Bloch functions on the unit ball of an infinite dimensional Hilbert space

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The Bloch space on the unit disk

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Theorem [Timoney 1980, Zhu 2005]. The spaces $X_i := \{f \in H(B_n) : \|f\|_{B_i} < \infty\}$ are equal for $i = 1, 2, 3$ and the three semi-norms are equivalent.

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Remark. $\|\cdot\|_{B_i}$ is not invariant by automorphisms of B_n: $\|f \circ \varphi\|_{B_i} \neq \|f\|_{B_i}$.

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Theorem [Timoney, 1980]. $f \in \mathcal{B}(B_n)$ if and only if $f \in X_1$, that is,

$$\|f\|_B < \infty \text{ if and only if } \|f\|_{B_1} := \sup_{z \in B_n} (1 - \|z\|^2) \|\nabla f(z)\| < \infty.$$

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For any $a \in B_n$, we consider the analytic map $m_a : B_n \longrightarrow \mathbb{C}^n$ given by

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To define the analogues of Möbius transformations on B_n, we consider $P_a : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, the orthogonal projection along the one-dimensional subspace spanned by a, that is,

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The biholomorphic automorphisms of B_n are compositions of φ_a with unitary transformations U of $\mathbb{C}^n$.

Zhu's results. Connection with the invariant gradient

Definitions.

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$$\|f\|_{Bloch} = |f(0)| + \|f\|_B \longrightarrow (\mathcal{B}(B_n), \| \cdot \|_{Bloch}) \text{ is a Banach space.}$$

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- Given a holomorphic function $f : B_E \to \mathbb{C}$ and $x \in B_E$, we will denote $f'(x) = \nabla f(x) = \left(\frac{\partial f}{\partial x_k}(x)\right)_{k\in\Gamma} \in E^*$ such that for $f'(x) \in E^*$ we have

$$f'(x)(z) = \sum_{k\in\Gamma} \frac{\partial f}{\partial x_k}(x) z_k = \langle z, \overline{\nabla f(x)} \rangle.$$

Definition of $\mathcal{B}(B_E)$ and Main Questions

Definition. The Bloch space $\mathcal{B}(B_E)$ is the space of holomorphic functions $f : B_E \to \mathbb{C}$:

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Other equivalent norms for $\mathcal{B}(B_E)$

1. Let $\mathcal{R}(f)(x) = \langle \nabla f(x), \bar{x} \rangle = \sum_{k \in \Gamma} x_k \frac{\partial f}{\partial x_k}(x)$ for any $x \in B_E$.

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The automorphisms of the unit ball B_E turn to be compositions of such analogues Möbius transformations with unitary transformations U of E.

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Remark. For any $f \in \mathcal{B}_{inv}(B_E)$ and an automorphism φ of B_E, we have

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Tools to prove first inequality $\|f\|_{\mathcal{B}(B_E)} \leq \|f\|_{inv}$

Proposition. If $f : B_E \to \mathbb{C}$ is a holomorphic function then

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Lema. Let $f : B_E \to \mathbb{C}$ holomorphic s.t. $\|f\|_{\mathcal{B}(B_E)} < \infty$. Let $x \in B_E$ and $y \in E$, $\|y\| = 1$, such that $\langle x, y\rangle = 0$. Then

$$|\langle \nabla f(x), y\rangle|\sqrt{(1 - \|x\|^2)} \leq \frac{\sqrt{31}}{2}\|f\|_{\mathcal{B}(B_E)}.$$

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$$\sqrt{\frac{|\langle \varphi_x'(0)(\nabla f(x)), \varphi_x'(0)^{-1}(w)\rangle|}{\|\varphi_x'(0)^{-1}(w)\|}} \leq |\langle \nabla f(x), \frac{x}{\|x\|}\rangle|(1 - \|x\|^2) + |\langle \nabla f(x), \frac{y}{\|y\|}\rangle|\sqrt{1 - \|x\|^2}.$$

Lema. Let $f : B_E \to \mathbb{C}$ holomorphic s.t. $\|f\|_{\mathcal{B}(B_E)} < \infty$. Let $x \in B_E$ and $y \in E$, $\|y\| = 1$, such that $\langle x, y\rangle = 0$. Then

$$|\langle \nabla f(x), y\rangle|\sqrt{(1 - \|x\|^2)} \leq \frac{\sqrt{31}}{2}\|f\|_{\mathcal{B}(B_E)}.$$

Final of proof. $\|\widetilde{\nabla} f(x)\| \leq \|f\|_{\mathcal{B}(B_E)} + \frac{\sqrt{31}}{2}\|f\|_{\mathcal{B}(B_E)} \leq \left(1 + \frac{\sqrt{31}}{2}\right)\|f\|_{\mathcal{B}(B_E)}$.

Re-definition of the Bloch space

The Bloch space $\mathcal{B}(B_E)$ is endowed with the norm

$$\|f\|_{Bloch} := |f(0)| + \|f\|_{inv}$$

and $(\mathcal{B}(B_E), \|\cdot\|_{Bloch})$ is a Banach space whose corresponding semi-norm $\|\cdot\|_{inv}$ is invariant by automorphisms of B_E.

The inclusion $H^\infty(B_E) \subset \mathcal{B}(B_E)$ and the use of Schwarz lemma

The space $H^\infty(B_E)$ is given by $\{f : B_E \to \mathbb{C} : f \textit{ is holomorphic and bounded}\}$.

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Corollary. The inclusion $i : H^\infty(B_E) \longrightarrow \mathcal{B}(B_E)$ is a linear operator satisfying

$$\|f\|_{\mathcal{B}(B_E)} \leq \|f\|_\infty.$$

The "log" function in this case

Remark. Let $E = \ell_2$. Define for $x = \sum_{k=1}^{\infty} x_k e_k \in B_{\ell_2}$, the function

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Proof. Let $0 < r < 1$ and $\|x\| \leq r$. The series $\sum_{n=0}^{\infty} \frac{1}{n+1} (\sum_{k=1}^{n} x_k^2)^n$ converges uniformly in $\|x\| \leq r$. Hence f is holomorphic on the unit ball.

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Finally we observe that selecting $x = (z, 0, \cdots,)$ we have $f(x) = \log(\frac{1}{1-z^2})$ and therefore $f \notin H^\infty(B_{\ell_2})$.

General constructions

Proposition. Let $f \in H^\infty(B_E)$ with $\|f\|_\infty = 1$ and $\varphi \in \mathcal{B}$. Then $g = \varphi \circ f \in \mathcal{B}(B_E)$ and $\|g\|_{\mathcal{B}(B_E)} \leq \|\varphi\|_{\mathcal{B}}$.

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Proposition. Let $(P_k)_{k=1}^\infty$ be a sequence of 2^k−homogeneous polynomials on $E = \ell_2$ with

$$M = \sup_{k \in \mathbb{N}, y \in B_E} |P_k(y)| < \infty.$$

Then $f(x) = \sum_{k=0}^\infty P_k(x) \in \mathcal{B}(B_E)$.

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