A new radius for Dirichlet series

Manuel Maestre

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Definition –The Dirichlet-Bohr radius

Given a x in $\mathbb{N}$, the *Dirichlet-Bohr radius* L_x is the best $r = r(x) \geq 0$ such that for every $a_1, \ldots a_x$ in $\mathbb{C}$, we have

$$\sum_{n \leqslant x} |a_n| r^{\Omega(n)} \leqslant \sup_{\mathrm{Re}s>0} \left| \sum_{n \leqslant x} a_n n^{-s} \right|,$$

where $\Omega(n)$ denotes the number of prime divisors of $n \in \mathbb{N}$ (counted with multiplicities).

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Theorem D. Carando, A. Defant, D. García, M. M. and P. Sevilla, 2014

There exist $A, B > 0$ such that

$$A \frac{\sqrt[4]{\log x}}{x^{1/8}} \leqslant L_x \leqslant B \frac{\sqrt[4]{\log x}}{x^{1/8}},$$

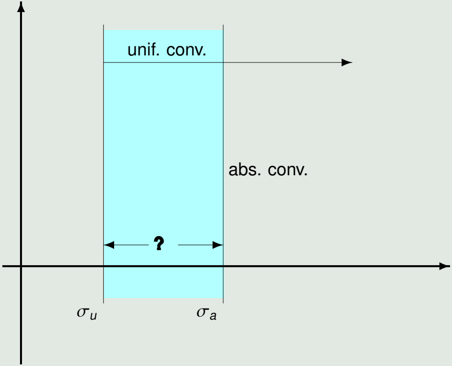
for every $x \geqslant 2$.

Dirichlet series

$$D = \sum_{n=1}^{\infty} a_n \frac{1}{n^s}$$

with coefficients $a_n \in \mathbb{C}$ and variable $s \in \mathbb{C}$

Convergence of Dirichlet series $\sum_n a_n \frac{1}{n^s}$



Dirichlet series convergence

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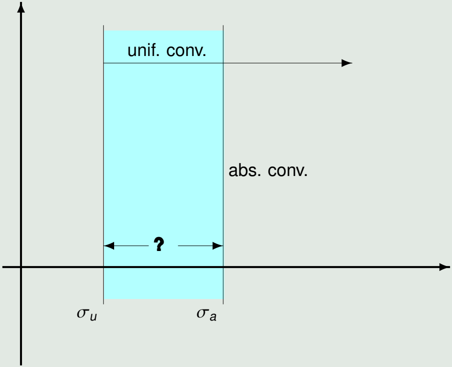

Definition

$$S := \sup\left\{\sigma_a(D) - \sigma_u(D) \quad : \quad D = \sum_n a_n \frac{1}{n^s} \text{ Dirichlet series}\right\}$$

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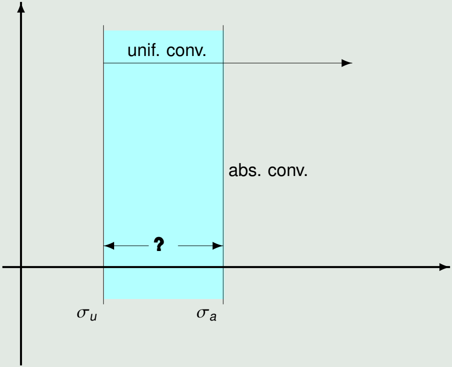

Bohr's absolute convergence problem

$$S = ?$$

Dirichlet series convergence

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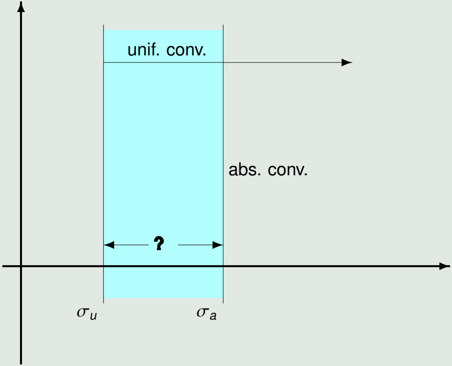

Bohnenblust-Hille Theorem (1931 Annals of Math.)

$$S = \frac{1}{2}$$

Theorem

Let X be a complex Banach space, and let $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ be a Dirichlet series in X. i.e. a_n belongs to X for all n.Then

$$S(X) = \sup_{\sum \frac{a_n}{n^s}} \{\sigma_a - \sigma_u\}$$

Vector valued Dirichlet series

Theorem

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Theorem. A. Defant, D. García, M. M., D. Pérez (Math. Annalen 2008)

