Parseval quasi-dual frames

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Frames for Hilbert spaces

$\mathcal{H}$ denotes a complex, separable, infinite dimensional Hilbert space.
$\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ is a **frame** for $\mathcal{H}$ if $\exists$ **bounds** $A, B > 0$

$$A\,\|f\|^2 \leq \sum_{n\in\mathbb{N}} |\langle f, f_n\rangle|^2 \leq B\,\|f\|^2\,,\ \ \forall f \in \mathcal{H}\,.$$

In this case we consider:

- $T_\mathcal{F} \in B(\ell^2(\mathbb{N}), \mathcal{H})$, the **synthesis operator** of $\mathcal{F}$:

$$T_\mathcal{F}\,(c_n)_{n\in\mathbb{N}} = \sum_{n\in\mathbb{N}} c_n\, f_n \in \mathcal{H}\,;$$

- $T_\mathcal{F}^* \in B(\mathcal{H}, \ell^2(\mathbb{N}))$, the **analysis operator** of $\mathcal{F}$:

$$T_\mathcal{F}^* f = (\langle f\,, f_n\rangle)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})\,;$$

- $S_\mathcal{F} = T_\mathcal{F} T_\mathcal{F}^* \in B(\mathcal{H})^+$, the **frame operator** of $\mathcal{F}$:

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$\implies$ $S_{\mathcal{F}}$ is **invertible**: hence $S_{\mathcal{F}}^{-1} \in B(\mathcal{H})^+$ and

$$f = S_{\mathcal{F}}(S_{\mathcal{F}}^{-1} f) = \sum_{n\in\mathbb{N}} \langle S_{\mathcal{F}}^{-1} f,\, f_n\rangle\, f_n = \sum_{n\in\mathbb{N}} \langle f,\, S_{\mathcal{F}}^{-1} f_n\rangle\, f_n = \sum_{n\in\mathbb{N}} \langle f, f_n^{\#}\rangle\, f_n$$

where $f_n^{\#} = S_{\mathcal{F}}^{-1} f_n$, $n \in \mathbb{N}$ is the **canonical dual** of $\mathcal{F}$.

In general, a frame $\mathcal{G} = \{g_n\}_{n\in\mathbb{N}}$ for $\mathcal{H}$ is a **dual** of $\mathcal{F}$ if: $\forall f \in \mathcal{H}$

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Frames for Hilbert spaces (bounds and excess)

Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a frame for $\mathcal{H}$:

$$\mathcal{E}(\mathcal{F}) := \dim\ker T_{\mathcal{F}} \in \mathbb{N} \cup \{\infty\} - \text{ is the } \textbf{excess} \text{ of } \mathcal{F}\ .$$

Recall that $A,\ B > 0$ are bounds for $\mathcal{F}$ iff $\forall\, f \in \mathcal{H}$

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Hence, the optimal bounds for $\mathcal{F}$ are

$$A_{\mathcal{F}} := \min\sigma(S_{\mathcal{F}}) = \|S_{\mathcal{F}}^{-1}\|^{-1} \quad \text{and} \quad B_{\mathcal{F}} := \max\sigma(S_{\mathcal{F}}) = \|S_{\mathcal{F}}\|$$

We say that $\mathcal{F}$ is **Parseval** if $A_{\mathcal{F}} = B_{\mathcal{F}} = 1$ i.e. $S_{\mathcal{F}} = I$: in this case

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Existence of Parseval Dual frames

Consider

$$\mathcal{P}(\mathcal{H}) = \{\mathcal{G} = \{g_n\}_{n\in\mathbb{N}} : \ \mathcal{G} \ \text{ is a Parseval frame for } \mathcal{H}\} \ .$$

Theorem (D. Han (2008))

Let $\mathcal{F}$ be a frame for $\mathcal{H}$, with optimal lower frame bound $A_\mathcal{F}$ and frame operator $S_\mathcal{F}$. The following statements are equivalent:

- *There exists $\mathcal{G} \in \mathcal{P}(\mathcal{H})$ that is a dual frame of $\mathcal{F}$.*
- $A_\mathcal{F} \geq 1$ *and* $\dim \overline{R(S_\mathcal{F} - I)} \leq \mathcal{E}(\mathcal{F})$.

In terms of operators, the analysis operator $T^*_\mathcal{G}$ of $\mathcal{G}$ solves the system:

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What can be done if there is no solution to the system (1)?

