The Bishop-Phelps-Bollobás property for operators between spaces of continuous functions

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Let X, Y be Banach spaces.
$B_X = \{x \in X : \|x\| \leq 1\}$
$S_X = \{x \in X : \|x\| = 1\}$
$\mathcal{L}(X, Y)$ *the space of all bounded and linear operators from X into Y*
X^* *the topological dual of X.*

- *E. Bishop and R.R. Phelps, A proof that every Banach space is subreflexive, Bull. Amer. Math. Soc. (N.S.)* **67** *(1961), 97-98.*

Theorem (Bishop-Phelps Theorem)

Let X be a Banach space , let $x^* \in X^*$ and $\epsilon > 0$ be arbitrary. Then $\exists \varphi \in X^*, \|\varphi\| = 1$ with the following two properties:

- *φ attains its norm. That is, $\exists y \in X, \|y\| = 1$, such that $\varphi(y) = \|\varphi\| = 1$.*
- *$\|\varphi - x^*\| < \varepsilon$.*

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Linear Operator version of Bishop-Phelps Theorem

Theorem (Lindenstrauss 1963)

Let X and Y be Banach spaces and let $T \in \mathcal{L}(X, Y)$. Let $\epsilon > 0$. Then $\exists\, S \in \mathcal{L}(X, Y)$ with following properties:

- $\|S - T\| < \epsilon$.
- *The double transpose $S^{**} \in \mathcal{L}(X^{**}, Y^{**})$ attains its norm.*

If X is reflexive, then we have the exact analogy of Bishop-Phelps Theorem
$\forall T \in \mathcal{L}(X, Y), \forall \epsilon > 0, \exists S \in \mathcal{L}(X, Y)$ *such that* $\|S - T\| < \epsilon$ *and* S *attains its norm.*

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Example (J. Lindenstrauss)

Let $X = Y \oplus Z$, where $Y = c_0$, $Z \simeq c_0$ such that Z is strictly convex. The norm in X is given by $\|(y,z)\| = max\{\|y\|, \|z\|\}$. Suppose $T_0 : c_0 \longrightarrow Z$ is an isomorphism and consider $T : X \longrightarrow X$ given by $T(y,z) = (0, T_0(y)$ Then, T cannot be approximated by a norm-attaining $S \in \mathcal{L}(X,X)$.

- *B. Bollobás, An extension to the theorem of Bishop and Phelps, Bull. London Math. Soc.* **2** *(1970), 181–182.*

Theorem (Bishop-Phelps-Bollobás Theorem)

Let X be a Banach space and $0 < \varepsilon < 1/2$. Given $x \in B_X$ and $x^* \in S_{X^*}$ with $|1 - x^*(x)| < \dfrac{\varepsilon^2}{2}$ Then , $\exists\, y \in S_X$ and $\exists\, y^* \in S_{X^*}$ with the following three properties:

- $y^*(y) = 1$,
- $\|y - x\| < \varepsilon$
- $\|y^* - x^*\| < \varepsilon$.

(Extensions of the Bishop-Phelps-Bollobás Theorem for Operators)

- *M.D. Acosta, R.M. Aron, D. García and M. Maestre, The Bishop-Phelps-Bollobás theorem for operators, J. Funct. Anal.* **254** *(11) (2008), 2780–2799.*

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Definition

Let X and Y be Banach spaces and $\mathcal{M}$ a linear subspace of $\mathcal{L}(X, Y)$. We say that $\mathcal{M}$ satisfies the *Bishop-Phelps-Bollobás property* if given $\varepsilon > 0$, there is $\eta(\varepsilon) > 0$ such that for any $T \in S_{\mathcal{M}}$ if $x_0 \in S_X$ satisfies that $\|Tx_0\| > 1 - \eta(\varepsilon)$, then there exist a point $u_0 \in S_X$ and an operator $R \in S_{\mathcal{M}}$ satisfying the following conditions:

$$\|Ru_0\| = 1, \quad \|u_0 - x_0\| < \varepsilon, \quad \text{and} \quad \|R - T\| < \varepsilon.$$

In the case that $\mathcal{M} = \mathcal{L}(X, Y)$ satisfies the previous property it is said that the pair (X, Y) has the *Bishop-Phelps-Bollobás property for operators* (shortly *BPBp for operators*).

