R (g(\rho)\rho^s)^p \, \frac{d\rho}{\rho} < \infty.$$

Theorem. Let $s > 0$, $s' < 0$, $s + s' > 0$, and $1 \le p \le \infty$. For $\rho > 0$, put $A_\rho := \{x \in \mathbb{R}^n ; |x - x_0| < \rho\}$. Then, for $f \in \mathcal{S}'(\mathbb{R}^n)$, we have that $f \in B^{s,s'}_{p,p}(x_0)$ if and only if $f = f_1 + f_2$, where

$$f_1 \in \dot{B}^s_{p,p}(\mathbb{R}^n) \quad \text{and} \quad \|f_2 \,|\, B^{s+s'}_{p,p}(A_\rho)\| = \mathcal{O}^{(p)}(\rho^{-s'}).$$

Summary: The idea of **Moritoh-Yamada (2004)** is as follows:

$$f \in \dot{B}^s_{p,q}(\mathbb{R}^n) \overset{\textbf{Fact}}{\Longleftrightarrow} \{C_{j,k}\} \in \dot{b}^s_{p,q} \quad (\textbf{seq. sp.})$$

$$\downarrow \text{ Extension}$$

$$f \in B^{s,s'}_{p,q}(x_0) \overset{\textbf{def}}{\Longleftrightarrow} \left\{(1+2^j|2^{-j}k - x_0|)^{s'} C_{j,k}\right\} \in \dot{b}^s_{p,q}$$

Main Theorem: Such an f has a **good decomposition**; the error term represents the singularities of the function f at x_0.

Moritoh-Tanaka (2013).

Further generalization: **singularities along the line** [MT].

The **uncertainty function** of Bony-Lerner [BoL, Section 9.1],

$$\lambda = 1 + |\xi_1| + |x_2||\xi| \quad (x = (x_1, x_2) \in \mathbb{R}^2,\ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2),$$

is considered. Then, **new two-microlocal Besov spaces** are defined.

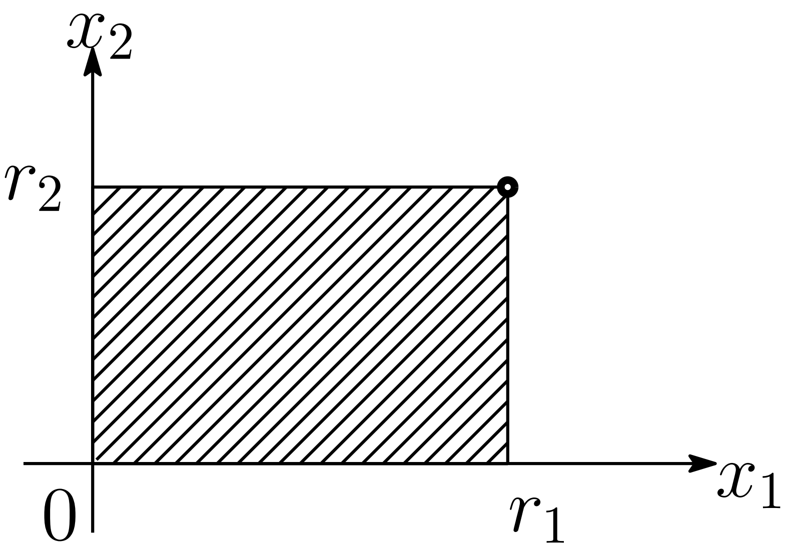
Such a λ stands for the uncertainty principle in **quantum mechanics** (**Weyl-Hörmander calculus**).

Remark. (1980$\sim$) Kashiwara, Laurent, Sjöstrand, Lebeau, Melrose, Ritter, Beals, ...

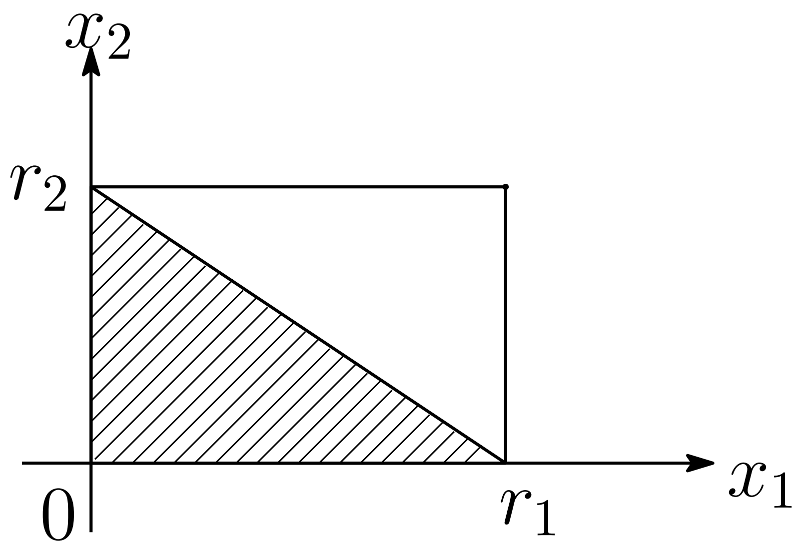
Our function spaces are two-microlocal version of the function spaces with **dominating mixed smoothness**