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- Compute $\alpha(\mathcal{F}) = \inf_{\mathcal{G}\in\mathcal{P}(\mathcal{H})} \|T_{\mathcal{F}}\, T_{\mathcal{G}}^* - I\|$, that is

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  $\alpha(\mathcal{F})$ is the optimal lower bound for the (normalized) worst case reconstruction error with $\mathcal{F}$ and any Parseval frame.

- Describe the set of Parseval "quasi-dual" frames of $\mathcal{F}$

$$\mathcal{X}(\mathcal{F}) = \{\mathcal{G} \in \mathcal{P}(\mathcal{H}) : \alpha(\mathcal{F}) = \|T_{\mathcal{F}}\, T_{\mathcal{G}}^* - I\|\}$$

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- Describe the set of Parseval "quasi-dual" frames of $\mathcal{F}$

$$\mathcal{X}(\mathcal{F}) = \{\mathcal{G} \in \mathcal{P}(\mathcal{H}) : \alpha(\mathcal{F}) = \|T_{\mathcal{F}}\, T_{\mathcal{G}}^* - I\|\}$$

  if the infimum is attained (i.e. in case $\mathcal{X}(\mathcal{F}) \neq \emptyset$).

Parseval quasi-dual frames with $\mathcal{E}(\mathcal{F}) = \infty$

Theorem ($\mathcal{E}(\mathcal{F}) = \infty$)

Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a frame for $\mathcal{H}$ with $\mathcal{E}(\mathcal{F}) = \infty$ and optimal lower bound $A_{\mathcal{F}}$. Then,

- $\alpha(\mathcal{F}) = 1 - \min\{A_{\mathcal{F}}^{1/2}, 1\}$;
- *$\alpha(\mathcal{F})$ is attained i.e. $\mathcal{X}(\mathcal{F}) \neq \emptyset$;*
- *If $\mathcal{G} \in \mathcal{X}(\mathcal{F})$ then $T_{\mathcal{F}} T_{\mathcal{G}}^*$ is invertible in $B(\mathcal{H})$.*
- *We can choose $\mathcal{G} \in \mathcal{X}(\mathcal{F})$ such that $T_{\mathcal{F}} T_{\mathcal{G}}^* = \min\{A_{\mathcal{F}}^{1/2}, 1\} I$.*

Proof. If $A_{\mathcal{F}} < 1$ let $\mathcal{F}' = \{A_{\mathcal{F}}^{-1/2} f_n\}$: then $\mathcal{E}(\mathcal{F}') = \infty$ and $A_{\mathcal{F}'} = 1$. By D. Han's result $\exists\, \mathcal{G} \in \mathcal{P}(\mathcal{H})$ that is a dual for $\mathcal{F}'$ i.e.

$$I = T_{\mathcal{F}'} T_{\mathcal{G}}^* = A_{\mathcal{F}}^{-1/2} T_{\mathcal{F}} T_{\mathcal{G}}^* \implies T_{\mathcal{F}} T_{\mathcal{G}}^* = A_{\mathcal{F}}^{1/2} I$$

and hence $\alpha(\mathcal{F}) \leq \|T_{\mathcal{F}} T_{\mathcal{G}}^* - I\| = (1 - A_{\mathcal{F}}^{1/2})$.

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Proof. ($A_\mathcal{F} < 1$) Let $\mathcal{G} \in \mathcal{P}(\mathcal{H})$, $\|T_\mathcal{F} T_\mathcal{G}^* - I\| \leq r < 1 - A_\mathcal{F}^{1/2} < 1$

$$\exists H = (T_\mathcal{F} T_\mathcal{G}^*)^{-1} \in B(\mathcal{H})\ ,\ \|H\| \leq \frac{1}{1-r} \implies (HT_\mathcal{F})T_\mathcal{G}^* = I$$

so $\mathcal{G} \in \mathcal{P}(\mathcal{H})$ is a dual frame of $\tilde{\mathcal{F}} = \{H f_n\}_{n\in\mathbb{N}}$, since $T_{\tilde{\mathcal{F}}} = H\, T_\mathcal{F}$;

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Parseval quasi-dual frames - slicing the problem

Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a frame for $\mathcal{H}$ with $\mathcal{E}(\mathcal{F}) < \infty$.
Let $\mathcal{M} \subset \ell^2(\mathbb{N})$ be a closed subspace with $\dim \mathcal{M} = \infty$

$$\text{Set } C_{\mathcal{M}} = \{\mathcal{G} \in \mathcal{P}(\mathcal{H}) : \ R(T_{\mathcal{G}}^*) = \mathcal{M}\} \ \ni \mathcal{G}_{\mathcal{M}} \text{ (fixed)}$$

If $\mathcal{G} \in C_{\mathcal{M}}$ then $T_{\mathcal{G}}\, T_{\mathcal{G}}^* = I,\ T_{\mathcal{G}}^*\, T_{\mathcal{G}} = P_{\mathcal{M}}$ and

$$\begin{aligned}
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Then, if $\mathcal{U}(\mathcal{M}) \subset B(\mathcal{M})$ is the group of unitary operators

$$\begin{aligned} \inf_{\mathcal{G} \in C_{\mathcal{M}}} \|T_{\mathcal{F}} T_{\mathcal{G}}^* - I\| &= \inf_{\mathcal{G} \in C_{\mathcal{M}}} \|(T_{\mathcal{G}_{\mathcal{M}}}^* T_{\mathcal{F}})_{\mathcal{M}} - (T_{\mathcal{G}_{\mathcal{M}}}^* T_{\mathcal{G}})_{\mathcal{M}}\|_{B(\mathcal{M})} \\ &= \inf_{U \in \mathcal{U}(\mathcal{M})} \|(T_{\mathcal{G}_{\mathcal{M}}}^* T_{\mathcal{F}})_{\mathcal{M}} - U\| =: d_{\mathcal{U}(\mathcal{M})}((T_{\mathcal{G}_{\mathcal{M}}}^* T_{\mathcal{F}})_{\mathcal{M}}) \end{aligned}$$

$$\text{Since } \ \mathcal{P}(\mathcal{H}) = \bigcup_{\mathcal{M} \subset \ell^2(\mathbb{N}),\, \dim \mathcal{M} = \infty} C_{\mathcal{M}} \ \text{ then}$$

$$\begin{aligned} \alpha(\mathcal{F}) = \inf_{\mathcal{G} \in \mathcal{P}(\mathcal{H})} \|T_{\mathcal{F}} T_{\mathcal{G}}^* - I\| &= \inf_{\mathcal{M} \subset \ell^2(\mathbb{N}),\, \dim \mathcal{M} = \infty} \ \inf_{\mathcal{G} \in \mathcal{C}_{\mathcal{M}}} \|T_{\mathcal{F}} T_{\mathcal{G}}^* - I\| \\ &= \inf_{\mathcal{M} \subset \ell^2(\mathbb{N}),\, \dim \mathcal{M} = \infty} d_{\mathcal{U}(\mathcal{M})}((T_{\mathcal{G}_{\mathcal{M}}}^* T_{\mathcal{F}})_{\mathcal{M}}) \end{aligned}$$

where $\mathcal{G}_{\mathcal{M}} \in \mathcal{P}(\mathcal{H})$ is such that $R(T_{\mathcal{G}_{\mathcal{M}}}^*) = \mathcal{M}$.

Parseval quasi-dual frames - slicing the problem

Let $\mathcal{M} \subset \ell^2(\mathbb{N})$ be a closed subspace with $\dim \mathcal{M} = \infty$

Set $C_{\mathcal{M}} = \{\mathcal{G} \in \mathcal{P}(\mathcal{H}) : \ T_{\mathcal{G}}^* T_{\mathcal{G}} = P_{\mathcal{M}}\} \ \ni \mathcal{G}_{\mathcal{M}}$ (fixed) .

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Parseval quasi-dual frames - slicing the problem

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Parseval quasi-dual frames - slicing the problem

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Parseval quasi-dual frames - slicing the problem

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\end{aligned}$$

where $\mathcal{G}_{\mathcal{M}} \in \mathcal{P}(\mathcal{H})$ is such that $R(T_{\mathcal{G}_{\mathcal{M}}}^*) = \mathcal{M}$.

