Algebras of Lorch Analytic Mappings

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Definition

A mapping $f : U \to E$ is Lorch analytic if given any $z_0 \in U$ there exists $r > 0$ and there exist (unique) elements $a_n \in E$, such that $B_r(z_0) \subset U$ and $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$, for all $z \in B_r(z_0)$.

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- If $f : U \to E$ is Lorch analytic in U then it is continuous and Fréchet differentiable in U and hence it is a holomorphic mapping in the usual sense . The converse is not true.

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- Glickfeld presented a generalization of the Mittag-Leffler's Theorem .

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- For each $n \in \mathbb{N}, \quad P \in \mathcal{P}_L(^nE)$ if and only if there exists $a \in E$ such that $P(z) = az^n$ for all $z \in E$.
- So, for each $n \in \mathbb{N}, \quad \mathcal{P}_L(^nE)$ is isometrically isomorphic to E.

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For $f : U \to F$ where $F =$ complex Banach space,

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- E separable $\Rightarrow (\mathcal{H}_d(U, F), \tau_d)$ is a Fréchet space (Dineen and Venkova, for $F = \mathbb{C}$)

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Well known:

- $\mathcal{M}(\mathcal{A})$ coincides with the set of the closed maximal ideals of $\mathcal{A}$.
- As a consequence:

$$\mathcal{R}(\mathcal{A}) = \bigcap_{\varphi \in \mathcal{M}(\mathcal{A})} \varphi^{-1}(0).$$

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Proposition

E is semi-simple if and only if $\mathcal{H}_L(E)$ is semi-simple.

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- For each $\phi \in \mathcal{M}(E)$, f_ϕ is defined in $\Delta_r(\phi(z_0)) = \phi(B_r(z_0))$ by

$$f_\phi(\lambda) = \sum_{n=0}^{\infty} \phi(a_n)(\lambda - \phi(z_0))^n.$$

Lemma

$\psi(P_{e,1}) \in \Delta_r(\psi_0(z_0))$ for every $\psi \in \mathcal{M}(\mathcal{H}_L(B_r(z_0), E))$.

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Clearly $f \in \mathcal{H}_L(B_r(z_0), E)$ and

$$\begin{aligned} \psi(f) &= \sum_{n=0}^{\infty} \psi_0 \left(\frac{e}{r^n \exp(in\theta)} \right) (\psi(P_{e,1}) - \psi_0(z_0))^n \\ &= \sum_{n=0}^{\infty} \frac{1}{r^n \exp(in\theta)} \lambda^n \exp(in\theta) = \sum_{n=0}^{\infty} \left(\frac{\lambda}{r} \right)^n. \end{aligned}$$

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Since this series converges (to $\psi(f)$), we must have $\frac{\lambda}{r} < 1$ and hence $\lambda < r$. So, $|\psi(P_{e,1}) - \psi_0(z_0)| = \lambda < r$, i.e., $\psi(P_{e,1}) \in \Delta_r(\psi_0(z_0))$.

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Now, for $f = P_{w_0,0}$ we have $f_\phi(\lambda_0) = f_\phi \circ \phi(z) = \phi \circ P_{w_0,0}(z) = \phi(w_0) \neq 0$ and from this we infer that $\delta(\phi, \lambda_0) \neq 0$.

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- $\delta(\phi, \lambda_0)$ is continuous:

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So, δ is injective.

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Now $\psi_0(a) = \psi(P_{a,1})$ is an element of $\mathcal{M}(E), \psi(P_{e,1}) \in \Delta_r(\psi_0(z_0))$ and

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take $\psi \in \mathcal{M}(\mathcal{H}_L(B_r(z_0), E))$ arbitrary.

$f \in \mathcal{H}_L(B_r(z_0), E) \Rightarrow f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ for all $z \in B_r(z_0)$

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By definition,

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So, δ is onto.

The algebra $\mathcal{H}_L(B_r(z_0), E)$

Consider $\mathcal{M}(\mathcal{H}_L(B_r(z_0), E))$ and $\mathcal{M}(E)$ endowed with Gelfand topology τ_G and $\bigcup_{\phi \in \mathcal{M}(E)} \{(\phi, \lambda_0)\,;\, \lambda_0 \in \Delta_r(\phi(z_0))\}$ endowed with the topology τ_π induced by the product topology in $\mathcal{M}(E) \times \mathbb{C}$.

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Theorem

The mapping δ defined by $\delta(\phi, \lambda_0)(f) = f_\phi(\lambda_0)$ establishes a homeomorphism between

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- $\psi_\alpha(P_{e,1}) \to \psi(P_{e,1})$ in $\mathbb{C}$.

Hence $\delta^{-1}(\psi_\alpha) \to \delta^{-1}(\psi)$ in $\bigcup_{\phi \in \mathcal{M}(E)} \{(\phi, \lambda) ; \lambda \in \Delta_r(\phi(x_0))\}$.