$$S\dot{B}^{\boldsymbol{s}}_{\boldsymbol{p},\boldsymbol{q}}(\mathbb{R}^2),\ \boldsymbol{s} \in \mathbb{R}^2,\ \boldsymbol{p} \in (\mathbb{R}_+ \cup \{\infty\})^2,\ \boldsymbol{q} \in (\mathbb{R}_+ \cup \{\infty\})^2.$$

Schmeisser-Triebel [ST]. Cf. Nikol'skij ('62), Pietsch ('78).



d. m. s.



anisotropic

Differentiability

Notation. For $f \in \mathcal{S}'(\mathbb{R}^2)$,

$$f_{j_1,j_2} := \mathcal{F}^{-1}\Big(\varphi_{j_1}(\xi_1)\,\varphi_{j_2}(\xi_2)\,(\mathcal{F}f)(\xi_1,\xi_2)\Big)(x_1,x_2),$$

where $\{\varphi_j\}$ stands for **a smooth resolution of unity**.

Then, $f \in S\dot{B}^{\boldsymbol{s}}_{\boldsymbol{p},\boldsymbol{q}}(\mathbb{R}^2)$ **is defined by** f_{j_1,j_2}.

The wavelet decomposition of $f \in \mathcal{S}'(\mathbb{R}^2)$ is

$$f(x) = \sum_{\boldsymbol{j}\in\mathbb{Z}^2}\sum_{\boldsymbol{k}\in\mathbb{Z}^2} C_{\boldsymbol{j},\boldsymbol{k}}\,\psi_{j_1,k_1}(x_1)\,\psi_{j_2,k_2}(x_2).$$

Remark. The sums in $\boldsymbol{j}$ and $\boldsymbol{k}$ are the double sums in j_1, j_2, and k_1, k_2, respectively.

Fact ([Ba], [V])**:** $f \in S\dot{B}^{\boldsymbol{s}}_{\boldsymbol{p},\boldsymbol{q}}(\mathbb{R}^2)$ if and only if

$$\sum_{j_2\in\mathbb{Z}} 2^{j_2\tilde{s}_2 q_2}\left(\sum_{k_2\in\mathbb{Z}}\left(\sum_{j_1\in\mathbb{Z}} 2^{j_1\tilde{s}_1 q_1}\left(\sum_{k_1\in\mathbb{Z}} |C_{\boldsymbol{j},\boldsymbol{k}}|^{p_1}\right)^{q_1/p_1}\right)^{p_2/q_1}\right)^{q_2/p_2} < \infty,$$

where $\tilde{s}_i = s_i + 1/2 - 1/p_i \quad (i = 1, 2)$.

New **two-microlocal estimate**: Let $s_3 \in \mathbb{R}$. Then $f \in \mathcal{S}'(\mathbb{R}^2)$ is said to belong to the **two-microlocal Besov space with dominating mixed smoothness** $SB^{(s_1,s_2),s_3}_{\boldsymbol{p},\boldsymbol{q}}(\mathbb{R}_{x_1})$, if the estimate of the **fact** holds with

$$C_{\boldsymbol{j},\boldsymbol{k}} \to (1 + 2^{j_1} + 2^{-j_2} \, |k_2| \, 2^{j_1 \vee j_2})^{s_3} \, C_{\boldsymbol{j},\boldsymbol{k}}$$

for wavelet coefficients $C_{\boldsymbol{j},\boldsymbol{k}}$ of f. Here, $j_1 \vee j_2 = \max\{j_1, j_2\}$.

Remark. The **uncertainty function** of [BoL, Section 9.1],

$\lambda = 1 + |\xi_1| + |x_2||\xi| \quad (x = (x_1, x_2) \in \mathbb{R}^2, \, \xi = (\xi_1, \xi_2) \in \mathbb{R}^2)$,

corresponds to "$1 + 2^{j_1} + 2^{-j_2} \, |k_2| \, 2^{j_1 \vee j_2}$" in the definition.

[BoL] J.-M. Bony and N. Lerner, *Quantification asymptotique et microlocalisations d'ordre supérieur. I*, Ann. Sci. École Norm. Sup. (4) **22** (1989), no. 3, 377–433.