For every Banach space X

$$S(X) = \inf \left\{ \frac{1}{p'} \mid Y \text{ has cotype } p \right\} = 1 - \frac{1}{\mathrm{Cot}(X)}\,.$$

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For every Banach space X

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Definition

X has cotype p ($p \in [2, +\infty]$) if there exists a constant $K \geqslant 0$ such that

$$\Big(\sum_{k=1}^{n} \|x_k\|^p\Big)^{\frac{1}{p}} \leqslant K\Big(\int_{\Omega} \|\sum_{k=1}^{n} \varepsilon_k(\omega) x_k\|^2 d\omega\Big)^{\frac{1}{2}},$$

$$\mathrm{Cot}(X) := \inf\{2 \leqslant p \leqslant \infty \mid X \text{ has cotype } p\}$$

Recall

$$\mathrm{Cot}(\ell_p) = \begin{cases} 2 & \text{if } 1 \leq p \leq 2 \\ p & \text{if } 2 \leq p \leq \infty \end{cases},$$

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Corollary

$$S(\ell_p) = \begin{cases} \frac{1}{2}, & 1 \leq p \leq 2 \\ \\ 1 - \frac{1}{p}, & 2 \leqslant p \leqslant \infty \end{cases}$$

In particular,

$$S(\ell_\infty) = 1.$$

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In particular,

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Corollary

For every $t \in [\frac{1}{2}, 1]$ there is a Banach space X for which $t = S(X)$.

Content

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- Dirichlet series and complex analysis on polydiscs
-

Bohr's power series theorem, 1914

- $z \in \frac{1}{3}\mathbb{D}$

  $\Rightarrow \quad \forall \; f \in H_\infty(\mathbb{D}) : \quad \sum_n |c_n(f) z^n| \leqslant \|f\|_\infty$

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Definition – Nth Bohr radius

$$K_N := \sup \left\{ r \leqslant 1 \mid \forall f \in H_\infty(\mathbb{D}^N) : \sup_{z \in r \cdot \mathbb{D}^N} \sum |c_\alpha(f) z^\alpha| \leqslant \|f\|_\infty \right\}$$

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Bohr's power series theorem

$$K_1 = \frac{1}{3}$$

Problem

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Theorem Defant-Frerick-Ortega Cerdà-Ounaïes-Seip (2011 Annals of Math.)

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Theorem Bayart-Pellegrino-Seoane (2013)

$$K_N \sim \sqrt{\frac{\log N}{N}}$$

Why Bohr's thought about this radii?

$p =$ the sequence of prime numbers: $p_1 < p_2 < p_3 < \dots$

$p^\alpha = p_1^{\alpha_1} \times \cdots \times p_n^{\alpha_n}$ where $\alpha = (\alpha_1, \dots, \alpha_n, 0, \dots) \in \mathbb{N}_0^{(\mathbb{N})}$

A one to one correspondence (The Bohr transform):

Dirichlet series

$\mathfrak{D}$

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Dirichlet series		formal power series
$\mathfrak{D}$	$\longrightarrow$	$\mathfrak{P}$

A one to one correspondence (The Bohr transform):

$$\begin{array}{ccc} \text{Dirichlet series} & & \text{formal power series} \\ \mathfrak{D} & \xrightarrow{\hspace{2cm}} & \mathfrak{P} \\ \sum_n a_n \frac{1}{n^s} & \xrightarrow{a_n = a_{p^\alpha} = c_\alpha} & \sum_\alpha c_\alpha z^\alpha \end{array}$$

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Definition

$\mathcal{H}_\infty$:= the set of all those Dirichlet series D which converge on $[\mathrm{Re} > 0]$ and it is bounded on this halfplane.

