# The Bishop-Phelps-Bollobás Property for Numerical Radius

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# Some notations

**Definition**

Let $X$ be a Banach space over a real or complex scalar filed $\mathbb{F}$ and $X^*$ be the dual space of $X$.

- $B_X$ is the unit ball of $X$. That is, $B_X = \{x \in X : \|x\| \leq 1\}$.
- $S_X$ is the unit sphere of $X$. That is, $S_X = \{x \in X : \|x\| = 1\}$.

**Definition**

Let $\mathcal{L}(X, Y)$ be the bounded linear operators from a Banach space $X$ to a Banach space $Y$. Then it is the Banach space equipped with

$$\|T\| = \sup\{\|Tx\| : x \in B_X\}.$$

A linear operator $T$ is bounded if and only if $T$ is continuous on $X$. It is well-knwon that $B_X$ is compact if and only if $X$ is finite dimensional.

# Norm-attaining functoinals

**Definition**

An element $x^* \in X^*$ is said to be *norm-attaining* if there is $x_0 \in S_X$ such that

$$|x^*(x_0)| = \|x^*\| := \sup\{|x^*(x)| : x \in B_X\}.$$

**Remark**

*In a finite dimensional space $X$, $B_X$ is compact and every bounded (continuous) linear functional attains its norm. That is, for all $x^* \in X^*$, there is $x_0 \in S_X$ such that*

$$|x^*(x_0)| = \max\{|x^*(x)| : x \in B_X\} = \|x^*\|.$$

# Bishop-Phelps Theorem and James Theorem

**Definition**

Let $X$ be a Banach space and let $\mathcal{N}\mathcal{A}(X^*)$ be the set of all bounded linear functionals attaining their norms.

**Theorem (Bishop-Phelps Theorem, Bull. Amer. Math. Soc., 1961)**

*$\mathcal{N}\mathcal{A}(X^*)$ is dense in $X^*$.*

**Theorem (James Theorem)**

*$\mathcal{N}\mathcal{A}(X^*) = X^*$ if and only if $X$ is reflexive, that is, $X^{**} = X$.*

# Bishop-Phelps Theorem

## Remark

- *A Bishop-Phelps Theorem implies the following:*
  *For each* $x^* \in X^*$ *and each* $\varepsilon > 0$, *there is an* $y^* \in X^*$ *such that* $\|y^*\| < \varepsilon$ *and* $x^* + y^*$ *attains it norm.*
- *Every bounded linear functional is approximated by norm-attaining ones.*

# Bishop-Phelps Property

## Definition

Let $\mathcal{L}(X, Y)$ be the space of all bounded linear operators from $X$ to $Y$. An operator $T \in \mathcal{L}(X, Y)$ is said to be *norm-attaining* if there is $x_0 \in S_X$ such that

$$\|Tx_0\| = \|T\| := \sup\{\|Tx\| : x \in B_X\}.$$

## Definition

A pair $(X, Y)$ is said to have the Bishop-Phelps property (BPp) if

$$\overline{NA(X, Y)} = \mathcal{L}(X, Y).$$

# Bishop-Phelps Problem

**Remark**

*The pair $(X, Y)$ has the BPp if and only if every bounded linear operator from $X$ to $Y$ can be approximated by norm-attaining operators.*

**Question (Bishop-Phelps problem, 1961)**

*Does any pair $(X, Y)$ of Banach spaces have the Bishop-Phelps property?*

Lindenstrauss gave a negative answer in 1963 and introduced the notion of property A.

**Theorem (Lindenstrauss, 1963)**

*There is a Banach space $Y$ such that $(L_1[0, 1], Y)$ does not have the BPp.*

# Lindenstrauss Property A and Reflexivity

**Definition (Lindenstrauss, 1963)**

A Banach space $X$ is said to have the *(Lindenstrauss) property A* if the pair $(X, Y)$ has the BPp for all Banach spaces $Y$. That is, for all Banach spaces $Y$,

$$\overline{NA(X, Y)} = \mathcal{L}(X, Y).$$

**Theorem (Lindenstrauss, 1963)**

- *Every reflexive space has the property A.*
- $L_1(\mu)$ *has the property A if and only if* $\mu$ *is purely atomic.*
- *Then* $C(K)$ *has the property A if and only if* $K$ *is finite.*

# Numerical Radius and Numerical Index

**Definition**

For a Banach space $X$, let $\mathcal{L}(X) := \mathcal{L}(X, X)$ and let

$$\Pi(X) = \{(x, x^*) : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}.$$

For each $T \in \mathcal{L}(X)$, the **numerical radius** $v$ is a semi-norm defined by

$$v(T) = \sup\{|x^* Tx| : (x, x^*) \in \Pi(X)\}$$

and the **numerical index** $n(X)$ of $X$ is defined by

$$n(X) = \sup\{v(T) : \|T\| = 1, T \in \mathcal{L}(X)\}.$$

Notice that if $n(X) > 0$, then the numerical radius $v(\cdot)$ is equivalent to the operator norm of $\mathcal{L}(X, Y)$. Moreover, if $n(X) = 1$, then the numerical radius is equal to the norm of $\mathcal{L}(X)$.