Summary: The idea of **Moritoh-Tanaka (2013)** is

$$f \in S\dot{B}^{\boldsymbol{s}}_{\boldsymbol{p},\boldsymbol{q}}(\mathbb{R}^2) \overset{\textbf{Fact}}{\Longleftrightarrow} \{C_{\boldsymbol{j},\boldsymbol{k}}\} \in S\dot{b}^{\boldsymbol{s}}_{\boldsymbol{p},\boldsymbol{q}} \quad (\textbf{seq. sp.})$$

$$\downarrow \text{ Extension}$$

$$f \in SB^{(s_1,s_2),s_3}_{\boldsymbol{p},\boldsymbol{q}}(\mathbb{R}_{x_1}) \overset{\textbf{def}}{\Longleftrightarrow} \left\{\left(1 + 2^{j_1} + 2^{-j_2}\left|k_2\right| 2^{j_1 \vee j_2}\right)^{s_3} C_{\boldsymbol{j},\boldsymbol{k}}\right\} \in S\dot{b}^{\boldsymbol{s}}_{\boldsymbol{p},\boldsymbol{q}}$$

Main Theorem: Such an f has a **good decomposition**; the error term represents the singularities of the function f along the line $\mathbb{R}_{x_1}$.

More precisely, $f = \sum_{i=1}^{5} f_i$, where f_1, f_2, f_3 satisfy a **global** Besov type condition, and f_4, f_5 a **local** condition in the neighborhood of the x_1-axis.

What is a ridgelet?

Ridgelet = a combination of **wavelet and Radon** transformations

1998, Candès' thesis [C].
1999, Candès and Donoho, Ridgelets: a key to higher-dimensional intermittency? [CD].

Moritoh's wavelet transform [Mo1] enables us to detect **directional singularities** because of its microlocal properties.

Def. (Helgason [H], Ehrenpreis [E])

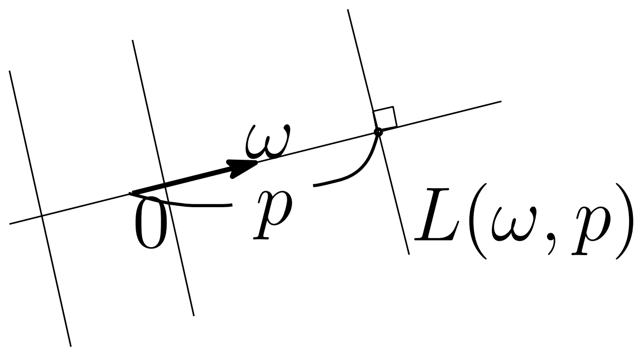
S^{n-1} ($\subset \mathbb{R}^n$): $(n-1)$-**dimensional unit sphere**.

$L(\omega, p) = \{x \in \mathbb{R}^n;\ x \cdot \omega = p\}$, where $\omega \in S^{n-1},\ p \in \mathbb{R}$, and $x \cdot \omega$ denotes the inner product of x and ω.

Then, for a function f on $\mathbb{R}^n$,

$$(\mathrm{R}f)(\omega, p) := \int_{L(\omega,p)} f(x)\, d\mu(x),$$

where $d\mu$ is the Lebesgue measure on $L(\omega, p)$.

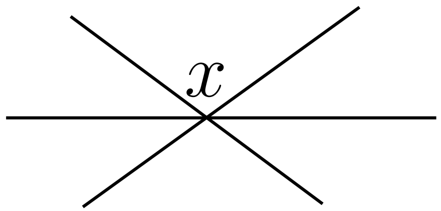


Def. Dual Radon transform of a function $g(\omega, p)$ on $S^{n-1} \times \mathbb{R}$ is defined as follows:

$$(\mathrm{R}^* g)(x) := \int_{S^{n-1}} g(\omega, x \cdot \omega)\, d\omega.$$

Then, we have, for some constant c, the following **reproducing formula**:

$$f(x) = c\,(-\Delta)^{(n-1)/2} (\mathrm{R}^* \mathrm{R} f)(x).$$



all hyperplanes through x

1. Notation.

$$\hat{g}(\omega, \hat{p}) := \mathcal{F}_{p \to \hat{p}} \left(g(\omega, p) \right).$$

2. Notation.

$$(\Lambda g)^{\wedge}(\omega, \hat{p}) := |\hat{p}|^{n-1} \hat{g}(\omega, \hat{p}).$$

3. Another form of the reproducing formula.

$$f(x) = c \, \mathrm{R}^{*}\big(\Lambda(\mathrm{R}f)\big)(x).$$

Projection-slice (PS) theorem: Let $\hat{f}(\xi)$ denote the n-dimensional Fourier transform of $f(x)$. Then

$$(\mathrm{R}f)^{\wedge}(\omega, \hat{p}) = \hat{f}(\hat{p}\,\omega).$$

Outline of the proof. Slice the whole space as $\mathbb{R}^n = \bigcup_{p \in \mathbb{R}} L(\omega, p)$. Then,

$$\begin{aligned} \hat{f}(\hat{p}\,\omega) &= \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\hat{p}\,\omega}\, dx \\ &= \int_{\mathbb{R}^1} \left[\int_{L(\omega,p)} f(x)\, d\mu(x) \right] e^{-ip\cdot\hat{p}}\, dp. \quad \square \end{aligned}$$

We defined our **wavelet transforms** from the viewpoint of microlocal analysis:

[Mo1] S. Moritoh, *Wavelet transforms in Euclidean spaces — their relation with wave front sets and Besov, Triebel-Lizorkin spaces —*, Tôhoku Math. J. 47 (1995), 555–565.