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\mathcal{H}_\infty & \longrightarrow & H_\infty(B_{c_0})
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Definition

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Educated guessing

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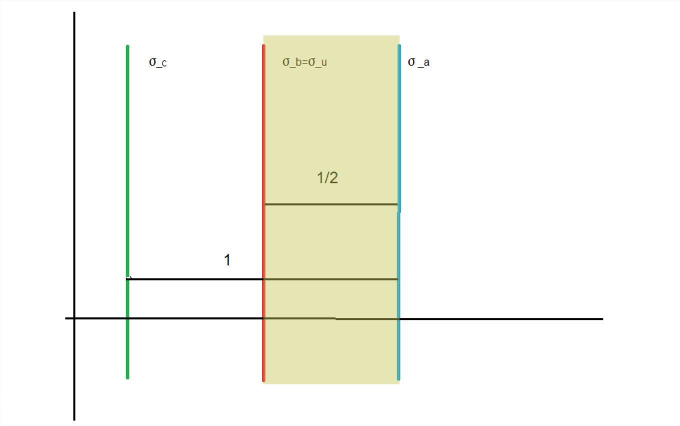
Assume that there would be a $C > 0$ such that

$$K_n \geqslant C$$

for all n. Then

$$\sigma_a = \sigma_u$$

for every Dirichlet series, and hence S=0!!! (Not true)


σ_c
σ_b=σ_u
σ _a
1/2
1

Let $D = \sum_{n=1}^{\infty} a_n \frac{1}{n^s}$ any bounded (by 1) and convergent Dirichlet series on $[\mathrm{Re}\, s > 0]$. Consider s with $\mathrm{Re}\, s > 0$ and define the sequence

$$z_0 = (p_1^{-s}, p_2^{-s}, \ldots, p_k^{-s}, \ldots)$$

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As converges to 0 there exists k_0 such that $|p_k^{-1}| < C$ for every $k \geqslant k_0$ and take $u = (u_k) = (0, 0, \dots, 0, p_{k_0}^{-s}, p_{k_0+1}^{-s} \dots)$.

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Hence

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |a_{p^\alpha} u^\alpha| = \sup_n \sup_{z \in \mathbb{D}^n} \sum_{\alpha \in \mathbb{N}_0^n} \left| a_{p^\alpha} u^\alpha \right| \leqslant \sup_n \left| \sup_{z \in \mathbb{D}^n} \sum_{\alpha \in \mathbb{N}_0^n} a_{p^\alpha} u^\alpha \right| \leqslant \|D\|_\infty \leqslant 1$$

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BUT BOHR HAD ALREADY PROVEN THAT THEN

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Definition – Nth Bohr radius (vector Valued)

let X a complex Banach space and $\lambda > 1$,

$$K_N(X, \lambda) := \sup \left\{ r \leqslant 1 \mid \forall f \in H_\infty(\mathbb{D}^N; X) : \sup_{z \in r \cdot \mathbb{D}^N} \sum \|c_\alpha(f) z^\alpha\| \leqslant \lambda \|f\|_\infty \right\}$$

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(O. Blasco), 2009

If we take $f : \mathbb{D} \to (\mathbb{C}^2, \|.\|_\infty)$ defined by $f(z) = (1, z) = e_1 + e_2 z$ for $z \in \mathbb{D}$.
We have

$$\|f(z)\|_\infty = \max\{1, |z|\} = 1,$$

for all $z \in \mathbb{D}$. But

$$\|e_1\|_\infty + \|e_2\|_\infty |z| = 1 + |z| > 1 = \|f\|_\infty,$$

for all $z \in \mathbb{D} \setminus \{0\}$. Hence

$$K_1(1, \ell^2_\infty) = 0$$

.

Theorem A. Defant, M.M. and U. Schwarting (2012 Advances in Math.)

Let X be a complex Banach space and $\lambda > 1$. With constants depending only on λ and X we have:

$$K_N(X,\lambda) \asymp \sqrt{\frac{\log N}{N}}, \quad \text{for every finite dimensional } X.$$

$$\frac{1}{N^{1-\frac{1}{\mathrm{Cot}(X)+\varepsilon}}} \prec K_N(X,\lambda) \prec \frac{1}{N^{1-\frac{1}{\mathrm{Cot}(X)}}} \quad \text{for every infinite dimensional } X.$$

In particular, if X has no finite cotype, then

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Corollary

With constants only depending on λ and $p \geqslant 2$ we have

$$K_N(\ell_p,\lambda) \asymp \frac{1}{N^{1-\frac{1}{p}}}.$$

Content

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- The Dirichlet-Bohr radius

Definition –The Dirichlet-Bohr radius

Given a subset J of $\mathbb{N}$, the *Dirichlet-Bohr radius* $L(J)$ of J is the best $r = r(J) \geq 0$ such that for every Dirichlet series $\sum_{n \in J} a_n n^{-s}$ convergent on the open half-plane $[\mathrm{Re}s > 0]$, we have

$$\sum_{n \in J} |a_n| r^{\Omega(n)} \leqslant \sup_{\mathrm{Re}s>0} \left| \sum_{n \in J} a_n n^{-s} \right|,$$

where $\Omega(n)$ denotes the number of prime divisors of $n \in \mathbb{N}$ (counted with multiplicities).