Distance to the unitaries

For $T \in B(\mathcal{H})$, consider

- $m(T) = \min \sigma(|T|)$ and $m_e(T) = \min \sigma_e(|T|)$,
- Fredholm index: $\operatorname{ind}(T) = \dim N(T) - \dim N(T^*) \in \mathbb{Z} \cup \{\pm\infty\}$

Theorem (Rogers (1977))

Let $T \in B(\mathcal{H})$ be semi-Fredholm such that $\operatorname{ind}(T)$ is defined . Then,

$$d_{\mathcal{U}(\mathcal{H})}(T) = \begin{cases} \max\{\|T\| - 1, 1 - m(T)\} & \textit{if } \operatorname{ind}(T) = 0 , \\ \max\{\|T\| - 1, 1 + m_e(T)\} & \textit{if } \operatorname{ind}(T) < 0 , \\ \max\{\|T\| - 1, 1 + m_e(T^*)\} & \textit{if } \operatorname{ind}(T) > 0 . \end{cases}$$

Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a frame for $\mathcal{H}$ with $\mathcal{E}(\mathcal{F}) < \infty$

$$\alpha(\mathcal{F}) = \inf_{\mathcal{G}\in\mathcal{P}(\mathcal{H})} \|T_{\mathcal{F}} T_{\mathcal{G}}^* - I\| = \inf_{\mathcal{M}\subset\ell^2(\mathbb{N}),\, \dim\mathcal{M}=\infty} d_{\mathcal{U}(\mathcal{M})}((T_{\mathcal{G}_{\mathcal{M}}}^* T_{\mathcal{F}})_{\mathcal{M}})$$

where $\mathcal{G}_{\mathcal{M}} \in \mathcal{P}(\mathcal{H})$, $R(T_{\mathcal{G}_{\mathcal{M}}}^*) = \mathcal{M}$ and $\operatorname{ind}[(T_{\mathcal{G}_{\mathcal{M}}}^* T_{\mathcal{F}})_{\mathcal{M}}]$ is defined!

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$$d_{\mathcal{U}(\mathcal{H})}(T) = \begin{cases} \max\{\|T\| - 1, 1 - m(T)\} & \textit{if } \text{ind}(T) = 0\,, \\ \max\{\|T\| - 1, 1 + m_e(T)\} & \textit{if } \text{ind}(T) < 0\,, \\ \max\{\|T\| - 1, 1 + m_e(T^*)\} & \textit{if } \text{ind}(T) > 0\,. \end{cases}$$

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Distance to the unitaries

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where $\mathcal{G}_{\mathcal{M}} \in \mathcal{P}(\mathcal{H})$, $R(T_{\mathcal{G}_{\mathcal{M}}}^*) = \mathcal{M}$ and $\text{ind}[(T_{\mathcal{G}_{\mathcal{M}}}^* T_{\mathcal{F}})_{\mathcal{M}}]$ is defined!

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$\mathcal{E}(\mathcal{F}) < \infty$ and $\operatorname{ind}[(T^*_{\mathcal{G}_\mathcal{M}} T_\mathcal{F})_\mathcal{M}] = 0$

For $T \in B(\ell^2(\mathbb{N}), \mathcal{H})$ define

$$s_k(T) = \sup\{\min \sigma(|T|_\mathcal{N}) \ : \ \mathcal{N} \subset \ell^2(\mathbb{N}), \ \dim(\mathcal{N}) = k\} \ \text{ for } \ k \geq 1.$$

Proposition

Let $\mathcal{F}$ be a frame for $\mathcal{H}$ with $\mathcal{E}(\mathcal{F}) = n$ and optimal lower bound $A_\mathcal{F}$.

1. *If $\mathcal{G} \in \mathcal{P}(\mathcal{H})$ with $R(T^*_\mathcal{G}) = \mathcal{M} \subset \ell^2(\mathbb{N})$ then*

$$\operatorname{ind}[(T^*_\mathcal{G} T_\mathcal{F})_\mathcal{M}] = 0 \ \text{ iff } \ \dim \mathcal{M}^\perp = \mathcal{E}(\mathcal{G}) = n\,.$$

2. *If we let $C_\mathcal{F} = s_{n+1}(T_\mathcal{F})$ then*

$$\inf\{\|T_\mathcal{F} T^*_\mathcal{G} - I\|, \ \mathcal{G} \in \mathcal{P}(\mathcal{H}), \ \mathcal{E}(\mathcal{G}) = n\} = \max\{1 - A_\mathcal{F}^{1/2}, \ C_\mathcal{F} - 1\}\,.$$

3. *There is a closed $\mathcal{M} \subset \ell^2(\mathbb{N})$, with $\dim \mathcal{M}^\perp = n$ such that*

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$\mathcal{E}(\mathcal{F}) < \infty$ and $\operatorname{ind}[(T^*_{\mathcal{G}_\mathcal{M}} T_\mathcal{F})_\mathcal{M}] \neq 0$

For $T \in B(\ell^2(\mathbb{N}), \mathcal{H})$ recall that $m_e(T) = \min \sigma_e(|T|)$.