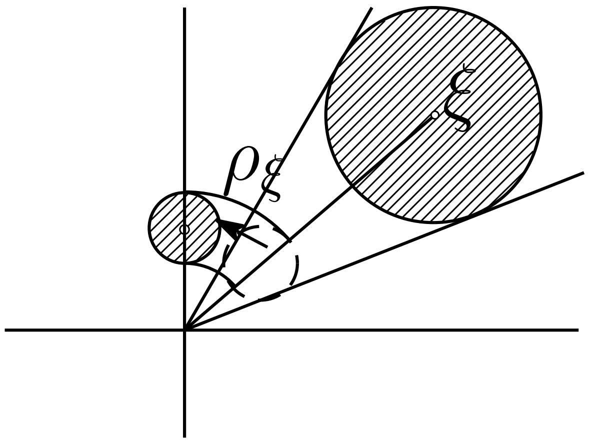
By using the wavelet $\psi(x)$, we define our **ridgelet function** $\varphi(\omega, p)$ on $S^{n-1} \times \mathbb{R}$ as follows:

$$\varphi(\omega, p) := \Lambda^{1/2}(\mathrm{R}\psi)(\omega, p). \tag{1}$$

Another representation:

$$\hat{\varphi}(\omega, \hat{p}) = |\hat{p}|^{(n-1)/2}\, \hat{\psi}(\hat{p}\,\omega).$$

For every $\xi \in \mathbb{R}^n - \{0\}$, rotate and dilate the wavelet ψ to define ψ_ξ. Define the ridgelet function φ_ξ similarly.



ψ_ξ**: rotation and dilation of** ψ

$$\hat{\psi}_\xi(\hat{x}) = \hat{\psi}(|\xi|^{-1}\rho_\xi \hat{x})$$

Remark. Our wavelet function ψ has **rotational invariance**. This invariance is essential for our definition of wavelet transformation, and plays an important role in detecting **directional singularities** of a function f, denoted by $WF(f)$, or $SS(f)$.

Def. Microlocal ridgelet transform of a function f is defined as follows:

$$(\mathcal{R}_\varphi f)(\omega, p; \xi) := \int_{\mathbb{R}^n} f(x) \overline{\varphi_\xi(\omega, x \cdot \omega - p)}\, dx. \qquad (2)$$

Another representation with Radon transformation:

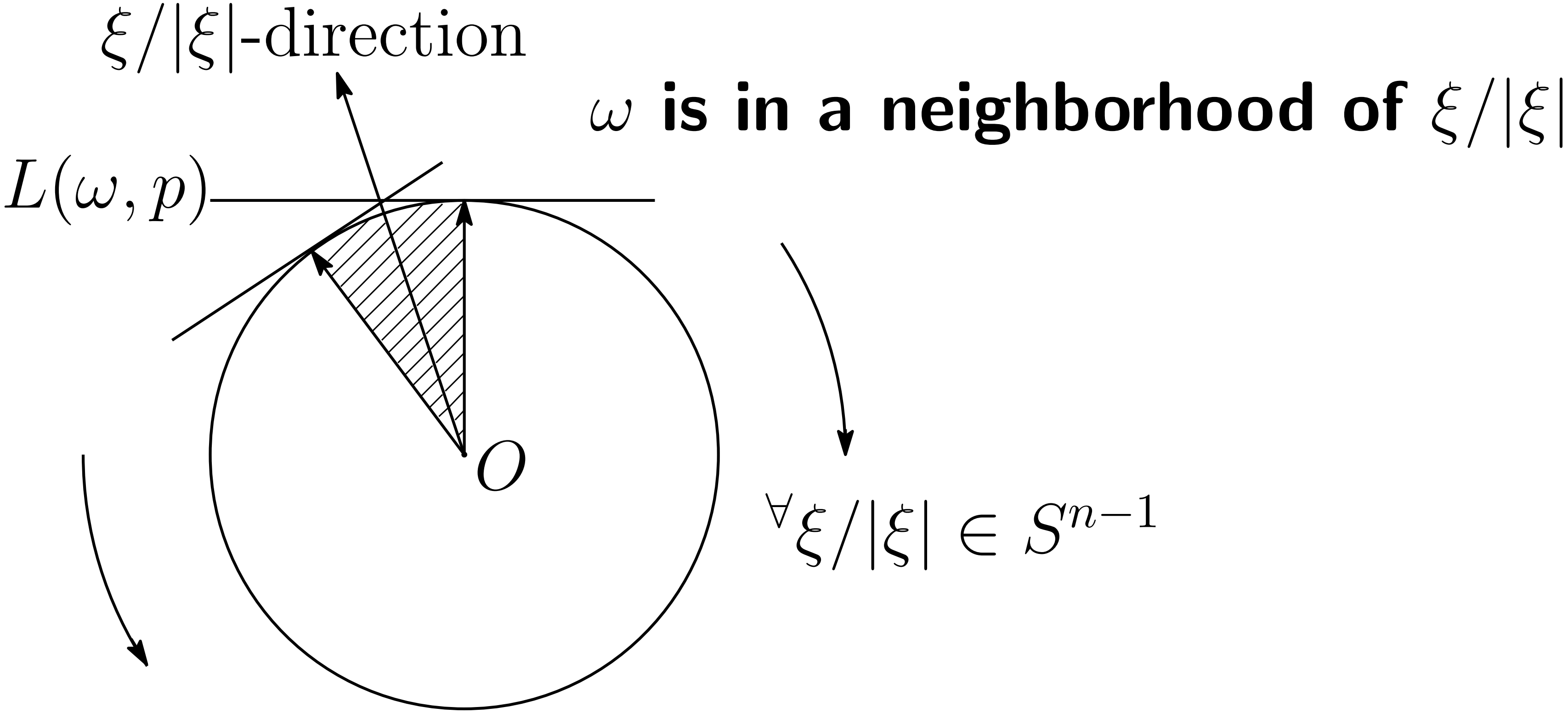
$$(\mathcal{R}_\varphi f)(\omega, p; \xi) = \int_{\mathbb{R}} (\mathrm{R}f)(\omega, q)\overline{\varphi_\xi(\omega, q-p)}\, dq. \qquad (3)$$