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Examples

1. $(X, \mathbb{C})$ *has the BPBp, for all* X *Banach space.*
2. *The pair* (l_1, Y) *has the BPBp for operators, when the Banach space* Y *is:*
   - *finite dimensional space,*
   - *uniformly convex space.*
3. (X, Y) *has the BPBp for operators for every Banach space* Y *when* X *is uniformly convex.*

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Example (R. Aron, Y. Choi, S. Kim, J. Lee, M. Martin *The Bishop-Phelps-Bollobás version of Lindenstrauss properties A and B*, preprint)

- *There exists a reflexive Banach space X such that the pair (X, X) does not satisfy the BPBp for operators.*
  Indeed, let Y be reflexive strictly convex space which is not uniformly convex and consider the reflexive space

$$X = l_1^2 \oplus_1 Y$$

- *Open Problem : In the real case it is not known whether or not the pair (c_0, l_1) has the BPBp for operators*

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(M Acosta,J, Becerra-Guerrero,Y.S. Choi, M. Ciesielski S. K. Kim, H.. Lee, M.L. Lourenço and M. Martín,)

The Bishop-Phelps-Bollobás property for operators between spaces of continuous functions, Nonlinear Analysis **95** *(2014),323-332.*

Theorem

Let K and S be compact Hausdorff topological spaces. Then the pair $(C(K), C(S))$ has the Bishop-Phelps-Bollobás property for operators.

Here $C(K)$ is the space of real valued continuous functions on K

This result improves a Theorem of [J. Johnson and J. Wolfe, Norm attaining operators, Studia Math. **65** *(1979), 7–19.*

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Proof: *Let us fix* $\frac{2}{3} < r < 1$. *Given* $0 < \varepsilon < 2$ *let us choose* $0 < \delta < \varepsilon^{\frac{1-r}{2}}$. *Assume that* $T_0 \in S_{\mathcal{L}(C(K),C(S))}$ *and* $f_0 \in S_{C(K)}$ *satisfy that* $\| T_0(f_0)\| > 1 - \frac{\delta^2}{12}$. *Then,* $\exists\, s_1 \in S$ *such that* $\big|\big[T_0(f_0)\big](s_1)\big| > 1 - \frac{\delta^2}{12}$.
We may assume that $T_0(f_0)(s_1) > 1 - \frac{\delta^2}{12}$.

For this operator T_0, *we can associated the* w^**-continuous function* $\mu_0 : S \longrightarrow M(K)$ $\mu_0(s) = T_0^*(\delta_s)$ *for every* $s \in S$. *and we can apply the following Lemma:*

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Lemma (1)

Let S and K be compact Hausdorff spaces Let $\mu_0 : S \longrightarrow M(K)$ be a w^*-continuous function satisfying $\|\mu_0\| = 1$ and $0 < \delta < 1$. Suppose that $s_0 \in S$ and $f_0 \in S_{C(K)}$ satisfy $\int_K f_0 \, d\mu_0(s_0) > 1 - \frac{\delta^2}{12}$. Then there exist a w^*-continuous mapping $\mu_1 : S \longrightarrow M(K)$, an open set U in S, an open set V of K and $h_0 \in C(K)$ satisfying the following conditions:

i) $|\mu_1(s)|(V) = 0$ *for every* $s \in U$.

ii) $\int_K h_0 \, d\mu_1(s) \geq \|\mu_1\| - \delta$ *for every* $s \in U$.

iii) $\|h_0 - f_0\| < \delta$.

iv) $\|h_0\| = 1$ *and* $|h_0(t)| = 1 \ \ \forall t \in K \setminus V$.

v) $\|\mu_1 - \mu_0\| < \delta$.

Lemma (J. Johnson and J. Wolfe)

Let S and K be compact Hausdorff spaces Let $\mu : S \longrightarrow M(K)$ be w^*-continuous and $\delta > 0$. Suppose there is an open set $U \subset S$, an open set $V \subset K$, $s_0 \in U$ and $h_0 \in C(K)$ with $\|h_0\| = 1$ such that

i) *if $s \in U$, then $|\mu(s)|(V) = 0$,*
ii) $\int_K h_0 \, d\mu(s_0) \geq \|\mu\| - \delta$,
iii) $|h_0(t)| = 1$ *for* $t \in K \setminus V$.

Then, for any $\frac{2}{3} < r < 1$ there exist a w^*-continuous function $\mu' : S \longrightarrow M(K)$ and a point $s_1 \in U$ such that

i) *if $s \in U$, then $|\mu'(s)|(V) = 0$,*
ii) $\int_K h_0 \, d\mu'(s_1) \geq \|\mu'\| - r\delta$,
iii) $\|\mu' - \mu\| \leq r\delta$.