Examples

- For $J = P = \{p \,:\, \text{prime}\}$, $L(P) = 1$.
  Well-known: for every $\sum_{\text{p}} a_p p^{-s}$ convergent in $[\text{Re}s > 0]$,

$$\sum_{p \text{ prime}} |a_p| = \sup_{\text{Re}s>0} \Big| \sum_{p \text{ prime}} a_p p^{-s} \Big|.$$

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- $L\left(\left\{2^k \mid k \in \mathbb{N}\right\}\right) = \frac{1}{3}$.
- $L\left(\left\{p_\ell^k \,\middle|\, k, \ell \in \mathbb{N}\right\}\right) = \frac{1}{3}$

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- $L\Big(\Big\{p_\ell^k \,\Big|\, k, \ell \in \mathbb{N}\Big\}\Big) = \frac{1}{3}$
- Let P_k be finite sets of primes of maximum lenght N. Assume that the P_k are pairwise disjoint. If
  $$J = \bigcup_{k=1}^{\infty} \Big\{ n = p^\alpha \,|\, \alpha_j = 0, \text{ if } p_j \notin P_k \Big\},$$
  Then
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The Dirichlet-Bohr radius

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  Then
$$L(J) = K_N$$
- $L(\mathbb{N}) = 0$

The Dirichlet-Bohr radius

For any natural number x, we write

$$L_x = L\Big(\Big\{n \in \mathbb{N} \,\Big|\, 1 \leqslant n \leqslant x\Big\}\Big),$$

i.e.

$$L_x = \max\Big\{r \geqslant 0 \;:\; \sum_{n \leqslant x} |a_n| r^{\Omega(n)}\Big\} \leqslant \sup_{\mathrm{Re}s>0} \Big|\sum_{n \leqslant x} a_n n^{-s}\Big|\Big\},$$

for every $\sum_{n \leqslant x} a_n n^{-s}$ and call this number the *x-th Dirichlet-Bohr radius*.

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Theorem D. Carando, A. Defant, D. García, M. M. and P. Sevilla, 2014

There exist $A, B > 0$ such that

$$A\,\frac{\sqrt[4]{\log x}}{x^{1/8}} \leqslant L_x \leqslant B\,\frac{\sqrt[4]{\log x}}{x^{1/8}},$$

for every $x \geqslant 2$. in particular,

$$\sum_{n=1}^{x} |a_n| \left(\frac{A\,\sqrt[4]{\log x}}{x^{1/8}} \right)^{\Omega(n)} \leqslant \sup_{\mathrm{Re} s > 0} \left| \sum_{n=1}^{x} a_n n^{-s} \right|,$$

for every x and every finite Dirichlet poynomial $\sum_{n=1}^{x} a_n n^{-s}$

Reduction Theorem, D. Carando, A. Defant, D. García, M. M. and P. Sevilla, 2014

If we denote

$$\mathcal{H}_\infty^{(x,m)} := \left\{ \sum_{n=1}^{x} a_n \frac{1}{n^s} \;\middle|\; a_n \neq 0 \quad \text{only if} \quad n \leqslant x, \Omega(n) = m \right\}.$$

and for $m \in \mathbb{N}$ we define the m-homogeneous x-th Dirichlet-Bohr radius by

$$L_{x,m} := \sup \left\{ 0 \leqslant r \leqslant 1 \;\middle|\; \forall D \in \mathcal{H}_\infty^{(x,m)} : \sum_{n=1}^{x} |a_n| \leqslant r^{-m} \|D\|_\infty \right\}.$$

Then,

$$\frac{1}{3} \inf_m L_{x,m} \;\leqslant\; L_x \leqslant\; \inf_m L_{x,m} \quad \text{for all} \quad x \in \mathbb{N}$$

The Tools for the Dirichlet-Bohr radius II

Bohr's fundamental Lemma, 1913

For every finite Dirichlet polynomial $\sum_{n=1}^{x} a_n \frac{1}{n^s}$ we have

$$\sup_{t\in\mathbb{R}} \left| \sum_{n=1}^{x} a_n n^{-it} \right| = \sup_{z\in\mathbb{D}^{\pi(x)}} \left| \sum_{\substack{\alpha\in\mathbb{N}_0^{\pi(x)} \\ 1\leqslant p^\alpha\leqslant x}} a_{p^\alpha} z^\alpha \right|.$$

Here, π denotes the *prime counting function*, i.e., $\pi(x)$ is the number of prime numbers less than or equal to x.