Proposition

Let $\mathcal{F}$ be a frame for $\mathcal{H}$ with $\mathcal{E}(\mathcal{F}) = n$. Then

$$\inf\{\|T_\mathcal{F} T^*_\mathcal{G} - I\|,\ \mathcal{G} \in \mathcal{P}(\mathcal{H}),\ \mathcal{E}(\mathcal{G}) \neq n\} = 1 + m_e(T_\mathcal{F})\ .$$

Moreover, there is a closed $\mathcal{M} \subset \ell^2(\mathbb{N})$, $\dim \mathcal{M}^\perp \neq n$ such that

$$\inf\{\|T_\mathcal{F} T^*_\mathcal{G} - I\|,\ \mathcal{G} \in \mathcal{P}(\mathcal{H}),\ T^*_\mathcal{G} T_\mathcal{G} = P_\mathcal{M}\} = 1 + m_e(T_\mathcal{F})$$

$\mathcal{E}(\mathcal{F}) < \infty$ and $\mathrm{ind}[(T^*_{\mathcal{G}_\mathcal{M}} T_\mathcal{F})_\mathcal{M}] \neq 0$

For $T \in B(\ell^2(\mathbb{N}), \mathcal{H})$ recall that $m_e(T) = \min \sigma_e(|T|)$.

Proposition

Let $\mathcal{F}$ be a frame for $\mathcal{H}$ with $\mathcal{E}(\mathcal{F}) = n$. Then

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Computation of $\alpha(\mathcal{F})$ in case $\mathcal{E}(\mathcal{F}) < \infty$.

Theorem

Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a frame for $\mathcal{H}$ with $\mathcal{E}(\mathcal{F}) = n$ and optimal lower bound $A_{\mathcal{F}}$. If $C_{\mathcal{F}} = s_{n+1}(T_{\mathcal{F}})$ then

$$\alpha(\mathcal{F}) = \min\{\max\{1 - A_{\mathcal{F}}^{1/2}, C_{\mathcal{F}} - 1\},\ 1 + m_e(T_{\mathcal{F}})\}$$

Moreover, there exists a closed subspace $\mathcal{M} \subset \ell^2(\mathbb{N})$ such that

$$\begin{aligned}\alpha(\mathcal{F}) &= \inf\{\|T_{\mathcal{F}}T_{\mathcal{G}}^* - I\|,\ \mathcal{G} \in \mathcal{P}(\mathcal{H}),\ T_{\mathcal{G}}^*T_{\mathcal{G}} = P_{\mathcal{M}}\}\\ &= d_{\mathcal{U}(\mathcal{M})}((T_{\mathcal{G}_{\mathcal{M}}}^* T_{\mathcal{F}})_{\mathcal{M}})\end{aligned}$$

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Parseval quasi-dual frames with $\mathcal{E}(\mathcal{F}) < \infty$

Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a frame for $\mathcal{H}$ with $\mathcal{E}(\mathcal{F}) < \infty$ and optimal lower bound $A_{\mathcal{F}}$.

In our approach, the existence (and characterization) of Parseval quasi-dual frames of $\mathcal{F}$ i.e. elements of

$$\mathcal{X}(\mathcal{F}) = \{\mathcal{G} \in \mathcal{P}(\mathcal{H}) : \alpha(\mathcal{F}) = \|T_{\mathcal{F}} T_{\mathcal{G}}^* - I\|\}$$

relies on the existence of unitary approximants in Rogers' result i.e.

$$\alpha(\mathcal{F}) = d_{\mathcal{U}(\mathcal{M})}((T_{\mathcal{G}}^* T_{\mathcal{F}})_{\mathcal{M}}) = \|(T_{\mathcal{G}}^* T_{\mathcal{F}})_{\mathcal{M}} - U^{(\mathcal{M})}\|$$

where $U^{(\mathcal{M})} \in \mathcal{U}(\mathcal{M})$ - for appropriate $\mathcal{M} \subset \ell^2(\mathbb{N})$.

Yet, in all examples considered, $\mathcal{X}(\mathcal{F}) \neq \emptyset$.

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Some references for this talk

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Thank you!