Dynamical Sampling

Ursula Molter
Universidad de Buenos Aires
IMAS - CONICET

WIdaBA 2014

Collaborators:

Collaborators:

- Akram Aldroubi - Vanderbilt University

Collaborators:

- Akram Aldroubi - Vanderbilt University
- Carlos Cabrelli - UBA - IMAS

Collaborators:

- Akram Aldroubi - Vanderbilt University
- Carlos Cabrelli - UBA - IMAS
- Sui Tang - Vanderbilt University

Objective:

$f : H \longrightarrow \mathbb{C}$ is a signal. We can measure $f(x_i)$

and want to recover f.

Objective:

$f : H \longrightarrow \mathbb{C}$ is a signal. We can measure $f(x_i)$ and want to recover f.

When is that possible?

Theorem of Shanon - Whitacker:

A bandlimited function can be perfectly reconstructed from a countable sequence of samples if the bandlimit B is no greater than half *the sampling rate* (samples per second).

Theorem of Shanon - Whitacker:

A bandlimited function can be perfectly reconstructed from a countable sequence of samples if the bandlimit B is no greater than half *the sampling rate* (samples per second).

So, if f is such that $\hat{f} \subseteq \left[-\frac{1}{2}, \frac{1}{2}\right]$ we can recover f if we know $f(n)$, $n \in \mathbb{Z}$.

What happens if we only know $f(2k)$?

Theorem of Shanon - Whitacker:

A bandlimited function can be perfectly reconstructed from a countable sequence of samples if the bandlimit B is no greater than half ***the sampling rate*** (samples per second).

So, if f is such that $\hat{f} \subseteq \left[-\frac{1}{2}, \frac{1}{2}\right]$ we can recover f if we know $f(n)$, $n \in \mathbb{Z}$.

What happens if we only know $f(2k)$?

Not possible to recover f?

Dynamical Sampling!!

If in addition I am given

$Af(2k)$ when A is some operator

Can we then recover f?

We seek conditions on:

A, Ω such that any function in a given class can be recovered from the measurements:

$$Y = \{ f(x_i), Af(x_i), A^2 f(x_i), \dots ; x_i \in \Omega \}$$

Or even simpler:

If $\hat{f} \subseteq [-\frac{1}{2}, \frac{1}{2}]$ and $X \subseteq \mathbb{Z}$,

we seek conditions on A and X

such that f can be "recovered from

$$Y = \{ f(i), Af(i), A^2 f(i), \dots ; i \in X \}$$

Or even simpler:

If $\hat{f} \subseteq [-\frac{1}{2}, \frac{1}{2}]$ and $X \subseteq \mathbb{Z}$,

we seek conditions on A and X such that f can be "recovered from

$$Y = \{ f(i), Af(i), A^2 f(i), \dots ; i \in X \}$$

equivalently

Or even simpler:

If $\hat{f} \subseteq [-\frac{1}{2}, \frac{1}{2}]$ and $X \subseteq \mathbb{Z}$,

we seek conditions on A and X

such that f can be "recovered from

$$Y = \{ f(i), Af(i), A^2 f(i), \dots ; i \in X \}$$

equivalently

$$c_1 \|f\|_2^2 \leq \sum_{j \in \mathbb{N}} \sum_{i \in X} |\langle f, A^{*j} e_i \rangle|^2 \leq c_2 \|f\|_2^2$$

or:

$$\left\{ A^{*l} e_i : l = 0, \dots, L,\ i \in X \right\}$$ (L could be ∞)

is a frame for $\mathcal{H}$

or:

$$\{ A^{*\ell} e_i : \ell = 0, \dots, L,\ i \in X \}$$ (L could be ∞)

is a frame for $\mathcal{H}$

equivalently, if $A^* = P C P^{-1}$

$$\{ C^\ell b_i : i \in X,\ \ell = 0, \dots L \}$$ with $b_i = P^{-1} e_i$

is a frame for $\mathcal{H}$

Case I: finite dimension.