From $\mu_1 : S \longrightarrow M(K)$ be w^*-continuous we have a sequence $\mu_n : S \longrightarrow M(K)$ of w^*-continuous functions and a sequence $\{s_n\}$ in U satisfying

$$\|\mu_{n+1} - \mu_n\| \leq r^n\delta, \ \|\mu_n\| \leq \int_K h_0 \, d\mu_n(s_n) + r^n\delta \text{ and } |\mu_n(s)|(V) = 0$$

for every $s \in U$ and $n \in \mathbb{N}$.
Associated with the function $\{\mu_n\}$ we have bounded linear operator $T_n \in \mathcal{L}(C(K), C(S)), \forall n$

$$\|T_{n+1} - T_n\| \leq r^n\delta \qquad \text{and} \qquad \|T_n\| \leq \|T_n(h_0)\| + r^n\delta. \tag{1}$$

the sequence (T_n) is Cauchy sequence , so it converges to an operator $T \in \mathcal{L}(C(K), C(S))$

$$\left\| \frac{T}{\|T\|} - T_0 \right\| < \varepsilon.$$

and

$$\|T\| = \|T(h_0)\|$$

$\|h_0 - f_0\| < \delta < \varepsilon$,
so the pair $(C(K), C(S))$ *satisfies the BPBp for operators.*

Theorem

$\mathcal{K}(C_0(L), Y)$ satisfies the BPBp for any locally compact Hausdorff topological space L and a uniformly convex Banach space Y.

$C_0(L)$ the space of continuous either real or complex valued functions on L vanishing at infinity.

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1. *M.D, Acosta, The Bishop-Phelps-Bollobás property for operators on $C(K)$, preprint May 2014.*

Theorem

The space $\mathcal{WC}(C_0(L), Y)$ satisfies the Bishop-Phelps-Bollobás property for any locally compact Hausdorff space L and any $\mathbb{C}$-uniformly convex space Y.

Corollary

In the complex case the pair $(C_0(L), L_p(\mu))$ has the BPBp for operator for every Hausdorff locally compact space L, every positive measure μ and $1 \leq p < \infty$.

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Theorem

The space $\mathcal{K}(X, Y)$ has the Bishop-Phelps-Bollobás property when X is an arbitrary Banach space and Y is a predual of an L_1-space in both the real and complex case.

- *J. Johnson and J. Wolfe, Norm attaining operators, Studia Math.* **65** *(1) (1979), 7–19.*
  1. *The set of norm-attaining finite-rank operators from an arbitrary Banach space into a predual of an L_1-space is dense in the space of compact operator.*

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  1. *The set of norm-attaining finite-rank operators from an arbitrary Banach space into a predual of an L_1-space is dense in the space of compact operator.*

Proof: *For $0 < \varepsilon < 1$ we take a convenient $\eta(\varepsilon) > 0$,*
Let $T \in S_{\mathcal{K}(X,Y)}$ and $x_0 \in S_X$ satisfying $\|Tx_0\| > 1 - \eta(\varepsilon)$
In view of a resutl of

(Lazar and Lindenstrauss (Banach spaces whose duals are L_1 spaces and their representing matrices, Acta Math 126 (1971), 165-193))

there is a subspace $E \subset Y$ isometric to ℓ_∞^n for some natural number n.
Let $P : Y \longrightarrow Y$ be a norm one projection onto E, and consider the operator $R : Y \longrightarrow E$ gven by i $R = \frac{P \circ T}{\|P \circ T\|}$. R satisfies $\|R(x_0)\| > 1 - \eta\left(\frac{\varepsilon}{2}\right)$.
Since E is isometric to ℓ_∞^n,

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Lemma

For every positive integer n and every Banach space X, the pair (X, ℓ_∞^n) has the BPBp for operators.

there exist an operator $R_1 \in \mathcal{L}(X, E) \subset \mathcal{L}(X, Y)$ with $\|R_1\| = 1$ and $z_0 \in S_X$ satisfying that

$$\|R_1 - R\| < \frac{\varepsilon}{2}, \qquad \|z_0 - x_0\| < \frac{\varepsilon}{2}, \quad \text{and} \quad \|R_1 z_0\| = 1.$$

It is possible to get

$$\|R_1 - T\| < \epsilon.$$

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GRACIAS!