Our **reproducing formula** reads as follows (analogue of **Calderón's formula**).

Theorem.

$$f(x) = C_\varphi \int_{\mathbb{R}^n} \left[\int_{S^{n-1}} \int_{\mathbb{R}} (\mathcal{R}_\varphi f)(\omega, p; \xi) \times \right.$$

$$\left. \times\, \varphi_\xi(\omega, x \cdot \omega - p)\, dp\, d\omega \right] d\xi / |\xi|^n.$$



Remark 1. Fix a $\xi \in \mathbb{R}^n - \{0\}$. Microlocal ridgelet transform $(\mathcal{R}_\varphi f)(\omega, p\,;\xi)$ of a function f has **its support** $\omega \sim \xi/|\xi|$, and captures the data of $f(x)$ in the neighborhood of $L(\omega, p)$.

Remark 2. From another representation, we see that the data of **the Radon transform** $(\mathrm{R}f)(\omega, q)$**, in the neighborhood of** $q = p$, are captured.

These remarks explain why our ridgelet transform $(\mathcal{R}_\varphi f)(\omega, p\,;\xi)$ can be said to be **microlocalization of Candès' ridgelet transform**. See [Mo2]. See also

[FJW] M. Frazier, B. Jawerth and G. Weiss, Littlewood-Paley Theory and the Study of Function Spaces, CBMS Regional Conference Series 79, AMS, Providence, Rhode Island, 1991.

Another example (F.B.I. transformations).

Córdoba-Fefferman [CF] is used.

[CF] A. Córdoba and C. Fefferman, *Wave packets and Fourier integral operators*, Comm. Partial Differential Equations **3** (1978), no. 11, 979–1005.

See also Palamodov [Pa]. Put

$$g_\xi(x) := |\xi|^{n/4} \exp(-|\xi||x|^2/2 + i\xi \cdot x).$$

Then, the **F.B.I. transform** of a function f is defined as

$$(Tf)(x, \xi) := (f * \widetilde{g_\xi})(x).$$

Equivalent representation:

$$(Tf)(x,\xi)=\int_{\mathbb{R}^n} f(t)|\xi|^{n/4}\overline{\exp(-|\xi||t-x|^2/2+i\xi\cdot(t-x))}\,dt.$$

On the **Fourier** side:

$$(Tf)^\wedge(\hat{x},\xi) = \hat{f}(\hat{x})\,|\xi|^{-n/4}\,\exp\bigl(-|\hat{x}-\xi|^2/(2|\xi|)\bigr).$$