$f \in \mathbb{C}^d$, $A \in \mathbb{C}^{d \times d}$, $X \subseteq \{1, \dots, d\}$

$\{A^{*l} e_i : l = 0, \dots, L, \ i \in X\}$ needs to span $\mathbb{C}^d$

$A^* = \overbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{bmatrix}}^{D}$ $\lambda_i \neq \lambda_j$, $i \neq j$ NO way! but

$A^* = U D U^*$, col.(1) $U = \begin{bmatrix} | \\ | \\ \vdots \\ | \end{bmatrix} \Longrightarrow$

$\{A^{*l} e_1 : l = 1, \dots, d\}$ are l.i.

$A^{*l} e_1 = \begin{bmatrix} \lambda_1^l \\ \vdots \\ \lambda_d^l \end{bmatrix}$ and so they span $\mathbb{C}^d$

$\{A^{*l} e_1 : l = 1, \ldots, d\}$ are l. i.

$$A^{*l} e_1 = \begin{bmatrix} \lambda_1^l \\ \vdots \\ \lambda_d^l \end{bmatrix}$$ and so they span $\mathbb{C}^d$

Note that $\{A^{*l} e_i : i \in X, 0 \le l \le L\}$ is a frame, and $A^* = PDP^{-1}$, iff $\{D^l b_i : i \in X, 0 \le l \le L\}$ is a frame

Moreover: Since every matrix in $\mathbb{C}^d$ can be written as $A = PJP^{-1}$ where J is the Jordan form of A, we will focus on the question: given $\{b_i\}_{i\in\Omega} \in \mathbb{C}^d$

and $J = D + \begin{bmatrix} N_1 & & \\ & \ddots & \\ & & N_l \end{bmatrix}$ N_i cyclic nilpotent

which are the conditions on $\{b_i\}$ such that:

$$\text{span}\left\{ J^l b_i : i \in \Omega, \ l = 0, \dots, \right\} = \mathbb{C}^d$$

which are the conditions on $\{b_i\}$ such that:

$$\operatorname{span}\{ J^{l} b_i : i \in \Omega,\ l = 0, \dots \} = \mathbb{C}^d$$

But: $\mathbb{C}^d = V_1 \oplus \dots \oplus V_n$, with V_s, J invariant,

and $V_s = V_s^{1} \oplus \dots \oplus V_s^{n_s}$, with V_s^{j} cyclic

If W_s = span of the cyclic vectors of V_s^{j}, $j = 1, \dots, n_s$ then:

which are the conditions on $\{b_i\}$ such that:

$$\text{span}\left\{ J^l b_i : i \in \Omega,\ l = 0, \dots \right\} = \mathbb{C}^d$$

But: $\mathbb{C}^d = V_1 \oplus \dots \oplus V_n$, with V_s, J invariant, and $V_s = V_s^1 \oplus \dots \oplus V_s^{a_s}$, with V_s^j cyclic

If W_s = span of the cyclic vectors of V_s^j, $j = 1, \dots, n_s$ then:

$*$ iff $\text{span}\{ P_{W_s} b_i : i \in \Omega \} = W_s$, $s = 1, \dots, n$.

Example: $J = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ work.

$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \; J\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \; J^2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 9 \end{bmatrix}$ are l.i.

Example: $J = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ work.

$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \; J\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \; J^2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 9 \end{bmatrix}$ are l.i.

Note that you need at least as many vectors b, as the largest number of cyclic components of any V_s.

Infinite dimensional case:

$\mathcal{H} = l_2(\mathbb{N})$. $A : l_2(\mathbb{N}) \to l_2(\mathbb{N})$,

bounded, selfadjoint and compact.

$$A = P^{-1} D P \text{, with}$$

$$D = \sum_j \lambda_j P_j, \quad \lambda_i \in \mathbb{R}$$

P_j orthogonal projections: $\sum P_i = I \wedge P_i P_k = 0,\ i \neq k$

Completeness:

$$\{ D^l b_i : i \in \Omega, l = 0, \dots, l_i \} \text{ is complete in } l_2$$

iff

$$\{ P_j (b_i) : i \in \Omega \} \text{ is complete in } P_j(l_2(\mathbb{N})) \ \forall j.$$

Minimality:

If Ω is finite, the set:

$\{ D^l b_i : i \in \Omega,\ l = 0, \ldots, l_i \}$ is never minimal!