# Numerical Radius Attaining operators

## Definition

An operator $T \in \mathcal{L}(X)$ is said to be **numerical radius attaining** if there is $(x, x^*) \in \Pi(X)$ such that

$$v(T) = |x^* Tx|.$$

## Remark

*Suppose that a Banach space $X$ has the numerical index 1 and that the set of numerical radius-attaining operators is dense in $\mathcal{L}(X)$. Then $(X, X)$ has the Bishop-Phelps property. That is,*

$$\overline{NA(X, X)} = \mathcal{L}(X, X)$$

# Denseness of numerical radius attaining operators

**Question (Sims, 1972)**

*Is the set of all numerical radius-attaining operators dense in $\mathcal{L}(X)$?*

**Theorem (Berg and Sims, 1984)**

*The set of all numerical radius-attaining operators is dense in $\mathcal{L}(X)$ if $X$ is uniformly convex.*

**Theorem (Cardassi, 1985)**

*The set of all numerical radius-attaining operators is dense in $\mathcal{L}(X)$ if $X$ is either $\ell_1$, $c_0$, $L_1(\mu)$, uniformly smooth or $C(K)$ if $K$ is a compact metric space.*

# Denseness of numerical radius attaining operators

**Theorem (Acosta and Payá 1989)**

*The set of all numerical radius-attaining operators is dense in $\mathcal{L}(X)$ if $X$ has the Radon-Nikodým property.*

**Theorem (Acosta, 1990)**

*The set of all numerical radius-attaining operators is dense in $\mathcal{L}(X)$ if $X$ is a $C(K)$ space for a compact Hausdorff space $K$.*

**Theorem (Payá, 1992)**

*There is a Banach space $X$ such that $X$ is isomorphic to $c_0$ and the set of all numerical radius-attaining is not dense in $\mathcal{L}(X)$.*

The above result gave a negative answer to Sims' question.

# Bishop-Phelps-Bollobás Theorem

**Theorem (Bollobás, 1970)**

*Given* $\varepsilon > 0$, *if* $x^* \in S_{X^*}$ *and* $x \in S_X$ *satisfy* $|x^*(x)| > 1 - \frac{\varepsilon^2}{4}$, *then there exist* $y^* \in S_{X^*}$ *and* $y \in S_X$ *such that*

$$|y^*(y)| = 1, \quad \|x - y\| < \varepsilon \ \text{ and } \ \|x^* - y^*\| < \varepsilon.$$

A Bishop-Phelps-Bollobás theorem says that whenever a pair $(x, x^*) \in S_X \times S_{X^*}$ is close 1, there exist $(y, y^*)$ such that $y$ and $y^*$ are close to $x$ and $x^*$ respectively, and $y^*$ attains its norm at $y$.

# Bishop-Phelps-Bollobás Property

**Definition (Acosta, Aron, García and Maestre, 2008)**

The pair $(X, Y)$ has the *Bishop-Phelps-Bollobás property (BPBp)* if, given $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that if $T \in S_{L(X,Y)}$ and $x \in S_X$ satisfy $\|Tx\| > 1 - \delta(\varepsilon)$, then there exist $S \in S_{L(X,Y)}$ and $y \in S_X$ such that

$$\|Sy\| = 1, \quad \|x - y\| < \varepsilon \text{ and } \|S - T\| < \varepsilon.$$

It is clear that if $(X, Y)$ has the BPBp, then $(X, Y)$ has the BPp.

**Theorem (Acosta, Aron, García and Maestre, 2008)**

*Let $X$ be a strictly convex Banach space. Then $(\ell_1^{(2)}, X)$ has the BPBp if and only if $X$ is uniformly convex.*

# Bishop-Phelps-Bollobás Property for Numerical Radius

Guirao and Kozhushikina introduced the notion of the Bishop-Phelps-Bollobás property for numerical radius.