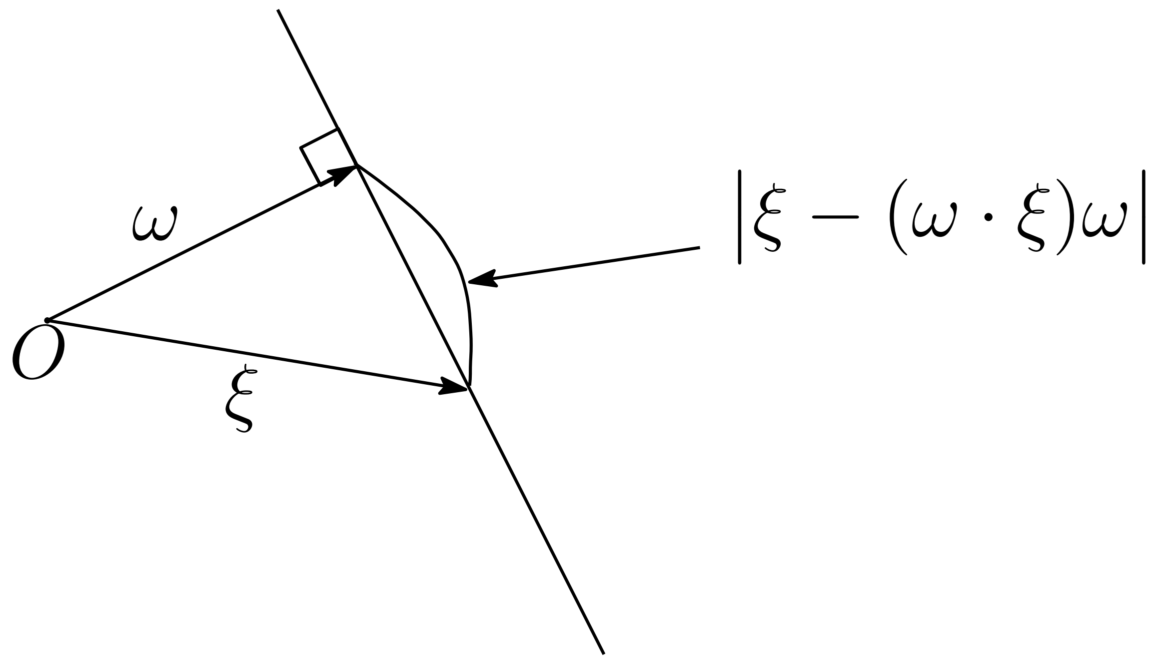
The **"almost inversion formula"** for the F.B.I. transformation is as follows:

$$f = \int_{\mathbb{R}^n} f * \widetilde{g_\xi} * g_\xi \; d\xi/|\xi|^n \,+\, Ef,$$

where the symbol of E belongs to the Hörmander class $S^{-1}_{1,0}$.

Now, **the Radon transform of the wave packet** is calculated as follows [Pa]:

$$(\mathrm{R}g_\xi)(\omega, p) = C\, \exp\Bigl(-[|\xi|p^2 + |\xi|^{-1}|\xi - (\omega\cdot\xi)\omega|^2]/2\Bigr).$$



Summary (1∼5)

1. wavelet series

$$f(x) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{jk}\, \psi(2^j x - k) = \sum_{j,k} c_{jk}\, \psi_{jk}$$

Analogue of Parseval: $$\int_{\mathbb{R}} |f(x)|^2\, dx = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |c_{jk}|^2.$$

2. Two-microlocal analysis of a function f measures **pointwise regularities** of f. The objects, whose pointwise oscillations are very rapid, are well described by two-microlocal analysis. Cf. turbulence.

Philosophy

Parseval:

$$f \in L_2(\mathbb{R}^n) \iff \text{wavelet coefficients } \{C_{j,k}\} \in l_2 \quad (\textbf{seq. sp.})$$

Extension:

$$f \in \dot{B}^s_{p,q}(\mathbb{R}^n) \overset{\textbf{Fact}}{\iff} \text{wavelet coeff. } \{C_{j,k}\} \in \dot{b}^s_{p,q} \quad (\textbf{seq. sp.})$$

$\downarrow$ Further extension

New function spaces $\overset{\textbf{def}}{\iff}$ more general conditions on $C_{j,k}$

Desired theorem: Every f belonging to the new function space has **a good decomposition**. **The error term** in this decomposition describes the very **singular part** of the function f.

3 and **4** are good illustrations of the **Philosophy**.

3. The idea of **Moritoh-Yamada (2004)** is

$$f \in \dot{B}^s_{p,q}(\mathbb{R}^n) \overset{\textbf{Fact}}{\Longleftrightarrow} \{C_{j,k}\} \in \dot{b}^s_{p,q} \quad (\textbf{seq. sp.})$$

$$\downarrow \text{ Extension}$$

$$f \in B^{s,s'}_{p,q}(x_0) \overset{\textbf{def}}{\Longleftrightarrow} \left\{(1+2^j|2^{-j}k - x_0|)^{s'} C_{j,k}\right\} \in \dot{b}^s_{p,q}$$

Main Theorem: Such an f has a **good decomposition**; the error term represents the singularities of the function f **at** x_0.