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In order to show the continuity of δ, take any net $((\phi_\alpha, \lambda_\alpha))_{\alpha \in L}$ in $\bigcup_{\phi \in \mathcal{M}(E)} \{(\phi, \lambda) ; \lambda \in \Delta_r(\phi(z_0))\}$ and $(\phi_0, \lambda_0) \in \bigcup_{\phi \in \mathcal{M}(E)} \{(\phi, \lambda) ; \lambda \in \Delta_r(\phi(z_0))\}$ such that $(\phi_\alpha, \lambda_\alpha) \stackrel{\tau_\pi}{\to} (\phi_0, \lambda_0)$.

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- $\phi_\alpha \overset{\tau_G}{\to} \phi_0$ and $\lambda_\alpha \to \lambda_0 \Rightarrow$ there exists $\alpha_\epsilon \in L$ such that $|(\phi_0 - \phi_\alpha)(f(z))| < \epsilon/2$, $|\lambda_0^\alpha - \lambda_0| = |\phi_\alpha(z) - \phi_0(z)| < \delta_\epsilon/2$ and $|\lambda_\alpha - \lambda_0| < \delta_\epsilon/2$ for all $\alpha \geq \alpha_\epsilon$.

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Hence $\lambda_\alpha \in \Delta_{\delta_\epsilon}(\lambda_0^\alpha) = \phi_\alpha(B_{\delta_\epsilon}(z))$ for all $\alpha \geq \alpha_\epsilon$ and so there exists $w_\alpha \in B_{\delta_\epsilon}(z)$ such that $\phi_\alpha(w_\alpha) = \lambda_\alpha$.

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$|f_{\phi_\alpha}(\lambda_\alpha) - f_{\phi_0}(\lambda_0)| \leq \left|f_{\phi_\alpha}(\lambda_\alpha) - f_{\phi_\alpha}(\lambda_0^\alpha)\right| + \left|f_{\phi_\alpha}(\lambda_0^\alpha) - f_{\phi_0}(\lambda_0)\right| < \epsilon/2 + \epsilon/2 = \epsilon$ for all $\alpha \geq \alpha_\epsilon$.

The algebra $\mathcal{H}_L(B_r(z_0), E)$

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Theorem

Let E be a commutative Banach algebra with a unit element e. The spectrum $\mathcal{M}(\mathcal{H}_L(B_E))$ is homeomorphic to $\mathcal{M}(E) \times \Delta$ by the mapping

$$\delta : \mathcal{M}(E) \times \Delta \longrightarrow \mathcal{M}(\mathcal{H}_L(B_E))$$

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As a consequence of the description of the spectrum of $\mathcal{H}_L(B_r(z_0), E)$ we get:

Proposition

E is semi-simple if and only if $\mathcal{H}_L(B_r(z_0), E)$ is semi-simple.

The algebra $\mathcal{H}_L(B_r(z_0), E)$

Proof: For $\mathcal{A} = E$ or $\mathcal{H}_L(B_r(z_0), E)$, $\mathcal{A}$ is semi-simple $\Leftrightarrow \mathcal{R}(\mathcal{A}) = \{0\}$ where $\mathcal{R}(\mathcal{A}) = \bigcap_{\varphi \in \mathcal{M}(\mathcal{A})} \varphi^{-1}(0)$.

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$f \in \mathcal{R}\left(\mathcal{H}_L(B_r(z_0), E)\right) \Leftrightarrow f_\phi(\lambda) = \delta(\phi, \lambda)(f) = 0 \;\; \forall \phi \in \mathcal{M}(E), \;\; \forall \lambda \in \Delta_r(\phi(z_0))$.

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Since $\phi(z) \in \Delta_r(\phi(z_0)) \ \ \forall\, z \in B_r(z_0)$ and $\forall\, \phi \in \mathcal{M}(E)$, this gives

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$\Rightarrow f(z) = 0$ for every $z \in B_r(z_0)$.

$\Rightarrow f \equiv 0$ whenever $f \in \mathcal{R}\left(\mathcal{H}_L(B_r(z_0), E)\right)$.

The algebra $\mathcal{H}_L(B_r(z_0), E)$

Proof: For $\mathcal{A} = E$ or $\mathcal{H}_L(B_r(z_0), E)$, $\mathcal{A}$ is semi-simple $\Leftrightarrow \mathcal{R}(\mathcal{A}) = \{0\}$ where $\mathcal{R}(\mathcal{A}) = \bigcap_{\varphi \in \mathcal{M}(\mathcal{A})} \varphi^{-1}(0)$.

Suppose that E is semi-simple.

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$\Rightarrow \mathcal{H}_L(B_r(z_0), E)$ is semi-simple.

The algebra $\mathcal{H}_L(B_r(z_0), E)$

Suppose that $\mathcal{H}_L(B_r(z_0), E)$ is semi-simple and take any $a \in \mathcal{R}(E)$.

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$\psi \in \mathcal{M}(\mathcal{H}_L(B_r(z_0), E)) \Rightarrow \quad \exists(\phi, \lambda) \in \mathcal{M}(E) \times \Delta_r(\phi(z_0))$ such that $\psi(f) = f_\phi(\lambda)$ for every $f \in \mathcal{H}_L(B_r(z_0), E)$.

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$\Rightarrow \quad E$ is semi-simple.

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