**Definition**

A Banach space $X$ is said to have the **Bishop-Phelps-Bollobás property for numerical radius** (with a function $\eta(\varepsilon)$) if, given $0 < \varepsilon < 1$, there is $\eta(\varepsilon) > 0$ such that whenever $T \in \mathcal{L}(X)$ and $(x, x^*) \in \Pi(X)$ satisfy that $v(T) = 1$ and $|x^* Tx| > 1 - \eta(\varepsilon)$, we have $S \in \mathcal{L}(X)$ and $(y, y^*) \in \Pi(X)$ such that

$$v(S) = |y^* Sy| = 1, \quad \|T - S\| < \varepsilon, \quad \|x - y\| < \varepsilon, \quad \text{and} \quad \|x^* - y^*\| < \varepsilon.$$

**Theorem (Guirao and Kozhushikina)**

*$\ell_1$ and $c_0$ have the nBPBp.*

**Theorem (Avilés, Guirao and Rodríguez, 2014)**

*A real $C(K)$ has the nBPBp for compact metrizable spaces $K$.*

**Theorem (Falcó, 2014)**

*$L_1(\mathbb{R})$ has the nBPBp.*

**Theorem (Kim, L and Martín, 2014)**

*If the space $X$ satisfies the one of the following conditions, then it has the nBPBp.*

- *$X$ is finite-dimensional.*
- *$X$ is a uniformly convex and uniformly smooth space with $n(X) > 0$.*
- *$X$ is an $L_1(\mu)$ space.*

**Remark**

*It is well-known that every complex Banach space has a strictly positive numerical index. However, for a real Hilbert space $H$ with dimension $\geq 2$, $n(H) = 0$. Very recently, we proved that the real Hilbert space $H$ has the nBPBp even though $n(H) = 0$.*

**Theorem (Kim, L and Martín, 2014)**

*If $L_1(\mu) \oplus_1 X$ has the nBPBp, then the pair $(L_1(\mu), X)$ has the BPBp for operators.*

The above result gives the following which was previously proved by Choi, Kim, Lee, Martín.

**Corollary (Choi, Kim, L and Martín, 2014)**

*A pair $(L_1(\mu), L_1(\nu))$ has the BPBp for measures $\mu$ and $\nu$.*

**Proof.**

Notice that $L_1(\mu) \oplus_1 L_1(\nu)$ is isometrically isomorphic to $L_1(\tilde{\mu})$ for some measure $\tilde{\mu}$. So it has the nBPBp. Therefore $(L_1(\mu), L_1(\nu))$ has the BPBp. □

# Some counterexamples

**Question**

*Is it true that $X$ has the nBPBp if the set of all numerical radius attaining operators is dense in $\mathcal{L}(X)$?*

The answer is no. From now on, we consider only real Banach spaces.

**Example**

Let $Y$ be a strictly convex, not uniformly convex reflexive space. Then Acosta, Aron, García and Maestre showed that $(\ell_1^{(2)}, Y)$ does not have the BPBp. So $X = \ell_1^{(2)} \oplus_1 Y$ does not have the nBPBp. However it is reflexive. So the set of numerical radius attaining operators is dense in $\mathcal{L}(X)$.

# Some conterexamples

Moreover, we obtained the following stronger result.

**Theorem**

*Every infinite-dimensional separable Banach space can be renormed to fail the nBPBp.*

In the proof, we used the following lemma which was shown to us by Vladimir Kadets.

**Lemma**

*Let $Y$ be an infinite-dimensional separable Banach space. Then the set of equivalent norms on $Y$ which are strictly convex and are not (locally) uniformly convex is dense in the set of all equivalent norms on $Y$ (with respect to the Banach-Mazur distance).*

# Continued Proof.

**Lemma (Finet, Martín and Payá, 2003)**

*The mapping from the set of all equivalent norms to its numerical index is continuous with respect to the Banach-Mazur distance and the image is a nontrivial interval.*

**Proof.**

Let $Y$ be a subspace of $X$ with codimension 2. Then there is a renorming $|\cdot|$ of $Y$ such that $n(Y, |\cdot|) > 0$ and $Y$ is not uniformly convex, but strictly convex. Now, the space $\widetilde{X} = \ell_1^{(2)} \oplus_1 (Y, |\cdot|)$ is an equivalent renorming of $X$ which does not have the nBPBp. Otherwise, $(\ell_1^{(2)}, (Y, |\cdot|))$ has the BPBp. This is a contradiction to the result of Acosta, Aron, García and Maestre. □

Thank You !