4. The idea of **Moritoh-Tanaka (2013)** is

$$f \in S\dot{B}^{\boldsymbol{s}}_{\boldsymbol{p},\boldsymbol{q}}(\mathbb{R}^2) \overset{\textbf{Fact}}{\iff} \{C_{\boldsymbol{j},\boldsymbol{k}}\} \in S\dot{b}^{\boldsymbol{s}}_{\boldsymbol{p},\boldsymbol{q}} \quad (\textbf{seq. sp.})$$

$$\downarrow \text{ Extension}$$

$$f \in SB^{(s_1,s_2),s_3}_{\boldsymbol{p},\boldsymbol{q}}(\mathbb{R}_{x_1}) \overset{\textbf{def}}{\iff} \left\{(1+2^{j_1}+2^{-j_2}|k_2|\,2^{j_1 \vee j_2})^{s_3}\, C_{\boldsymbol{j},\boldsymbol{k}}\right\} \in S\dot{b}^{\boldsymbol{s}}_{\boldsymbol{p},\boldsymbol{q}}$$

Main Theorem: Such an f has a **good decomposition**; the error term represents the singularities of the function f **along the line** $\mathbb{R}_{x_1}$.

5. Ridgelet transformation based on Radon

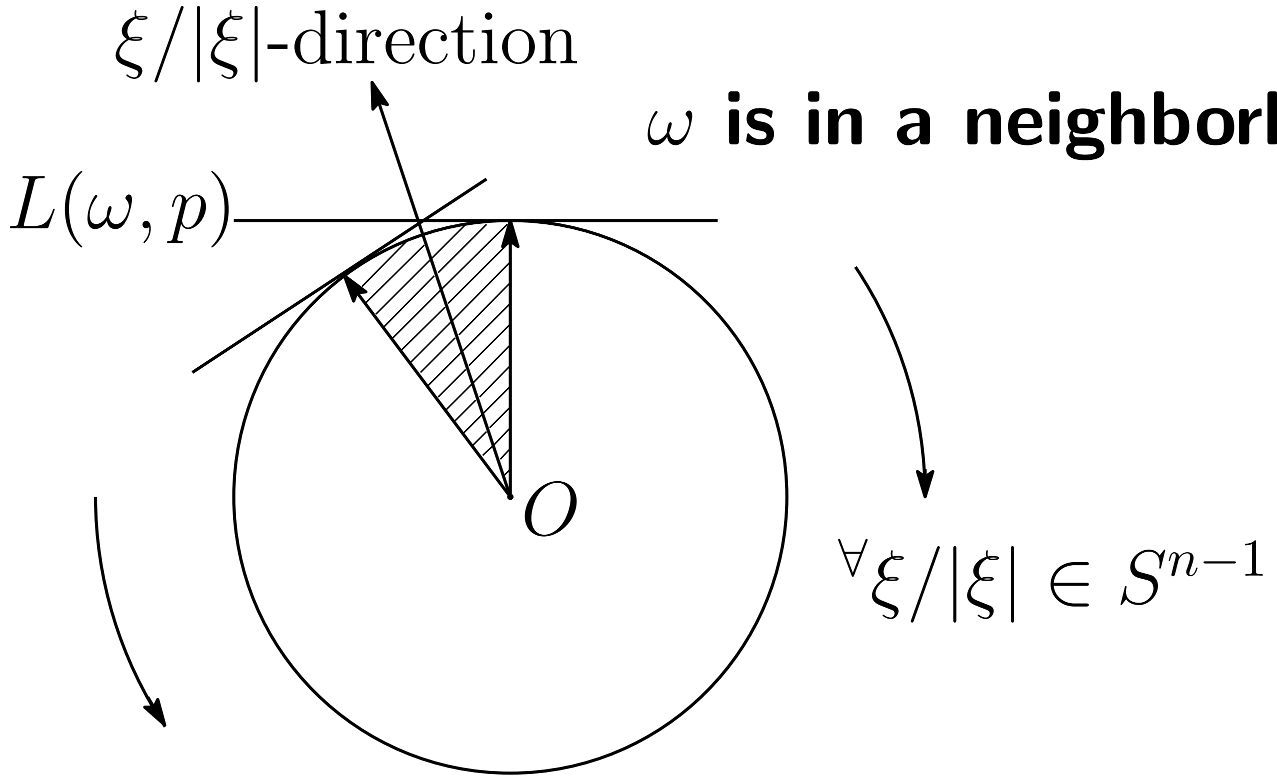
Reproducing formula (analogue of **Calderón**)

$$f(x) = C_\varphi \int_{\mathbb{R}^n} \left[\int_{S^{n-1}} \int_{\mathbb{R}} (\mathcal{R}_\varphi f)(\omega, p; \xi) \times \right.$$

$$\left. \times\, \varphi_\xi(\omega, x \cdot \omega - p)\, dp\, d\omega \right] d\xi / |\xi|^n.$$

Remark 1. Fix a $\xi \in \mathbb{R}^n - \{0\}$. Microlocal ridgelet transform $(\mathcal{R}_\varphi f)(\omega, p\,; \xi)$ of a function f has **its support** $\omega \sim \xi/|\xi|$, and captures the data of $f(x)$ in the neighborhood of $L(\omega, p)$.