$l_i = \infty$ or $l_i = \deg q^D_{b_i} \rightarrow D$-annihilator of b_i

Minimality:

If Ω is finite, the set:

$$\{ D^l b_i : i \in \Omega, \ l = 0, \dots, l_i \} \text{ is never minimal!}$$

$l_i = \infty$ or $l_i = \deg q^D_{b_i} \to$ D-annihilator of b_i

Theorem of Müntz-Szász: Let $0 \leq m_1 \leq m_2 \leq \dots \in \mathbb{N}$

x^{m_k} is complete in $C[0,1]$ iff $\sum_{k=2}^{\infty} \frac{1}{m_k} = \infty$ $(m_1 = 0)$

Corollary: If b is such that $l = \infty$ then $\{D^l b : l \in \mathbb{N}\}$ is NOT minimal.

Pf: $\overline{\text{span}}\{D^l b,\ l \in \mathbb{N}\} =$ (Müntz - Szász)

$$\overline{\text{span}}\left\{D^{n_i} b,\ \sum \frac{1}{n_i} = \infty\right\}.$$

$$\left\{ D^l b_i : i \in \Omega,\ l = 0, \dots, l_i \right\},\ \bigcup_{i \in \Omega} \overline{\operatorname{span}} \{ P_i (b_s))$$

is complete $\to$ never minimal.

Could it be a frame?

$$\left\{ D^l b_i : i \in \Omega,\ l = 0, \dots, l_i \right\},\quad \bigcup_{i \in \Omega} \overline{\operatorname{span}} \{ P_i(b_s) \}$$

is complete $\to$ never minimal.

Could it be a frame?

If $\exists\ 0 < C_1 \leq C_2 < \infty$:

$$C_1 \leq \inf \{ \| D^l b_i \| \} \leq \sup \{ \| D^l b_i \| \} \leq C_2$$

then NO frame!

Why?

Feichtinger conjecture: (equivalent to Kadison-Singer)

Feichtinger ~~conjecture~~ theorem: (equivalent to Kadison - Singer)

Every norm-bounded sequence in $\mathcal{H}$ that is a frame is the finite union of Riesz sequences

R.S. $:=$ minimal + Bessel - but D'b never minimal

So, the only chance to be a frame, is if

$\inf\{\|D^\ell b_i\|\}_i = 0$. (the upper bound has to hold)

However:

So, the only chance to be a frame, is if

$\inf \{ \|D^l b_i\| \}_i = 0$. (the upper bound has to hold)

However:

Theorem: If $1 \notin \overline{\{D_{ii}\}} \Rightarrow \{D^l b_i\}$ NOT frame.

Pf: $\cdot\ \lambda_j > 1$, $P_j(b_i) \neq 0$, $l_i = \infty \Rightarrow$

$$\|D^l b_i\|^2 \geq |\lambda_j|^{2l} \|P_j b_i\|^2$$

$\cdot\ \lambda_j < 1$ (1 not accumulation point!)

then $\underbrace{\ \ }_{\text{for } \varepsilon > 0}$ we can find $f : \sum_{i \in \Omega} \sum_{l=0}^{l_i} |\langle f, D^l b_i \rangle|^2 < \varepsilon$

Thank You!

Friday, July 18

Surprise!

bear with me a bit:

Jorge Antezana points out Carleson's Thm:

and relates the property of being a frame, to the property of $\{D_{x_i}\}$ to be an interpolating sequence for $H^2(D) = \{ f \in H(D) : f(z) = \sum a_n z^n : a_n \in l_2 \}$

holomorphic functions on D

So we have:

If $D_{ii} = \lambda_i < 1$. $b_i = 1 - \lambda_i^2$, $b = \{b_i\} \in l_2$

$\{D^l b : l = 0, \dots\}$ frame for $l_2(\mathbb{N})$ iff

$$\inf_n \prod_{k \neq n} \frac{|\lambda_n - \lambda_k|}{|1 - \overline{\lambda_n} \lambda_k|} \geq \delta > 0 .$$

Example: $\lambda_k^2 = 1 - \frac{1}{2^k}$.

Thank You

very much !!