The Tools for the Dirichlet-Bohr radius II

Bohr's fundamental Lemma, 1913

For every finite Dirichlet polynomial $\sum_{n=1}^{x} a_n \frac{1}{n^s}$ we have

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Here, π denotes the *prime counting function*, i.e., $\pi(x)$ is the number of prime numbers less than or equal to x.

Unusual notation for a Polynomial in $\mathbb{C}^n$

For $m, n \in \mathbb{N}$ we put

$$\mathcal{J}(m,n) = \{\mathbf{i} = (i_1, \ldots, i_m) \in \{1, \ldots, n\}^m : \ 1 \leqslant i_1 \leqslant \ldots \leqslant i_m \leqslant n\},$$

which allows to represent every m-homogeneous polynomial $P(z) = \sum_{\alpha\in\mathbb{N}^n} c_\alpha z^\alpha$, $z \in \mathbb{C}^n$ uniquely in the form

$$P(z) = \sum_{\mathbf{i}\in\mathcal{J}(m,n)} c_{\mathbf{i}} z_{j_1} \cdot \ldots \cdot z_{j_m}, \ z \in \mathbb{C}^n.$$

Theorem F. Bayart, A.Defant, F. Leonard, M.M. and P. Sevilla 2014

Let $n \geqslant 1$, let $m \geqslant l \geqslant 1$ and let $\kappa > 1$. There exists $C(\kappa) > 0$ such that, for any for any m-homogeneous polynomial P in $\mathbb{C}^n$ with coefficients $(c_{\mathbf{j}})_{\mathbf{j}}$, we have

$$\Bigg[\sum_{\mathbf{j} \in \mathcal{J}(l,n)} \Bigg(\sum_{\substack{\mathbf{i} \in \mathcal{J}(m-l,n) \\ i_{m-l} \leqslant j_1}} |c_{(\mathbf{i},\mathbf{j})}|^2 \Bigg)^{\frac{1}{2} \times \frac{2l}{l+1}} \Bigg]^{\frac{l+1}{2l}} \leqslant C(\kappa) \Bigg[\kappa \Big(1 + \frac{1}{l} \Big) \Bigg]^m \|P\|_\infty \,,$$

where $\|P\|_\infty = \sup\{|P(z)| : z \in \mathbb{D}^n\}$.

The Tools for the Dirichlet-Bohr radius, III

Theorem F. Bayart, A.Defant, F. Leonard, M.M. and P. Sevilla 2014

Let $n \geqslant 1$, let $m \geqslant l \geqslant 1$ and let $\kappa > 1$. There exists $C(\kappa) > 0$ such that, for any for any m-homogeneous polynomial P in $\mathbb{C}^n$ with coefficients $(c_{\mathbf{j}})_{\mathbf{j}}$, we have

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where $\|P\|_\infty = \sup\{|P(z)| : z \in \mathbb{D}^n\}$.

Theorem, Balasubramanian, Calado, and Queffélec, Studia Math. 2006

Let $m \geqslant 2$ and $\kappa > 1$. There exists $C(\kappa) > 0$ such that for every m-homogeneous Dirichlet polynomial $D = \sum_{n=1}^{x} a_n n^{-s}$ in $\mathcal{H}_\infty^{(x,m)}$ we have

$$\sum_{n=1}^{x} |a_n| \frac{(\log n)^{\frac{m-1}{2}}}{n^{\frac{m-1}{2m}}} \leqslant C(\kappa) m^{m-1} (2\kappa)^m \|D\|_\infty .$$

The Tools for the Dirichlet-Bohr radius, III

Theorem F. Bayart, A.Defant, F. Leonard, M.M. and P. Sevilla 2014

Let $n \geqslant 1$, let $m \geqslant l \geqslant 1$ and let $\kappa > 1$. There exists $C(\kappa) > 0$ such that, for any for any m-homogeneous polynomial P in $\mathbb{C}^n$ with coefficients $(c_{\mathbf{j}})_{\mathbf{j}}$, we have

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$$\sum_{p \leqslant x} p^{-\alpha} \ll \frac{1}{1-\alpha} \frac{x^{1-\alpha}}{\log x} \quad \textit{for every} \quad 0 < \alpha < 1$$

MUCHAS GRACIAS!!!!!!