Remark 2. We also see that the data of **the Radon transform** $(\mathrm{R}f)(\omega, q)$**, in the neighborhood of** $q = p$, are captured.



References

[Ba] D. B. Bazarkhanov, *Wavelet representations and equivalent normings for some function spaces of generalized mixed smoothness*, (Russian), Mat. Zh. **5** (2005), no. 2 (16), 12–16.

[Bo] J.-M. Bony, *Second microlocalization and propagation of singularities for semilinear hyperbolic equations*, Hyperbolic equations and related topics (Katata/Kyoto, 1984), 11–49, Academic Press, Boston, MA, 1986.

[BoL] J.-M. Bony and N. Lerner, *Quantification asymptotique et microlocalisations d'ordre supérieur. I*, Ann. Sci. École Norm. Sup. (4) **22** (1989), no. 3, 377–433.

[C] E. Candès, *Ridgelets: theory and applications*, Ph. D. thesis, Department of Statistics, Stanford University, 1998.

[CD] E. J. Candès and D. L. Donoho, *Ridgelets: a key to higher-dimensional intermittency?*, R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci. **357** (1999), no. 1760, 2495–2509.

[CF] A. Córdoba and C. Fefferman, *Wave packets and Fourier integral operators*, Comm. Partial Differential Equations **3** (1978), no. 11, 979–1005.

[D] I. Daubechies, *Orthonormal bases of compactly supported wavelets*, Comm. Pure Appl. Math. **41** (1988), no. 7, 909–996.

[FJW] M. Frazier, B. Jawerth and G. Weiss, Littlewood-Paley Theory and the Study of Function Spaces, CBMS Regional Conference Series 79, AMS, Providence, Rhode Island, 1991.

[E] L. Ehrenpreis, The universality of the Radon transform, with an appendix by P. Kuchment and E. T. Quinto, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2003.

[H] S. Helgason, The Radon Transform, Second Edition, Birkhäuser, Boston, Basel, Berlin, 1999.

[JM] S. Jaffard and Y. Meyer, *Wavelet methods for pointwise regularity and local oscillations of functions*, Mem. Amer. Math. Soc. **587** (1996).

[Mo1] S. Moritoh, *Wavelet transforms in Euclidean spaces — their relation with wave front sets and Besov, Triebel-Lizorkin spaces —*, Tôhoku Math. J. 47 (1995), 555–565.

[Mo2] S. Moritoh, *Radon transform and its application* (in Japanese), Archives of Mathematics, Nara-WU, Vol. 1 (2004), 42–45.

[MT] S. Moritoh and Y. Tanaka, *Microlocal Besov spaces and dominating mixed smoothness*, preprint (2013) .

[MY] S. Moritoh and T. Yamada, *Two-microlocal Besov spaces and wavelets*, Rev. Mat. Iberoamericana **20** (2004), no. 1, 277–283.

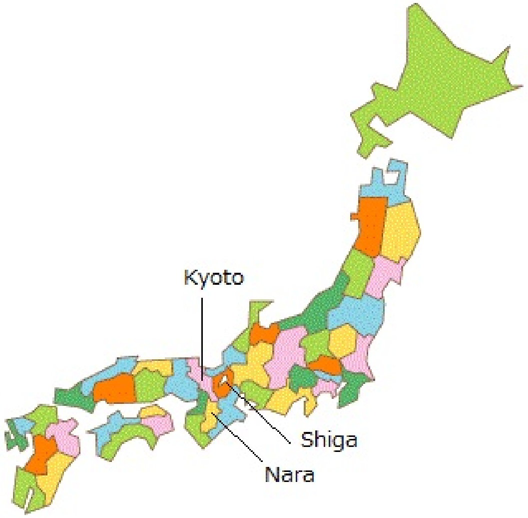
[Pa] V. Palamodov, Reconstructive integral geometry, Monographs in Mathematics, 98. Birkhuser Verlag, Basel, 2004.

[Pe] J. Peetre, New thoughts on Besov spaces, Duke University Mathematics Series, No. 1, Mathematics Department, Duke University, Durham, N.C., 1976.

[ST] H.-J. Schmeisser and H. Triebel, Topics in Fourier analysis and function spaces, A Wiley-Interscience Publication. John Wiley & Sons, Ltd., Chichester, 1987.

[V] J. Vybiral, *Function spaces with dominating mixed smoothness*, Dissertationes Math. (Rozprawy Mat.) **436** (2006).

Thanks for your attention!



Biwako Lake, Shiga (near my house)

Kohfukuji Temple, Nara (near my univ.)